The q-Zeilberger Algorithm and the q-WZ Method for Infinite Series

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The Main Result

The main result of this paper is to extend the q-WZ method to prove nonterminating basic hypergeometric series. The essential ingredient is the q-Gosper algorithm. The motivation is the firm believe that the q-WZ method should be applicable to infinite series. The discovery is based on an observation that the Andrews-Warnnar identities have telescoping proofs.

Some Notation

We always assume $|q| < 1$. The q-shifted factorials $(a;q)_n$ and $(a;q)_\infty$ are defined by

$$
\begin{aligned}
(a;q)_n &= \begin{cases} 1, & \text{if } n = 0, \\ (1-a)\cdots(1-aq^{n-1}), & \text{if } n \geq 1, \end{cases} \\
(a;q)_{-n} &= \frac{1}{(aq^{-n};q)_n}, \qquad \text{if } n \geq 1, \\
(a;q)_\infty &= (1-a)(1-aq)(1-aq^2)\cdots,
\end{aligned}
$$

and $(a_1,\ldots,a_k;q)_n = (a_1;q)_n \cdots (a_k;q)_n$.

Some Notation

An ${}_r\phi_s$ *basic hypergeometric series* is defined by

$$
{}_r\phi_s\left[\begin{matrix} a_1, a_2, \ldots, a_r \\ b_1, b_2, \ldots, b_s \end{matrix}; q, z\right]
$$

$$
:= \sum_{n=0}^{\infty} \frac{(a_1, a_2, \ldots, a_r; q)_n}{(q, b_1, \ldots, b_s; q)_n} \left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r} z^n, \qquad (1)
$$

where $q \neq 0$ when $r > s+1$.

Some Notation

An ${}_r\psi_s$ *bilateral basic hypergeometric series* is defined by

$$
{}_r\phi_s\left[\begin{matrix} a_1,\ldots,a_r \\ b_1,\ldots,b_s \end{matrix};q,x\right]
:= \sum_{k=0}^{\infty} \frac{(a_1;q)_k\cdots(a_r;q)_k}{(b_1;q)_k\cdots(b_s;q)_k}\frac{x^k}{(q;q)_k}\left((-1)^k q^{\binom{k}{2}}\right)^{s-r+1}. \quad (2)
$$

It is assumed that q, z and the parameters are such that each term of the series is well-defined.

Hypergeometric Term

We say that t_k is a hypergeometric term if $\frac{t_{k+1}}{t_k}$ is a rational function in k, i.e.,

$$\frac{t_{k+1}}{t_k} = \frac{P(k)}{Q(k)},$$

where $P(k)$ and $Q(k)$ are polynomials in k.
For example,

$$t(k) = k!,\ a^k,\ \frac{(3k+1)!}{(5k+4)!}.$$

q-hypergeometric Term

Let $x = q^k$. We say that t_k is a q-hypergeometric term if

$$\frac{t_{k+1}}{t_k} = r(x),$$

where $r(x)$ is a rational function in x.
For example,

$$t_k = \frac{(a;q)_k}{(q;q)_k} z^k, \qquad k > 0.$$

where $(b;q)_k = (1-b)(1-bq)\cdots(1-bq^{k-1})$.

The Gosper's Algorithm

- The Gosper's algorithm is a milestone for proving hypergeometric identities. Zeilberger built up a powerful machinery based on the Gosper's algorithm.

  R.W. Gosper, Decision procedure for indefinite hypergeometric summation, Proc. Natl. Acad. Sci. USA **75(1)** (1978) 40–42.

- The q-Gosper algorithm is q-analogue of the Gosper's algorithm, which was introduced by Koornwinder.

  T.H. Koornwinder, On Zeilberger's algorithm and its q-analogue, J. Comput. Appl. Math. **48** (1993) 91–111.

The Zeilberger's Algorithm and the WZ-Method

- H.S. Wilf and D. Zeilberger developed the Zeilberger's algorithm and the WZ-method for proving identities on hypergeometric series.

  They were awarded the Steele Prize for Seminal Contribution (1998).

- Böing-Koepf and T.H. Koornwinder gave detailed discussions about the q-analogues.

The Package

There are many packages to implement the Gosper's algorithm, the Zeilberger's algorithm, the WZ-method and their q-analogues. We use

the package *qsum6.mpl* to implement the q-Gosper algorithm, which is maintained at the following site

http://www.mathematik.uni-kassel.de/ koepf/Publikationen/index.html#down

Zeilberger's Algorithm

The Zeilberger's algorithm uses the Gosper's algorithm to find polynomials $a_i(n)$ which are free of k and a hypergeometric term $G(n,k)$ such that

$$a_0(n)F(n,k) + a_1(n)F(n+1,k) + \cdots + a_d(n)F(n+d,k) \\ = G(n,k+1) - G(n,k).$$

Summing over k, we obtain a recursion:

$$a_0(n)f(n) + \cdots + a_d(n)f(n+d) = 0$$

where $f(n) = \sum_k F(n,k)$.

Applications of Zeilberger's algorithm

The identities are verified by finding recurrences and comparing initial values.

- Chinese identity (Li, Shan-Lan 1811–1882):

$$\sum_{k=0}^{n} \binom{n}{k}^2 \binom{m+2n-k}{2n} = \binom{m+n}{n}^2,$$

- Its q-analogue (Shi)

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}^2 \begin{bmatrix} m+2n-k \\ 2m \end{bmatrix} q^{k^2} = \begin{bmatrix} m+n \\ n \end{bmatrix}^2,$$

It is a special case of the q-Saalschütz identity.

Andrews-Warnaar Identities

We utilize the telescoping method to prove the following two identities

$$\left(\sum_{n=0}^{\infty}(-1)^n a^n q^{\binom{n}{2}}\right)\left(\sum_{n=0}^{\infty}(-1)^n b^n q^{\binom{n}{2}}\right)$$
$$= (q,a,b;q)_\infty \sum_{n=0}^{\infty} \frac{(abq^{n-1};q)_n}{(q,a,b;q)_n} q^n, \qquad (3)$$

and

Andrews-Warnaar Identities

$$1+\sum_{n=1}^{\infty}(-1)^n q^{\binom{n}{2}}(a^n+b^n)$$

$$= (a,b,q;q)_\infty \sum_{n=0}^{\infty} \frac{(ab/q;q)_{2n}}{(q,a,b,ab;q)_n} q^n, \qquad (4)$$

which were proved by Andrews and Warnaar.

G.E. Andrews and S. Ole Warnaar, The product of partial theta functions. Adv. Applied Math. **39** (2007) 116–120.

Telescoping for (3) and (4)

Proof. Let $g(a)$ and $f(a)$ denote the left side and right side of (3), respectively. Note that the left hand side contains two factors, one is concerned with a, and the other is involved with b. Comparing the coefficients of a^i, $i = 0, 1, 2, \ldots$, yields the iteration relation:

$$g(a) = (1 - a)g(aq) + aqg(aq^2). \tag{5}$$

However, it is not clear how to derive the same iteration relation for the right hand side, because it involves both a and b.

Telescoping for (3) and (4)

We will first verify that $f(a)$ satisfies the same iteration relation as $g(a)$ by constructing a telescoping relation, which gives a hint of the possibility that one can use the q-Gosper algorithm to achieve the same goal. This incidental observation was in fact the starting point of this paper. Let

$$f(a) = \sum_{n=0}^{\infty} D_n(a), \qquad D_n(a) = (q, a, b; q)_\infty \frac{(abq^{n-1}; q)_n q^n}{(q, a, b; q)_n}.$$

Telescoping for (3) and (4)

If we can find u_n satisfying

$$D_n(a) - (1-a)D_n(aq) - aqD_n(aq^2) = u_{n+1} - u_n$$

and

$$\lim_{n\to+\infty} u_n = u_0,$$

then sum n over the non-negative integers and show that $f(a)$ satisfies the same recurrence relation as $g(a)$. u_n can be solved by applying the q-Gasper algorithm to

$$D_n(a) - (1-a)D_n(aq) - aqD_n(aq^2).$$

Telescoping for (3) and (4)

In fact, we obtain

$$u_n = -\frac{(1-q^n)(1-bq^{n-1})(abq^n;q)_n(q,a,b;q)_\infty aq^n}{(1-aq^n)(1-abq^{2n-1})(q,a,b;q)_n}.$$

Let $G(a) = g(a) - f(a)$, then $G(a)$ satisfies the recurrence relation

$$G(a) = (1-q)G(aq) + aqG(aq^2).$$

Iterating the above identity, we find

$$G(a) = A_n G(aq^{n+1}) + B_n G(aq^{n+2}),$$

Telescoping for (3) and (4)

where

$$A_0 = (1-q), \qquad A_1 = (1-a)(1-aq) + aq,$$

$$B_0 = aq, \qquad B_1 = (1-a)aq^2,$$

$$A_{n+1} = (1-aq^{n+1})A_n + aq^{n+1}A_{n-1}, \qquad B_{n+1} = aq^{n+2}A_n \qquad n \geq 1.$$

We can rewrite the recurrence relation of A_n as

$$A_{n+1} - A_n = -aq^{n+1}(A_n - A_{n-1}).$$

Telescoping for (3) and (4)

Iterating the above identity, we have

$$|A_{n+1} - A_n| = |-aq^{n+1}(A_n - A_{n-1})| = \cdots$$

$$= |(-1)^n a^n q^{(n+1)!}(A_1 - A_0)|$$

$$\leq |(-1)^n a^n q^{(n+1)!}|(|A_1| + |A_0|).$$

So, for fixed a, q, the limit $\lim\limits_{n\to+\infty} A_n$ exists. Since $B_{n+1} = aq^{n+2}A_n$, then the limit $\lim\limits_{n\to+\infty} B_n$ exists.

Telescoping for (3) and (4)

It is easy to verify that $G(0) = g(0) - f(0) = 0$. So we can deduce that

$$G(a) = G(0)\left(\lim_{n\to+\infty} A_n + \lim_{n\to+\infty} B_n\right) = 0. \quad \blacksquare$$

Using a similar method, we can prove the identity (4).

The Key Idea

For the identities (3) and (4), the recurrence relations of their left hand sides are easy to establish. Although their right hand sides contain infinite q-shifted factorials, we can still use the q-Gosper algorithm to solve u_n.

The key idea is that we can deal with infinite q-shifted factorials. If we go through the procedure of the q-Gosper algorithm, we will see that the q-shifted factorials will result in rational functions after divisions.

The Key Idea

For example, for the identity (3), $D_n(q)$ contains infinite q-shifted factorials $(q, a, b; q)_\infty$, but the ratio

$$\frac{D_{n+1}(a) - (1-a)D_{n+1}(aq) - aqD_{n+1}(aq^2)}{D_n(a) - (1-a)D_n(aq) - aqD_n(aq^2)}$$

is a rational function in q^n, it follows that $D_n(q)$ is a q-hypergeometric term. Therefore, we can directly use the q-Gosper to solve u_n.

The Key Idea

As will be seen later, even if the recurrence relation of any side of an identity is not known, Chen, Hou and Mu provided a method to get the recurrence relations of both sides of the identity.

Examining the telescoping proofs of (3) and (4), we find that the q-Gosper algorithm does apply to q-hypergeometric terms which contain infinite q-shifted factorials. Consequently, the q-Zeilberger algorithm should not be afraid of infinite q-shifted factorials. This is indeed the idea for the q-WZ method for infinite q-series.

Nonterminating Series

Note that the q-Zeilberger algorithm can not be directly used to prove nonterminating basic hypergeometric identities. The reason is that the summand is not a bivariate q-hypergeometric term in the strict sense.

For nonterminating hypergeometric identities:

- the Gauss' summation formula (I.M. Gessel)
- Saalschütz summation formula (T.H. Koornwinder)

Chen-Hou-Mu's Method

Recently, Chen, Hou and Mu provided a systematic method for proving nonterminating basic hypergeometric identities by means of the q-Zeilberger algorithm.

W.Y.C. Chen, Q.H. Hou and Y.P. Mu, Nonterminating basic hypergeometric series and the q-Zeilberger algorithm, Proc. Edinburgh Math. Soc. to appear.

Chen-Hou-Mu's Method

Chen, Hou and Mu's method can be stated as follows. Let

$$f(x_1, x_2, \ldots, x_l) = \sum_{k=0}^{\infty} t_k(x_1, x_2, \ldots, x_l),$$

where $t_k(x_1, x_2, \ldots, x_l)$ is a q-hypergeometric term. They first set one or more parameters $x_1, x_2, \ldots, x_l$ to $x_1q^n, x_2q^n, \ldots, x_lq^n$. This is the key step, it makes the summand $t_k(x_1q^n, x_2q^n, \ldots, x_lq^n)$ become a bivariate q-hypergeometric term.

Chen-Hou-Mu's Method

Then the q-Zeilberger algorithm is utilized to obtain the recurrence relation. Once they obtain the recurrence relations of both sides, the following theorem is used to prove the identity.

Chen-Hou-Mu's Method

Theorem (Chen, Hou and Mu, 2007) *Suppose that*

$$f(z) = a_1(z)f(zq) + a_2(z)f(zq^2) + \cdots + a_d(z)f(zq^d),$$

and there exist $w_1, \ldots, w_d \in \mathbb{C}$ *and* $M > 0$ *such that*

$$|a_i(z) - w_i| \leq M|z|, \quad 1 \leq i \leq d,$$

and

$$|w_d| + |w_{d-1} + w_d| + \cdots + |w_2 + \cdots + w_d| < 1.$$

Then $f(z)$ *is uniquely determined by* $f(0)$ *and the functions* $a_i(z)$.

Chen-Hou-Mu's Method for (3) and (4)

The Andrews-Warnnar identities (3) and (4) can be proved by Chen, Hou and Mu's method.

For (3), set a to aq^n, we can get that fact that both sides of (3) satisfy the same recurrence relation. Utilizing the above theorem, we can verifty this identity.

For (4), we first assume that $|a| < 1$, using the same method, we can prove that (4) holds for $|a| < 1$. By analytic continuation, we may drop the assumption that $|a| < 1$.

The q-WZ method for Infinite Series

Now we have a version of the q-WZ method for infinite series, keeping in mind that the q-WZ method has the advantage of generating certificates to verify identities.

Our method can be described as follows. We aim to prove

$$\sum_{k=N_0}^{\infty} F_k(a_1, a_2, \ldots, a_l) = r(a_1, a_2, \ldots, a_l), \tag{6}$$

The q-WZ method for Infinite Series

where l is an positive integer, $N_0 = 0$ or $N_0 = -\infty$, $\sum_{k=N_0}^{\infty} F_k(a_1, a_2, \ldots, a_l)$ is the form of the right hand side of (1) or (2) and

$$r(a_1, a_2, \ldots, a_l) = \frac{\prod_{i=1}^{\gamma} (c_i(a_1, a_2, \ldots, a_l); q)_\infty}{\prod_{j=1}^{\lambda} (d_j(a_1, a_2, \ldots, a_l); q)_\infty},$$

$c_i(a_1, a_2, \ldots, a_l)$ and $d_j(a_1, a_2, \ldots, a_l)$ are functions decided by $a_1, a_2, \ldots, a_l$.

The q-WZ method for Infinite Series

First, we set some parameters $a_1, \ldots, a_p$, $1 \le p \le l$ (without loss of generality) to $a_1q^n, \ldots, a_pq^n$, i.e.,

$$\sum_{k=N_0}^{\infty} F_k(a_1q^n, \ldots, a_pq^n, a_{p+1}, \ldots, a_l)$$
$$= r(a_1q^n, \ldots, a_pq^n, a_{p+1}, \ldots, a_l).$$

We remark that the choice of the parameters $a_1, a_2, \ldots, a_p$ is made by human considerations. Nevertheless, there are not many choices for a few parameters.

The q-WZ method for Infinite Series

If $r(a_1q^n, \ldots, a_pq^n, a_{p+1}, \ldots, a_l) \neq 0$, then let

$$F(n,k) = \frac{F_k(a_1q^n, \ldots, a_pq^n, a_{p+1}, \ldots, a_l)}{r(a_1q^n, \ldots, a_pq^n, a_{p+1}, \ldots, a_l)}.$$

Otherwise, for $r(a_1q^n, \ldots, a_pq^n, a_{p+1}, \ldots, a_l) = 0$, put

$$F(n,k) = F_k(a_1q^n, \ldots, a_pq^n, a_{p+1}, \ldots, a_l).$$

The q-WZ method for Infinite Series

Once having established that

$$\sum_{k=N_0}^{\infty} F(n,k) = constant, \qquad n = 0, 1, 2, \ldots . \quad (7)$$

we can let $n = 0$ and select special cases of $a_1, a_2, \ldots, a_l$ to determine the constant. This completes the proof of identity (6).

The q-WZ method for Infinite Series

To use the q-WZ method, let $f(n)$ denote the left hand side of (7), i.e., $f(n) = \sum_{k=N_0}^{\infty} F(n,k)$. Then, we try to prove that $f(n+1) - f(n) = 0$ for every nonnegative integer n. One way to achieve this goal is find a function $G(n,k)$ such that

$$F(n+1,k) - F(n,k) = G(n,k+1) - G(n,k). \tag{8}$$

Then, sum both sides of (8) from $k = N_0$ to $+\infty$,

The q-WZ method for Infinite Series

under suitable hypotheses, we can show $f(n+1) - f(n) = 0$ for every nonnegative integer n. A pair of functions $(F(n,k), G(n,k))$ that satisfy (8) is called a *WZ pair*.

The question is how to find $G(n,k)$? In fact, it can be solved by applying the q-Gasper algorithm to $F(n+1,k) - F(n,k)$.

To do so, we shall show that $F(n+1,k) - F(n,k)$ is a q-hypergeometric term. If $r(a_1, \ldots, a_l) = 0$, obviously, $F(n+1,k) - F(n,k)$ is a q-hypergeometric term.

The q-WZ method for Infinite Series

If $r(a_1, \ldots, a_l) \neq 0$, then let

$$M_1 = \frac{r(a_1 q^{n+1}, \ldots, a_p q^{n+1}, a_{p+1}, \ldots, a_l)}{r(a_1 q^n, \ldots, a_p q^n, a_{p+1}, \ldots, a_l)},$$

$$M_2 = \frac{F_{k+1}(a_1 q^{n+1}, \ldots, a_p q^{n+1}, a_{p+1}, \ldots, a_l)}{F_k(a_1 q^{n+1}, \ldots, a_p q^{n+1}, a_{p+1}, \ldots, a_l)},$$

$$M_3 = \frac{F_{k+1}(a_1 q^n, \ldots, a_p q^n, a_{p+1}, \ldots, a_l)}{F_k(a_1 q^{n+1}, \ldots, a_p q^{n+1}, a_{p+1}, \ldots, a_l)},$$

$$M_4 = \frac{F_k(a_1 q^n, \ldots, a_p q^n, a_{p+1}, \ldots, a_l)}{F_k(a_1 q^{n+1}, \ldots, a_p q^{n+1}, a_{p+1}, \ldots, a_l)}.$$

The q-WZ method for Infinite Series

Since M_1 is a rational function in q^n and is independent on k, M_2, M_3, M_4 are rational functions in q^k, then

$$\frac{F(n+1,k+1)-F(n,k+1)}{F(n+1,k)-F(n,k)}=\frac{M_2-M_1M_3}{1-M_1M_4}$$

is a rational function in q^k, i.e.,
$F(n+1,k)-F(n,k)$ is a q-hypergeometric term.
So we can employ the q-Gosper algorithm to decide whether such a $G(n,k)$ exists or not.

The q-WZ method for Infinite Series

The following theorem is used to prove identities and discover new identities, which was provided by H.S. Wilf and D. Zeilberger.

H.S. Wilf and D. Zeilberger, Rational functions certify combinatorial identities, J. Amer. Math. Soc. **3(1)** (1990) 147–158.
In fact, we extension looks the same as the original q-WZ method. We just need to pretend to treat the infinite q-shifted factorials by the finite counterparts after some parameters a_i to $a_i q^n$.

The q-WZ method for Infinite Series

The following conditions are used in the theorem:

(C1) For each integer $n \geq 0$, $\lim_{k \to \pm\infty} G(n,k) = 0$.

(C2) For each integer k, the limit

$$f_k = \lim_{n \to \infty} F(n,k) \tag{9}$$

exists and is finite.

(C3) We have $\lim_{L \to \infty} \sum_{n \geq 0} G(n,-L) = 0$.

The q-WZ method for Infinite Series

Theorem **(H.S. Wilf and D. Zeilberger, 1990)** *Let* $(F(n,k), G(n,k))$ *satisfy* (8). *If (C1) holds then we have the identity*

$$\sum_k F(n,k) = constant, \qquad n = 0, 1, 2, \ldots. \tag{10}$$

If (C2) and (C3) hold then we have the identity (companion identity)

$$\sum_{n=0}^{\infty} G(n,k) = \sum_{j \le k-1} (f_j - F(0,j)), \tag{11}$$

where f_j *is defined by* (9).

q-Gauss sum

We give some examples.

Example 1. The q-Gauss sum is

$$\sum_{k=0}^{\infty} \frac{(a, b; q)_k}{(q, c; q)_k} \left(\frac{c}{ab}\right)^k = \frac{(c/a, c/b; q)_\infty}{(c, c/ab; q)_\infty},$$

where $|c/ab| < 1$.

q-Gauss sum

Consider the pair functions

$$F(n,k) = \frac{(aq^n, b; q)_k (cq^n, c/ab; q)_\infty}{(q, cq^n; q)_k (c/a, cq^n/b; q)_\infty} \left(\frac{c}{ab}\right)^k,$$

$$G(n,k) = -\frac{(a - aq^k)(aq^n, b; q)_k (cq^n, c/ab; q)_\infty}{(q, cq^n; q)_k (c/a, cq^n/b; q)_\infty (1 - aq^n)} \left(\frac{c}{ab}\right)^k q^n.$$

Since $|c/ab| < 1$, it is easy to verify that the two functions $(F(n,k), G(n,k))$ satisfy the relation (8) and conditions (C1), (C2) and $C(3)$.

q-Gauss sum

By (10), we have

$$\sum_{k=0}^{\infty} F(n,k) = constant, \qquad n = 0, 1, 2, \ldots .$$

In order to determine the constant, let $c = 0$ and $n = 0$, then the constant is 1, so we have

$$\sum_{k=0}^{\infty} F(0,k) = 1,$$

which is q-Gauss sum.

The Companion Identity of the q-Gauss sum

By (11), we obtain the companion identity of the q-Gauss sum is

$$\sum_{j=0}^{k}\frac{(a,b;q)_j}{(q,c;q)_j}\left(\frac{c}{ab}\right)^j=\frac{(c/b;q)_\infty}{(c;q)_\infty}\sum_{j=0}^{k}\frac{(b;q)_j}{(q;q)_j}\left(\frac{c}{ab}\right)^j$$

$$+\frac{(a,b;q)_{k+1}c^{k+1}}{(q;q)_k(c;q)_{k+1}a^kb^{k+1}}\sum_{n=0}^{\infty}\frac{(aq^{k+1},c/b;q)_n}{(a;q)_{n+1}(cq^{k+1};q)_n}q^n.$$

The ${}_6\phi_5$ Sum

Example 2.

The sum of a very-well-poised ${}_6\phi_5$ series is

$$\sum_{k=0}^{\infty} \frac{(1-aq^{2k})(a,b,c,d;q)_k}{(1-a)(q,aq/b,aq/c,aq/d)_k} \left(\frac{aq}{bcd}\right)^k$$

$$= \frac{(aq,aq/bc,aq/bd,aq/cd;q)_\infty}{(aq/b,aq/c,aq/d,aq/bcd;q)_\infty}, \qquad |aq/bcd| < 1. \tag{12}$$

The ${}_6\phi_5$ Sum

It is easy to check that when $|aq/bcd| < 1$ the two functions

$$F(n,k) = \frac{(1-aq^{n+2k})(aq^n, bq^n, c, d; q)_k}{(1-aq^n)(q, aq/b, aq^{n+1}/c, aq^{n+1}/d; q)_k} \times \frac{(aq/b, aq^{n+1}/c, aq^{n+1}/d, aq/bcd; q)_\infty}{(aq^{n+1}, aq/bc, aq/bd, aq^{n+1}/cd; q)_\infty} \left(\frac{aq}{bcd}\right)^k$$

The ${}_6\phi_5$ Sum

and

$$G(n,k) = \frac{(1-q^k)(c,d;q)_k(a/b,a/bcd;q)_\infty}{(aq^n/c,aq^n/d;q)_k(aq^n,aq^n/cd;q)_\infty} \times \frac{(aq^n,bq^n;q)_k(aq^n/c,aq^n/d;q)_\infty}{(a/bd,a/bc;q)_\infty(aq^{n+k}-c)(aq^{n+k}-d)} \times \frac{(a-bc)(a-bd)(aq^n-cd)}{(a/b,q;q)_k(a-bcd)(bq^n-1)}\left(\frac{aq}{bcd}\right)^k q^n$$

are a WZ-pair and satisfy the conditions (C1), (C2) and (C3).

The ${}_6\phi_5$ Sum

Then, by (10), $\sum_{k=0}^{\infty} F(n,k)$ is a constant. Let $n = 0$ and $a = 0$, so the constant is 1, then we have

$$\sum_{k=0}^{\infty} F(0,k) = constant = 1,$$

which completes the proof. By (11), we obtain the companion identity of (12)

The Companion Identity of ${}_6\phi_5$

$$\sum_{j=0}^{k}\frac{(1-aq^{2j})(a,b,c,d;q)_j}{(1-a)(q,aq/b,aq/c,aq/d;q)_j}\left(\frac{aq}{bcd}\right)^k$$

$$=\frac{(aq,aq/cd;q)_\infty}{(aq/c,aq/d;q)_\infty}\sum_{j=0}^{k}\frac{(c,d;q)_j}{(q,aq/b;q)_j}\left(\frac{aq}{bcd}\right)^j$$

$$+\frac{b(aq;q)_k(b,c,d;q)_{k+1}}{(q,aq/b;q)_k(aq/c,aq/d;q)_{k+1}}\left(\frac{aq}{bcd}\right)^{k+1}$$

$$\times\sum_{n=0}^{\infty}\frac{(aq^{k+1},bq^{k+1};q)_n(aq/cd;q)_n}{(b;q)_{n+1}(aq^{k+2}/c,aq^{k+2}/d;q)_n}q^n.$$

Ramanujan's ${}_1\psi_1$ Sum

Example 3. The Ramanujan's ${}_1\psi_1$ sum is

$$\sum_{k=-\infty}^{\infty} \frac{(a;q)_k}{(b;q)_k} z^k = \frac{(q, b/a, az, q/az; q)_\infty}{(b, q/a, z, b/a; q)_\infty},$$

where $|b/a| < |z| < 1$.

Ramanujan's ${}_1\psi_1$ Sum

The functions

$$F(n,k)=\frac{(aq^n;q)_k(bq^n,q^{1-n}/a,z,b/az;q)_\infty}{(bq^n;q)_k(q,b/a,azq^n,q^{1-n/az};q)_\infty}z^k,$$

$$G(n,k)=\frac{(aq^n;q)_k(bq^n,q^{-n}/a,z,b/az;q)_\infty(1-azq^n)}{(bq^n;q)_k(q,b/a,azq^n,q^{-n/az};q)_\infty(z-azq^n)}z^k$$

are a WZ pair. Since $|b/a|<|z|<1$, then we can verify that $G(n,k)$ satisfies the condition (C1), so by (10),

Ramanujan's ${}_1\psi_1$ Sum

$$\sum_{k=-\infty}^{\infty} F(n,k) = constant, \qquad n = 0, 1, 2, \ldots.$$

In order to determine the constant, let $n = 0$ and $b = q$ and utilize the q-binomial theorem, we get the constant is 1. Inserting the constant 1 and $n = 0$ into the above identity, we obtain the Ramanujan's ${}_1\psi_1$ sum.

Bailey's ${}_6\psi_6$ Sum

Example 4. The Bailey ${}_6\psi_6$ sum is

$$\sum_{k=-\infty}^{\infty} \frac{(1-aq^{2k})(b,c,d,e;q)_k}{(1-a)(aq/b,aq/c,aq/d,aq/e;q)_k} \left(\frac{a^2q}{bcde}\right)^k$$

$$= \frac{(aq,aq/bc,aq/bd,aq/be,aq/cd;q)_\infty}{(aq/b,aq/c,aq/d,aq/e,q/b;q)_\infty}$$

$$\times \frac{aq/ce,aq/de,q,q/a;q)_\infty}{(q/c,q/d,q/e,a^2q/bcde;q)_\infty}.$$

Bailey's ${}_6\psi_6$ Sum

The WZ-pair that works here is

$$F(n,k) = \frac{(1-aq^{n+2k})(aq/b, aq/c, aq^{n+1}/d; q)_\infty}{(1-aq^n)(aq^{n+1}, aq^{1-n}/bc, aq/be, aq/ce; q)_\infty}$$

$$\times \frac{(bq^n, cq^n, d, e; q)_k (aq^{n+1}/e, q^{1-n}/b; q)_\infty}{(aq/b, aq/c, aq^{n+1}/d, aq^{n+1}/e; q)_k (aq/bd; q)_\infty}$$

$$\times \frac{(q^{1-n}/c, q/d, q/e, a^2q/bcde; q)_\infty}{(aq/cd, aq^{n+1}/de, q, q^{1-n}/a; q)_\infty} \left(\frac{a^2q}{bcde}\right)^k$$

and

Bailey's ${}_6\psi_6$ Sum

$$G(n,k) = \frac{(bq^n, cq^n, d, e; q)_k (a/b, a/c; q)_\infty}{(1-bq^n)(1-cq^n)(a-ad)(1-e)(a^2-bcde)}$$
$$\times \frac{(aq^n/d, aq^n/e, q^{-n}/b, q^{-n}/c, a^2/bcde; q)_\infty}{(a/b, a/c; q)_k (a/bd, a/be, aq^n, aq^{-n}/bc; q)_\infty}$$
$$\times \frac{(-1+aq^n)(a-bd)(1/d, 1/e; q)_\infty}{(aq^{n+k}-d)(aq^n/d; q)_k (aq^{n+k}-e)(aq^n/e; q)_k}$$
$$\times \frac{(a-be)(a-cd)(a-ce)(aq^n-de)q^n}{(a/cd, a/ce, aq^n/de, q, q^{-n}/a; q)_\infty} \left(\frac{a^2 q}{bcde}\right)^k .$$

Bailey's ${}_6\psi_6$ Sum

Since $|a^2q/bcde| < 1$, we can verify that $G(n,k)$ satisfies the condition (C1), so by (10),

$$\sum_{k=-\infty}^{\infty} F(n,k) = constant, \qquad n = 0, 1, 2, \ldots . \quad (13)$$

In order to determine the constant, let $n = 0$ and $b = a$, by the the sum of a very-well-poised ${}_6\phi_5$ series (12), we have

Bailey's ${}_6\psi_6$ Sum

$$\sum_{k=0}^{\infty} \frac{(1-aq^{2k})(a,c,d,e;q)_k}{(1-a)(aq/c,aq/d,aq/e;q)_k}$$

$$\times \frac{(aq,aq/cd,aq/ce,aq/de;q)_\infty}{(aq/c,aq/d,aq/e,aq/cde;q)_\infty} \left(\frac{aq}{cde}\right)^k = 1.$$

Then, let $n = 0$, we have

$$\sum_{k=-\infty}^{\infty} F(0,k) = constant = 1,$$

which completes the proof.

Thank You