The g-Conjecture for Spheres

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The g-conjecture for spheres is a conjectured complete characterization of the possible number of i-dimensional faces, $0 \leq i \leq d-1$, of a triangulation of a $(d-1)$-dimensional sphere (or $(d-1)$-sphere). An abstract simplicial complex Δ is said to be a triangulation of a $(d-1)$-sphere $\mathbb{S}^{d-1}$ if its geometric realization (as defined in topology, e.g., Munkres [7]) is homeomorphic to $\mathbb{S}^{d-1}$. Let f_i denote the number of i-dimensional faces of Δ for $0 \leq i \leq d-1$, with $f_{-1} = 1$. The *h-vector* $h(\Delta) = (h_0, h_1, \ldots, h_d)$ of Δ is defined by

$$\sum_{i=0}^{d} h_i x^{d-i} = \sum_{i=0}^{d} f_{i-1}(x-1)^{d-i}.$$

The *Dehn-Sommerville equations* assert that $h_i = h_{d-i}$ for any triangulation of $\mathbb{S}^{d-1}$. The *g-vector* $g(\Delta) = (g_0, g_1, \ldots, g_{\lfloor d/2 \rfloor})$ of Δ is defined by

$$g_0 = 1, \quad g_i = h_i - h_{i-1},\ 1 \leq i \leq \lfloor d/2 \rfloor.$$

Define a *multicomplex* to be a set Γ of nonnegative integer vectors $(a_1, a_2, \ldots, a_n)$ (for some n) such that if $(a_1, \ldots, a_n) \in \Gamma$ and $0 \leq b_i \leq a_i$, then $(b_1, \ldots, b_n) \in \Gamma$. The *degree* of the vector $(a_1, \ldots, a_n)$ is defined to be $\sum a_i$.

The g-conjecture for spheres. A vector $(g_0, g_1, \ldots, g_{\lfloor d/2 \rfloor})$ is the g-vector of a triangulation of $\mathbb{S}^{d-1}$ if and only if there exists a multicomplex Γ with exactly g_i vectors of degree i, $0 \leq i \leq \lfloor d/2 \rfloor$.

There is a complicated numerical characterization of the vectors $(g_0, g_1, \ldots, g_{\lfloor d/2 \rfloor})$ appearing in the g-conjecture that we omit here, see Stanley [10]. An important special class of triangulations of spheres are *simplicial polytopes*. These are convex polytopes whose proper faces are simplices, so that their boundary is a geometric realization of a triangulated sphere. It is known that there are triangulations of $\mathbb{S}^{d-1}$ for $d \geq 4$ that are not polytopal, i.e., do not come from simplicial polytopes. McMullen [5] first formulated the g-conjecture for simplicial polytopes, a bold conjecture since there was so little evidence. He was aware of the possibility that it might hold for spheres but was reluctant to publish such a general conjecture. Stanley [9] was the first to state explicitly the g-conjecture for spheres, and even a slightly more general statement known as the g-conjecture for Gorenstein* complexes. The g-conjecture for simplicial polytopes was proved by Billera and Lee [3] (sufficiency of the conjectured conditions) and Stanley [9] (necessity). The sufficiency for simplicial polytopes shows also the sufficiency for spheres, so only necessity remains to be proved. The proof of necessity for simplicial polytopes uses deep tools from algebraic geometry; McMullen [6] later gave a more elementary proof though still very algebraic.

The theory of f-vectors remains an active research area of algebraic combinatorics. Some important recent work includes [2, 4, 8]. For additional reading, see [1] and [10].

References

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