Log-concavity and q-Log-convexity Conjectures on the Longest Increasing Subsequences of Permutations

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Abstract. Let $P_{n,k}$ be the number of permutations π on $[n] = \{1, 2, \ldots, n\}$ such that the length of the longest increasing subsequences of π equals k, and let $M_{2n,k}$ be the number of matchings on $[2n]$ with crossing number k. Define $P_n(x) = \sum_k P_{n,k} x^k$ and $M_{2n}(x) = \sum_k M_{2n,k} x^k$. We propose some conjectures on the log-concavity and q-log-convexity of the polynomials $P_n(x)$ and $M_{2n}(x)$. We also introduce the notions of ∞-q-log-convexity and ∞-q-log-concavity, and the notion of higher order log-concavity with respect to ∞-q-log-convex or ∞-q-log-concavity. A conjecture on the ∞-q-log-convexity of the Boros-Moll polynomials is presented. It seems that $M_{2n}(x)$ are log-concave of any order with respect to ∞-q-log-convexity.

Keywords: crossing number, log-concavity, longest increasing subsequences, matching, nesting number, q-log-concavity, q-log-convexity, strong q-log-convexity, Boros-Moll polynomials.

AMS Classification: 05A20, 05E99

1 The Conjectures

Let $P_n(x)$ and $M_{2n}(x)$ be defined as in the abstract. We propose the following conjectures.

Conjecture 1.1 *$P_n(x)$ is log-concave for $n \geq 1$.*

Conjecture 1.2 *$P_n(x)$ is ∞-log-concave for $n \geq 1$.*

Conjecture 1.3 *The polynomial sequence $\{P_n(x)\}$ is strongly q-log-convex.*

Conjecture 1.4 *The polynomial sequence $\{P_n(x)\}$ is ∞-q-log-convex.*

Conjecture 1.5 *$M_{2n}(x)$ is log-concave for $n \geq 1$.*

Conjecture 1.6 *$M_{2n}(x)$ is ∞-log-concave for $n \geq 1$.*

Conjecture 1.7 *The polynomial sequence* $\{M_{2n}(x)\}$ *is strongly q-log-convex.*

Conjecture 1.8 *The polynomial sequence* $\{M_{2n}(x)\}$ *is* ∞*-q-log-convex. Furthermore, the polynomials* $M_{2n}(x)$ *are log-concave of any order with respect to* ∞*-q-log-convexity.*

The following conjecture is concerned with the Boros-Moll polynomials [3, 4]. The log-concavity is established by Kauser and Paule [11].

Conjecture 1.9 *The sequence of the Boros-Moll polynomials is* ∞*-q-log-convex, and they are log-concave of any order with respect to* ∞*-q-log-convexity.*

2 The Background

The longest increasing subsequences of permutations have been extensively studied; see, for example, [1, 2, 6, 8, 14], in particular, the survey of Stanley [20]. Baik, Deift and Johansson [1] have shown that the limiting distribution of the coefficients of $P_n(x)$ is the Tracy-Widom distribution. The numbers $P_{n,k}$ can be computed by Gessel's theorem [8]. Let $\mathfrak{S}_n$ be the symmetric group on $[n]$, and let $\mathrm{is}(\pi)$ be the length of the longest increasing subsequences of π. Define

$$u_k(n) = \sharp\{w \in \mathfrak{S}_n : \mathrm{is}(w) \leq k\}, \tag{2.1}$$

$$U_k(x) = \sum_{n\geq 0} u_k(n)\frac{x^{2n}}{n!^2}, \quad k \geq 1, \tag{2.2}$$

$$I_i(2x) = \sum_{n\geq 0} \frac{x^{2n+i}}{n!(n+i)!}, \quad i \in \mathbb{Z}. \tag{2.3}$$

Theorem 2.1

$$U_k(x) = \det(I_{i-j}(2x))_{i,j=1}^{k}. \tag{2.4}$$

Since $P_{n,k} = u_k(n) - u_{k-1}(n)$ for $n \geq 1$, we can use Gessel's theorem to compute $P_{n,k}$ for small n. Here we list $P_n(x)$ for $1 \leq n \leq 18$:

$P_1(x) = x,$

$P_2(x) = x + x^2,$

$P_3(x) = x + 4x^2 + x^3,$

$P_4(x) = x + 13x^2 + 9x^3 + x^4,$

$$P_5(x) = x + 41x^2 + 61x^3 + 16x^4 + x^5,$$

$$P_6(x) = x + 131x^2 + 381x^3 + 181x^4 + 25x^5 + x^6,$$

$$P_7(x) = x + 428x^2 + 2332x^3 + 1821x^4 + 421x^5 + 36x^6 + x^7,$$

$$P_8(x) = x + 1429x^2 + 14337x^3 + 17557x^4 + 6105x^5 + 841x^6 + 49x^7 + x^8$$

$$P_9(x) = x + 4861x^2 + 89497x^3 + 167449x^4 + 83029x^5 + 16465x^6 + 1513x^7 + 64x^8 + x^9,$$

$$P_{10}(x) = x + 16795x^2 + 569794x^3 + 1604098x^4 + 1100902x^5 + 296326x^6 + 38281x^7 + 2521x^8 + 81x^9 + x^{10}.$$

$$P_{11}(x) = x + 58785x^2 + 3704504x^3 + 15555398x^4 + 14516426x^5 + 5122877x^6 + 874886x^7 + 79861x^8 + 3961x^9 + 100x^{10} + x^{11}$$

$$P_{12}(x) = x + 208011x^2 + 24584693x^3 + 153315999x^4 + 192422979x^5 + 87116283x^6 + 18943343x^7 + 2250887x^8 + 153341x^9 + 5941x^{10} + 121x^{11} + x^{12}$$

$$P_{13}(x) = x + 742899x^2 + 166335677x^3 + 1538907306x^4 + 2579725656x^5 + 1477313976x^6 + 399080475x^7 + 59367101x^8 + 5213287x^9 + 275705x^{10} + 8581x^{11} + 144x^{12} + x^{13}$$

$$P_{14}(x) = x + 2674439x^2 + 1145533650x^3 + 15743413076x^4 + 35098717902x^5 + 25191909848x^6 + 8312317976x^7 + 1508071384x^8 + 164060352x^9 + 11110464x^{10} + 469925x^{11} + 12013x^{12} + 169x^{13} + x^{14}$$

$$P_{15}(x) = x + 9694844x^2 + 8017098273x^3 + 164161815768x^4 + 485534447114x^5 + 434119587475x^6 + 172912977525x^7 + 37558353900x^8 + 4927007100x^9 + 410474625x^{10} + 22128576x^{11} + 766221x^{12} + 16381x^{13} + 196x^{14} + x^{15}$$

$$P_{16}(x) = x + 35357669x^2 + 56928364553x^3 + 1744049683213x^4 + 6835409506841x^5 + 7583461369373x^6 + 3615907795025x^7 + 927716186325x^8 + 143938455225x^9 + 14353045401x^{10} + 947236425x^{11} + 41662441x^{12} + 1203441x^{13} + 21841x^{14} + 225x^{15} + x^{16}$$

$$P_{17}(x) = x + 129644789x^2 + 409558170361x^3 + 18865209953045x^4 + 97966603326993x^5 + 134533482045389x^6 + 76340522760097x^7$$

$$+ 22904111472825x^8 + 4142947526101x^9 + 484748595081x^{10} + 38094121561x^{11} + 2043822961x^{12} + 74797417x^{13} + 1830561x^{14} + 28561x^{15} + 256x^{16} + x^{17}.$$

$$P_{18}(x) = x + 477638699x^2 + 2981386305018x^3 + 207591285198178x^4 + 1429401763567226x^5 + 2426299018270338x^6 + 1631788075873114x^7 + 568209449266202x^8 + 118504614869214x^9 + 16029615164446x^{10} + 1470147102730x^{11} + 93574631242x^{12} + 4166173834x^{13} + 128922442x^{14} + 2708305x^{15} + 36721x^{16} + 289x^{17} + x^{18}.$$

One can check that $P_n(x)$ are log-concave for $1 \le n \le 18$. We now recall the notion of k-log-concavity; see, [11]. Define the operator $\mathcal{L}$ which maps a sequence $\{a_i\}$ of nonnegative numbers to a sequence $\{b_i\}$ given by

$$b_i := a_i^2 - a_{i-1}a_{i+1}.$$

Then the log-concavity of the sequence $\{a_i\}$ is defined by the positivity of $\mathcal{L}\{a_i\}$, namely, b_i is nonnegative for all i. If the sequence $\mathcal{L}\{a_i\}$ is not only positive but also log-concave, then we say that $\{a_i\}$ is 2-log-concave. In general, we say that $\{a_i\}$ is k-log-concave if $\mathcal{L}^k\{a_i\}$ is nonnegative, and that $\{a_i\}$ is ∞-log-concave if $\mathcal{L}^k\{a_i\}$ is nonnegative for every $k \ge 1$.

In fact, when $n \le 18$, we can find the sequence $\{P_{n,k}\}_{1\le k\le n}$ is 4-log-concave. This evidence leads us to surmise that the sequence $\{P_{n,k}\}_{1\le k\le n}$ is ∞-log-concave.

The q-log-concavity of polynomials has been studied by many authors including Butler [5], Krattenthaler [12], Leroux [13], and Sagan [15, 16]. Notice that here we have use x instead of q for the polynomials $P_n(x)$ and $M_{2n}(x)$. Following the notation of Sagan [16], given two polynomials $f(q)$ and $g(q)$ in q, we write

$$f(q) \ge_q g(q)$$

if the difference $f(q) - g(q)$ has nonnegative coefficients as a polynomial of q. A sequence of polynomials $\{f_k(q)\}_{k\ge0}$ over the field of real numbers is called q-log-concave if

$$f_m(q)^2 \ge_q f_{m+1}(q)f_{m-1}(q), \quad \text{for all } m \ge 1. \tag{2.5}$$

Liu and Wang [21] introduced the notion of q-log-convexity. A polynomial sequence $\{f_n(q)\}_{n\ge0}$ is called q-log-convex if

$$f_{m+1}(q)f_{m-1}(q) \ge_q f_m(q)^2, \quad \text{for all } m \ge 1. \tag{2.6}$$

A stronger property, called strong q-log-convexity, is introduced by Chen, Wang and Yang [7]. A polynomial sequence $\{f_n(q)\}_{n\geq 1}$ is called strongly q-log-convex if

$$f_{m-1}(q)f_{n+1}(q) \geq_q f_m(q)f_n(q), \qquad \text{for all } 1 \leq m \leq n. \tag{2.7}$$

When $1 \leq n \leq 17$, we find $P_{m-1}(x)P_{n+1}(x) \geq_x P_n(x)P_m(x)$.

Motivated by the notion of ∞-log-concavity, we define the operator $\mathcal{H}$ which maps a polynomial sequence $\{A_i(q)\}_{i\geq 0}$ to a polynomial sequence $\{B_i(q)\}_{i\geq 0}$ given by

$$B_i(q) := A_{i-1}(q)A_{i+1}(q) - A_i(q)^2.$$

Then the q-log-convexity of the polynomial sequence $\{A_i(q)\}$ is defined by the q-positivity of $\mathcal{H}\{A_i(q)\}$, namely, the coefficients of B_i are nonnegative for all i. If the polynomial sequence $\{B_i(q)\}$ is q-log-convex, then we say that $\{A_i(q)\}$ is 2-q-log-convex. In general, we say that $\{A_i(q)\}$ is k-q-log-convex if the coefficients of $\mathcal{H}^k\{A_i(q)\}$ are nonnegative, and that $\{A_i(q)\}$ is ∞-q-log-convex if $\mathcal{H}^k\{A_i(q)\}$ is nonnegative for every $k \geq 1$.

When $1 \leq n \leq 16$, we find the polynomial sequence $\{P_n(x)\}$ log-concave of order 3 with respect to 3-q-log-convex. This leads us to surmise the polynomial sequence $\{P_n(x)\}$ is ∞-q-log-convex.

We now give a brief review on how to compute the polynomials $M_{2n}(x)$. The crossing number of a matching on $[2n]$ is the maximum number k such that there are k mutually intersecting edges in the standard representation of the matching; see [6]. Let $v_k(n)$ denote the number of matchings on $[2n]$ whose crossing number is not greater than k. For example, the crossing number of a noncrossing matching equals one, since by "noncrossing" we really mean 2-noncrossing. Note that here we have used a slightly different notation from that in [20]. Let

$$V_k(x) = \sum_{n\geq 0} v_k(n)\frac{x^n}{n!}. \tag{2.8}$$

Grabiner and Magyar [10] derived the following matching analogue of Gessel's Theorem. The same formula has also been obtained by Goulden [9].

Theorem 2.2

$$V_k(x) = \det(I_{i-j}(2x) - I_{i+j}(2x))_{i,j=1}^k. \tag{2.9}$$

Applying the above theorem, we can compute $v_k(n)$ when n is small. Since $M_{2n,k} = v_k(n) - v_{k-1}(n)$, we obtain

$M_2(x) = x$

$$M_4(x) = 2x + x^2$$

$$M_6(x) = 5x + 9x^2 + x^3$$

$$M_8(x) = 14x + 70x^2 + 20x^3 + x^4$$

$$M_{10}(x) = 42x + 552x^2 + 315x^3 + 35x^4 + x^5$$

$$M_{12}(x) = 132x + 4587x^2 + 4730x^3 + 891x^4 + 54x^5 + x^6$$

$$M_{14}(x) = 429x + 40469x^2 + 71500x^3 + 20657x^4 + 2002x^5 + 77x^6 + x^7$$

$$M_{16}(x) = 1430x + 377806x^2 + 1110174x^3 + 468650x^4 + 64960x^5 + 3900x^6 + 104x^7 + x^8$$

$$M_{18}(x) = 4862x + 3707054x^2 + 17850170x^3 + 10717004x^4 + 2005830x^5 + 167484x^6 + 6885x^7 + 135x^8 + x^9$$

$$M_{20}(x) = 16796x + 37958960x^2 + 298110266x^3 + 250367036x^4 + 61205916x^5 + 6681255x^6 + 376770x^7 + 11305x^8 + 170x^9 + x^{10}$$

$$M_{22}(x) = 58786x + 403068470x^2 + 5174115036x^3 + 6012729626x^4 + 1881276355x^5 + 258507711x^6 + 18770290x^7 + 766535x^8 + 17556x^9 + 209x^{10} + x^{11}$$

$$M_{24}(x) = 208012x + 4414995268x^2 + 93255969556x^3 + 148847198843x^4 + 58846367560x^5 + 9929079622x^6 + 892328976x^7 + 46525941x^8 + 1443112x^9 + 26082x^{10} + 252x^{11} + x^{12}$$

$$M_{26}(x) = 742900x + 49670294000x^2 + 1742677176125x^3 + 3801821241675x^4 + 1883667666025x^5 + 383697949650x^6 + 41553867355x^7 + 2657363995x^8 + 104687375x^9 + 2553850x^{10} + 37375x^{11} + 299x^{12} + x^{13}$$

$$M_{28}(x) = 2674440x + 571944706335x^2 + 33696177453720x^3 + 100188554780355x^4 + 61882893062850x^5 + 15038660453130x^6 + 1925587971450x^7 + 146942256825x^8 + 7060951170x^9 + 218017800x^{10} + 4296474x^{11} + 51975x^{12} + 350x^{13} + x^{14}$$

$$M_{30}(x) = 9694845x + 6721306583805x^2 + 672654700490610x^3 + 2722638343622385x^4 + 2089360244433195x^5 + 600751303879170x^6 + 89678011487445x^7 + 8005505867775x^8 + 456368933475x^9 + 17125516044x^{10} + 426120345x^{11} + 6929405x^{12} + 70470x^{13} + 405x^{14} + x^{15}.$$

It is easily verified that $\{M_{2n,k}\}_{1\leq k\leq n}$ is 4-log-concave for $1 \leq n \leq 15$ and $\{M_n(x)\}_{1\leq n\leq 15}$ is strongly q-log-convex.

If the coefficients of the $\mathcal{H}\{A_i(q)\}$ are not only positive but also log-concave, then we say that $\{A_i(q)\}$ is log-concave of order two with respect to the 2-q-log-convexity. In general, we say that $\{a_i\}$ is log-concave of order k with respect to the k-q-log-convexity, if the all polynomials in $\mathcal{H}^k\{A_i(q)\}$ are log-concave, and that $\{A_i(q)\}$ is q-∞-log-concave if all the polynomials in $\mathcal{H}^k\{A_i(q)\}$ are log-concave for every $k \geq 1$.

When $1 \leq n \leq 14$, we find the polynomial sequence $\{M_{2n}(x)\}$ is q-3-log-concave. This leads us to surmise the polynomial sequence $\{M_{2n}(x)\}$ is q-∞-log-convex.

Finally, we recall the definition of the Boros-Moll polynomials which are a class of the Jacobi polynomials, and are also denoted by $P_n(a)$ as in [11]:

$$P_n(a) = \sum_{i=0}^{n} d_i(n)a^i, \tag{2.10}$$

where

$$d_i(n) = 2^{-2n} \sum_{k=i}^{n} 2^k \binom{2n-2k}{n-k}\binom{n+k}{k}\binom{k}{i}. \tag{2.11}$$

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