Self-Avoiding Walks

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The lattice random walks or Pólya walks were introduced by George Pólya around 1920. Here, a random walker moves on a regular grid, usually taken to be the hypercubic lattice. A self-avoiding walk is a lattice random walk with one additional condition: no point may be revisited. Random walks and self-avoiding walks have considerable intrinsic mathematical interest, and their study involves a surprisingly broad range of areas of mathematics, biology, chemistry and physics.

An n-step self-avoiding walk ω on the d-dimensional integer lattice $\mathbb{Z}^d$ is an ordered set $\omega = (\omega(0), \omega(1), \ldots, \omega(n))$, with each $\omega(i) \in \mathbb{Z}^d$, $|\omega(i+1) - \omega(i)| = 1$ (Euclidean distance), and $\omega(i) \neq \omega(j)$ for $i \neq j$. We always take $\omega(0) = (0, 0, \ldots, 0)$.

Obviously, on a d-dimensional lattice, the number of n-step random walks is $(2d)^n$. Denote by $c_d(n)$ the number of n-step self-avoiding walks on $\mathbb{Z}^d$, by convention, $c_0 = 1$. A fundamental question is how big is c_n? What is the exact formula for it? In one dimension the problem becomes trivial. In two or more dimensions it seems to be a very difficult problem.

An excellent exposition can be found in Madras and Slade [9]. Even the computation of $c_d(n)$ for small values of n is a formidable computational problem. For the square lattice, Conway and Guttmann [3] have counted the number of self-avoiding walks up to 51 steps. Later, Jensen [6] gave the enumeration of self-avoiding walks up to and including 71 steps. A recent breakthrough is Hara and Slade's [5] determination of the asymptotic behavior of $c_d(n)$ for dimensions $d > 4$.

It is known that $\lim_{n\to\infty} [c_d(n)]^{1/n}$ exists. This limit is called the self-avoiding walk connective constant, and is denoted by μ_d.

The current best rigorous ranges for μ are:

$$\begin{aligned}
\mu_2 &\in [2.62002, 2.679192495] \\
\mu_3 &\in [4.572140, 4.7476] \\
\mu_4 &\in [6.742945, 6.8179] \\
\mu_5 &\in [8.828529, 8.8602] \\
\mu_6 &\in [10.874038, 10.8886].
\end{aligned}$$

For $d = 2$ and 3, there exists a positive constant γ such that

$$\lim_{n\to\infty} \frac{c_d(n)}{\mu_d^n n^{\gamma-1}}$$

exists and is nonzero [1, 2, 9]. For $d > 4$, the above limit is conjectured to exist, with the critical exponent $\gamma = 1$ [9]. For $d = 4$, the limit

$$\lim_{n\to\infty} \frac{c_d(n)}{\mu_d^n n^{\gamma-1}(\ln n)^{1/4}}$$

is also conjectured to exist and to be finite. Moreover, it has been conjectured that

$$\gamma = \begin{cases} 43/32 & d = 2, \\ 1.162... & d = 3, \\ 1 & d = 4. \end{cases}$$

Another fundamental question concerns the scaling limit of the two dimensional self-avoiding walk. It is believed to be given by the Schramm-Loewner evolution (SLE) with the parameter κ equal to 8/3, see [7] for further details.

A further question of interest is the computation of the mean square displacement over all n-step self-avoiding walks, defined as

$$s_d(n) \equiv \frac{1}{c_d(n)} \sum_{\omega} |\omega(n)|^2,$$

where the sum is over all n-step self-avoiding walks ω.

Like $c_d(n)$, the following limits are believed to exist and be finite:

$$\begin{cases} \lim\limits_{n\to\infty} \frac{s_d(n)}{n^{2\nu}} & d \neq 4, \\ \\ \lim\limits_{n\to\infty} \frac{s_d(n)}{n^{2\nu}(\ln n)^{1/4}} & d = 4. \end{cases} \tag{1}$$

where the critical exponent $\nu = 1/2$ for $d > 4$ ([9]). Moreover, it has been conjectured that [1, 8, 9]

$$\nu = \begin{cases} 3/4 & d = 2, \\ 0.59... & d = 3, \\ 1/2 & d = 4. \end{cases}$$

The critical exponents γ and ν are thought to be universal in the sense that they are lattice-independent (although dimension-dependent). However, no one has yet discovered a proof of their existence, let alone a proof of universality.

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