A Grammar of Dumont and a Theorem of Diaconis-Evans-Graham

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Abstract

We came across an unexpected connection between a remarkable grammar of Dumont for the joint distribution of $(\mathrm{exc}, \mathrm{fix})$ over S_n and a beautiful theorem of Diaconis-Evans-Graham on successions and fixed points of permutations. With the grammar in hand, we demonstrate the advantage of the grammatical calculus in deriving the generating functions, where the constant property plays a substantial role. On the grounds of left successions of a permutation, we present a grammatical treatment of the joint distribution investigated by Roselle. Moreover, we obtain a left succession analogue of the Diaconis-Evans-Graham theorem, exemplifying the idea of a grammar assisted bijection. The grammatical labelings give rise to an equidistribution of $(\mathrm{jump}, \mathrm{des})$ and $(\mathrm{exc}, \mathrm{drop})$ restricted to the set of left successions and the set of fixed points, where jump is defined to be the number of ascents minus the number of left successions.

Keywords: Context-free grammars, increasing binary trees, the Diaconis-Evans-Graham theorem, successions, fixed points.

AMS Classification: 05A15, 05A19

1 Introduction

This paper is concerned with a beautiful theorem of Diaconis-Evans-Graham [5] on the correspondence between successions and fixed points of permutations. Unlike a typical equidistribution property, an attractive feature of this theorem is that the bijection can be restricted to permutations with a specific set of successions and permutations with the same set of fixed points.

The topic of the enumeration of successions of permutations has a rich history. Dumont referred to the work of Roselle [9] on the joint distribution of the number of ascents and the number of successions. In fact, the grammar proposed by Dumont [6] is meant to deal with the joint distribution of the number of excedances, the number of drops and the number of fixed points of a permutation. His argument may be paraphrased in the language of a grammatical labeling of complete increasing binary trees. We will show that this grammar is related to the Diaconis-Evans-Graham theorem, even though it does not look so at first sight. It is worth mentioning that Dumont's citation to Roselle was not accurate; nevertheless, such an incident was somehow just to the point. Indeed, this work would not have come into being without the lucky pointer of Dumont.

First, we come to the realization that the grammar of Dumont can be adapted to a problem of Roselle. We just need to be more circumspect when it comes to the notion of a left succession, analogous to that of a left peak of a permutation. In the approach of Roselle, the consideration of a left succession at position 1 was considered informative for the computation of the generating function of interior successions. As for a left succession, one assumes that a zero is patched at the beginning of a permutation. In contrast to a left succession, a usual succession is called an interior succession.

Once the grammar is in place, a grammatical labeling is necessary in order to record a weighted counting of a combinatorial structure. A labeling scheme also makes it possible to carry out the grammatical calculus. We will show how the grammar of Dumont works for the joint distribution of $(\mathrm{exc}, \mathrm{fix})$. Furthermore, we give a different labeling scheme for permutations which shows that the same grammar of Dumont suits equally well for the joint distribution of $(\mathrm{jump}, \mathrm{lsuc})$, where jump and lsuc denote the number of jumps and the number of left successions of a permutation, respectively. It is no surprise that the constant property plays a substantial role in the grammatical calculus.

While the grammar is instrumental in establishing an equidistribution, it is not clear whether one can take a step forward in obtaining a Diaconis-Evans-Graham type theorem concerning a given set of left successions and the same set fixed points. Fortunately, the answer is yes. In fact, it is exactly where the idea of a grammar assisted bijection comes on the scene.

2 A grammar of Dumont

In this section, we recall a remarkable grammar of Dumont [6] for the joint distribution of the statistics $(\mathrm{exc}, \mathrm{drop}, \mathrm{fix})$ over S_n, the set of permutations of $[n] = \{1, 2, \ldots, n\}$, where $n \geq 1$. For a permutation $\sigma = \sigma_1 \cdots \sigma_n \in S_n$, an index $1 \leq i \leq n$ is called an excedance if $\sigma_i > i$, or a drop if $\sigma_i < i$, or a fixed point if $\sigma_i = i$. Clearly, n cannot be an excedance and 1 cannot be a drop. The number of excedances, the number of drops and the number of fixed points of σ are denoted by $\mathrm{exc}(\sigma)$, $\mathrm{drop}(\sigma)$ and $\mathrm{fix}(\sigma)$, respectively. A drop of a permutation is also called an anti-excedance.

The joint distribution of $(\mathrm{exc}, \mathrm{fix})$ was determined by Foata-Schützenberger [7], see also Shin-Zeng [10]. For $n \geq 1$, define

$$F_n(x, z) = \sum_{\sigma \in S_n} x^{\mathrm{exc}(\sigma)} z^{\mathrm{fix}(\sigma)}$$

and define $F_0(x, z) = 1$. Then

$$\sum_{n=0}^{\infty} F_n(x, z) \frac{t^n}{n!} = \frac{(1-x)e^{zt}}{e^{xt} - xe^t}. \tag{2.1}$$

Writing

$$F_n(x, y, z) = \sum_{\sigma \in S_n} x^{\mathrm{exc}(\sigma)} y^{\mathrm{drop}(\sigma)} z^{\mathrm{fix}(\sigma)}$$

and $F_0(x, y, z) = 1$, (2.1) can be converted into the homogeneous form

$$\sum_{n=0}^{\infty} F_n(x, y, z) \frac{t^n}{n!} = \frac{(y-x)e^{zt}}{ye^{xt} - xe^{yt}}. \tag{2.2}$$

Below are the first few values of $F_n(x, y, z)$:

$$\begin{aligned}
F_0(x, y, z) &= 1, \\
F_1(x, y, z) &= z, \\
F_2(x, y, z) &= xy + z^2, \\
F_3(x, y, z) &= 3xyz + xy^2 + x^2y + z^3, \\
F_4(x, y, z) &= 6xyz^2 + 4xy^2z + xy^3 + 4x^2yz + 7x^2y^2 + x^3y + z^4.
\end{aligned}$$

The grammar of Dumont reads

$$G = \{a \to az,\ z \to xy,\ x \to xy,\ y \to xy\}. \tag{2.3}$$

Let D be the formal derivative with respect to G, which can be expressed as a differential operator

$$az\frac{\partial}{\partial a}+xy\frac{\partial}{\partial z}+xy\frac{\partial}{\partial x}+xy\frac{\partial}{\partial y}.$$

Dumont [6] showed that the polynomials $F_n(x,y,z)$ can be generated by D.

Theorem 2.1 (Dumont)**.** *The following relation is valid for $n \geq 0$,*

$$D^n(a) = aF_n(x,y,z). \tag{2.4}$$

Dumont's argument can be understood as a description of the procedure of recursively generating permutations in the cycle notation. Recall that a cycle is written in such a way that the minimum element is at the beginning and the cycles of a permutation are arranged in the increasing order of the minimum elements. Here we give an explanation in the language of a grammatical labeling, which we call the (a,x,y,z)-labeling, both for permutations and for increasing binary trees.

Given a permutation σ of $[n]$, represent it in the cycle notation. Use a to signify the position where a new cycle may be formed. If i is in a 1-cycle, we label it by z. If (i,j) is an arc in the cycle notation, that is, $\sigma_i = j$, we label it by x if $i<j$, that is, i is an excedance, or by y if $i>j$, that is, i is a drop. Then an insertion of $n+1$ into σ can be formally described with the aid of the grammar rules.

For example, below is a permutation in the cycle notation, where the labels are placed after each element and the label a is placed at the end:

$$(1\,x\,8\,y\,4\,x\,9\,y\,6\,y)\ (2\,z)\ (3\,x\,5\,y)\ (7\,z)\ a. \tag{2.5}$$

Relying on the grammar, one can build a complete increasing binary tree to record the insertion process of generating a permutation of $[n+1]$ from a permutation of $[n]$, in the cycle notation, to be precise. To describe the procedure, we represent a cycle by arranging the minimum element at the beginning followed by a permutation of the remaining elements. Clearly, this permutation following the minimum element corresponds to a complete increasing binary tree, see, for example, Stanley [11, P. 23].

Now, we may represent a cycle by a planted complete increasing binary tree. First, designate the minimum element as the root. If the cycle contains only one element, then assign it a z-leaf. Otherwise, attach the complete increasing binary tree corresponding to the permutation following the minimum element as a subtree of the root. Note that the external leaves of the complete increasing binary tree comply with the (x,y)-labeling for the Eulerian polynomials, namely, a left leaf is labeled by x and a right leaf is labeled by y. For example, the cycles of the permutation in (2.5) are represented by the forest of planted complete increasing binary trees in Figure 1.

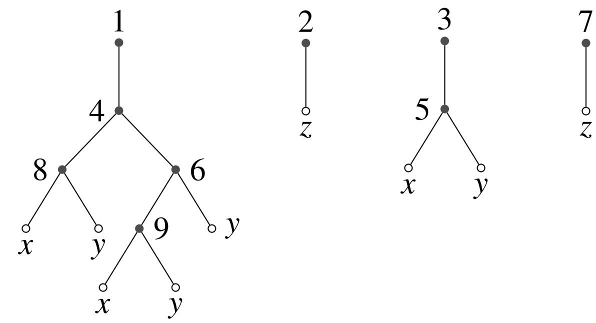

Figure 1: A forest of planted increasing binary trees.

As the last step, we can put together these planted increasing binary trees by drawing an edge between two roots next to each other to form a complete increasing binary tree with the (a,x,y,z)-labeling for which the root is 1 and the rightmost leaf is labeled by a.

For example, the forest in Figure 1 can be put together into a complete increasing binary tree with an (a,x,y,z)-labeling in Figure 2.

We observe the following properties.

- A z-leaf corresponds to a fixed point.
- An x-leaf corresponds to an excedance.
- A y-leaf corresponds to a drop.

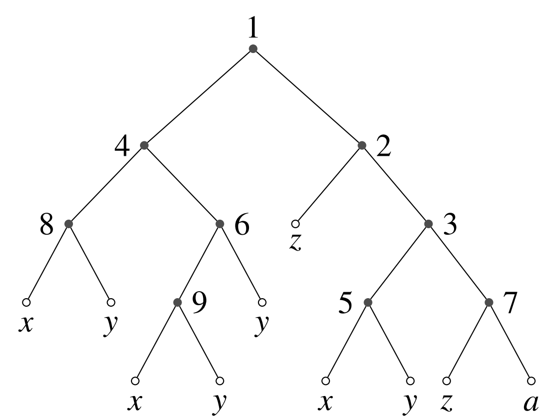

Figure 2: The (a,x,y,z)-labeling for $(\mathrm{exc}, \mathrm{drop}, \mathrm{fix})$.

To recover a permutation σ from a complete increasing binary tree T, we may decompose T into a forest of planted increasing binary trees by removing the edges from the root to the a-leaf and deleting the a-leaf.

The goal of this section is to show that the generating function of $F_n(x,y,z)$ can be easily derived by the grammatical calculus. A grammatical derivation of the generating function of the Eulerian polynomials $A_n(x,y)$ was given in [4]. The same reasoning can be carried over to the computation of the generating function of $F_n(x,y,z)$. Bear in mind that the generating function with respect to the formal derivative D is defined by

$$\mathrm{Gen}(w,t) = \sum_{n=0}^{\infty} D^n(w)\frac{t^n}{n!},$$

where w is a Laurent polynomial in the variables a,x,y,z. Note that the generating function with respect to D permits the multiplicative property, which is equivalent to the Leibniz rule, see [4] and references therein.

Theorem 2.2. *We have*

$$\mathrm{Gen}(a,t) = \frac{a(y-x)e^{zt}}{ye^{xt}-xe^{yt}}. \tag{2.6}$$

Proof. In virtue of the rules

$$x \to xy, \quad y \to xy,$$

we obtain the generating function

$$\mathrm{Gen}(x,t) = \frac{x-y}{1-yx^{-1}e^{(x-y)t}},$$

see [4]. Since $D(z-y) = xy - xy = 0$, i.e., $z-y$ is a constant relative to D, we deduce that

$$D^n(ax^{-1}) = D^{n-1}\left(ax^{-1}(z-y)\right) = ax^{-1}(z-y)^n,$$

and hence

$$\mathrm{Gen}(ax^{-1},t) = \sum_{n=0}^{\infty} D^n(ax^{-1})\frac{t^n}{n!} = ax^{-1}e^{(z-y)t}. \tag{2.7}$$

By the Leibniz rule or the product rule, we infer that

$$\mathrm{Gen}(a,t) = \mathrm{Gen}(x\cdot ax^{-1},t) = \mathrm{Gen}(x,t)\,\mathrm{Gen}(ax^{-1},t) = \frac{a(y-x)e^{zt}}{ye^{xt}-xe^{yt}},$$

as required. ∎

Putting $a=1$, we arrive at Equation (2.2). Furthermore, setting $z=0$ yields the generating function of the derangement polynomials; see Brenti [1].

3 The joint distribution of Roselle

In this section, we give an account of the generating function of Roselle [9] for the joint distribution of the number of ascents and the number of successions over S_n in a nutshell. Starting with recurrence relations, Roselle employed the symbolic method to accomplish the task of computation. Such an antiquate mechanism is rarely in demand these days, but perhaps it should not be completely forgotten, even though it seems obscure or dubious and even if its extinction may be inevitable.

3.1 The formulas of Roselle

Let us recall some definitions. Let $n \geq 1$, and let σ be a permutation of $[n]$. We assume that $\sigma_0 = 0$. An ascent or a rise of σ is an index $0 \leq i \leq n-1$ such that $\sigma_i < \sigma_{i+1}$. The number of ascents of σ is denoted by $\mathrm{asc}(\sigma)$. An index i $(1 \leq i \leq n-1)$ is called a descent of σ if $\sigma_i > \sigma_{i+1}$. In this definition, the index n is not counted as a descent. The number of descents of σ is denoted by $\mathrm{des}(\sigma)$. An index i $(1 \leq i \leq n-1)$ of σ is called a succession, or an interior succession, if $\sigma_i + 1 = \sigma_{i+1}$. We call an index i $(1 \leq i \leq n)$ a left succession if $\sigma_{i-1} + 1 = \sigma_i$. Mind the subtlety with respect to the range of indices for a left succession.

In order to single out ascents that are not left successions, we say that an index $1 \leq i \leq n$ of σ is a jump if $i-1$ is an ascent but i is not a left succession, that is, $\sigma_i \geq \sigma_{i-1} + 2$. The number of jumps of σ is denoted by $\mathrm{jump}(\sigma)$.

For $2 \leq i \leq n$, if i is a jump, then $i-1$ is called a big ascent by Ma-Qi-Yeh-Yeh [8], and the number of big ascents of σ is denoted by $\mathrm{basc}(\sigma)$. However, if 1 is a jump, it does not contribute to the counting of big ascents.

Let $P(n,r,s)$ denote the number of permutations of $[n]$ with r ascents and s (interior) successions. For example, $P(3,2,0) = 2$. The two permutations of $\{1,2,3\}$ with two ascents and no successions are 132, 213. Nevertheless, 132 has a left succession. While the term of a left succession is not manifestly put to use, one can find a clue through the generating function for the number of permutations of $[n]$ with r ascents and no left successions, see Roselle [9].

As to left successions, for $n \geq 1$, let $P^*(n,r,s)$ denote the number of permutations of $[n]$ with r ascents and s left successions. Define $P_0^*(x,z) = 1$ and define for $n \geq 1$,

$$P_n^*(x,z) = \sum_{r=1}^{n} \sum_{s=0}^{r} P^*(n,r,s) x^{r-s} z^s.$$

Since for $n \geq 1$ and for any permutation $\sigma \in S_n$.

$$\mathrm{asc}(\sigma) = \mathrm{jump}(\sigma) + \mathrm{lsuc}(\sigma),$$

we see that

$$P_n^*(x,z) = \sum_{\sigma \in S_n} x^{\mathrm{jump}(\sigma)} z^{\mathrm{lsuc}(\sigma)}.$$

The first few values of $P_n^*(x,z)$ are given below:

$$\begin{aligned}
P_0^*(x,z) &= 1, \\
P_1^*(x,z) &= z, \\
P_2^*(x,z) &= x + z^2, \\
P_3^*(x,z) &= x + x^2 + 3xz + z^3, \\
P_4^*(x,z) &= x + 7x^2 + x^3 + 4xz + 4x^2z + 6xz^2 + z^4.
\end{aligned}$$

Below is the generating function of $P_n^*(x,z)$.

Theorem 3.1 (Roselle)**.** *We have*

$$\sum_{n=0}^{\infty} P_n^*(x,z) \frac{t^n}{n!} = \frac{(1-x)e^{zt}}{e^{xt} - xe^t}. \tag{3.1}$$

Notice that this formula coincides with (2.1) for the joint distribution of $(\mathrm{exc}, \mathrm{fix})$. As will be seen, this is by no means a coincidence. We will encounter the same grammar in Section 3.2 and so we ought to have the same story.

Let us turn to the main theme of Roselle. Set $P_0(x,z) = 1$, and for $n \geq 1$ define

$$P_n(x,z) = \sum_{r=1}^{n} \sum_{s=0}^{r-1} P(n,r,s) x^r z^s, \tag{3.2}$$

or equivalently,

$$P_n(x,z) = \sum_{\sigma \in S_n} x^{\mathrm{asc}(\sigma)} z^{\mathrm{suc}(\sigma)}. \tag{3.3}$$

The first few values of $P_n(x,z)$ are given below:

$$\begin{aligned}
P_0(x,z) &= 1, \\
P_1(x,z) &= x, \\
P_2(x,z) &= x^2z + x, \\
P_3(x,z) &= x + 2x^2 + 2x^2z + x^3z^2, \\
P_4(x,z) &= x + 8x^2 + 2x^3 + 3x^2z + 6x^3z + 3x^3z^2 + x^4z^3.
\end{aligned}$$

As shown by Roselle, the polynomials $P_n^*(x,z)$ serve as a stepstone to compute $P_n(x,z)$.

Theorem 3.2 (Roselle). *For $n \geq 1$, we have*

$$P_n(x,z) = P_n^*(x,xz) + x(1-z)P_{n-1}^*(x,xz). \tag{3.4}$$

Combining the generating function of $P_n^*(x,z)$ and the above relation gives rise to the generating function of $P_n(x,z)$.

Corollary 3.3. *We have*

$$\sum_{n=0}^{\infty} P_{n+1}(x,z)\frac{t^n}{n!} = \frac{x(1-x)^2 e^{(xz+1)t}}{(e^{xt}-xe^t)^2}. \tag{3.5}$$

3.2 A grammatical labeling for left successions

As alluded by the grammar of Dumont, we tend to believe that the notion of a left succession should be considered as a legitimate object of the subject, but it does not seem to have gained enough recognition.

For $n \geq 1$, define

$$L_n(x,y,z) = \sum_{\sigma \in S_n} x^{\mathrm{jump}(\sigma)} y^{\mathrm{des}(\sigma)} z^{\mathrm{lsuc}(\sigma)}.$$

For $n = 0$, set $L_0(x,y,z) = 1$.

The following theorem shows that the polynomials $L_n(x,y,z)$ can be generated by the grammar G in (2.3) of Dumont, that is,

$$G = \{a \rightarrow az,\ z \rightarrow xy,\ x \rightarrow xy,\ y \rightarrow xy\}.$$

Theorem 3.4. *Let D be the formal derivative with respect to G. For $n \geq 0$, we have*

$$D^n(a) = aL_n(x,y,z). \tag{3.6}$$

The above theorem can be justified by a labeling scheme of permutations. Assume that $n \geq 1$ and σ is a permutation of $[n]$. Consider the position after each element σ_i for $i = 0,1,\ldots,n$, with $\sigma_0 = 0$. First of all, label the position after the maximum element n by a. Next, if $\sigma_n \neq n$, label the position after σ_n by y. For the remaining positions, if i is a jump, then label the position on the left of σ_i by x; if i is a left succession, then label the position on the left of σ_i by z, if i is a descent and $\sigma_i \neq n$, label the position on the right of σ_i by y. Below is an example:

$$0\ x\ 2\ x\ 6\ y\ 3\ z\ 4\ y\ 1\ x\ 5\ x\ 8\ z\ 9\ a\ 7\ y. \tag{3.7}$$

Write $*$ for the element $n+1$ to be inserted into σ. The change of labels can be described as follows. Assume that $*$ is to be inserted at the position between σ_i and σ_{i+1}, where $0 \leq i \leq n$.

1. If $*$ is inserted at a position a, that is, $\sigma_i = n$, then we get $n z * a \sigma_{i+1}$ in the neighborhood, this operation is captured by the rule $a \to az$.

2. If $*$ is inserted at a position x, then we see the update of σ: $\sigma_i x \sigma_{i+1} \to \sigma_i x * a \sigma_{i+1}$. In the meantime, the label a after n in σ, wherever it is, will be switched to y, because $*$ is not inserted after n. This change of labels is reflected by the rule $x \to xy$.

3. If $*$ is inserted at a position y, since $\sigma_i \neq n$, the update of σ can be described by $\sigma_i y \sigma_{i+1} \to \sigma_i x * a \sigma_{i+1}$. In the meantime, the label a after n in the labeling of σ, wherever it is, will be switched to y. This change of labels is governed by the rule $y \to xy$.

4. If $*$ is inserted at a position z, then we have the update $\sigma_i z \sigma_{i+1} \to \sigma_i x * a \sigma_{i+1}$. In the meantime, the label a after n in the labeling of σ, wherever it is, will be switched to y. This change of labels is in compliance with rule $z \to xy$.

We now have the same grammar for the two occasions. Thus we are furnished with an equidistribution.

Theorem 3.5. *For $n \geq 1$, the statistics* $(\mathrm{jump}, \mathrm{des}, \mathrm{lsuc})$ *and the statistics* $(\mathrm{exc}, \mathrm{drop}, \mathrm{fix})$ *are equidistributed over the set of permutations of* $[n]$.

In other words, the above theorem says that for $n \geq 0$,

$$F_n(x,y,z) = L_n(x,y,z). \tag{3.8}$$

In fact, we are going to pursue a stronger version of the above theorem, that is, a left succession analogue of the Diaconis-Evans-Graham theorem. While a grammar might be sufficient to guarantee an equidistribution of two sets of statistics, it does not tell us explicitly how to form a bijection. Nevertheless, there are occasions that the grammar can be a guideline for establishing a correspondence even under certain constraints. We will come back to this point in Section 4.

3.3 Back to interior successions

Returning to the original formulation of the joint distribution of Roselle, let $R_0(x,y,z) = 1$, and for $n \geq 1$, let

$$R_n(x,y,z) = \sum_{\sigma \in S_n} x^{\mathrm{jump}(\sigma)} y^{\mathrm{des}(\sigma)} z^{\mathrm{suc}(\sigma)}, \tag{3.9}$$

which we call the Roselle polynomials. The first few values of $R_n(x,y,z)$ are given below:

$$\begin{aligned}
R_0(x,y,z) &= 1,\\
R_1(x,y,z) &= 1,\\
R_2(x,y,z) &= xy+z,\\
R_3(x,y,z) &= xy+2xyz+xy^2+x^2y+z^2,\\
R_4(x,y,z) &= 3xyz+3xyz^2+xy^2+3xy^2z+xy^3+3x^2yz+x^2y+7x^2y^2+x^3y+z^3.
\end{aligned}$$

Using the same reasoning for the grammatical labeling for left successions together with a slight alternation of the grammar, a grammatical calculus can be carried out for the Roselle polynomials. Suppose that we are working with the grammar for left successions, but we would like to avoid 1 being counted as a left succession, which is labeled by z. This requirement can be easily met by turning to an additional label b as a substitute of the label z. That is to say, the rule $z \to xy$ should be recast as $b \to xy$. For example, we should start with the initial labeling $0b1a$ instead of $0z1a$. As for the original labels a,x,y,z, their roles will remain unchanged. Thus we meet with the mended grammar:

$$G = \{a \to az,\ b \to xy,\ x \to xy,\ y \to xy,\ z \to xy\}. \tag{3.10}$$

Let D be the formal derivative of G in (3.10). Then we have

$$D(ab) = abz + axy,$$

which is the sum of weights of the two permutations

$$0\ b\ 1\ z\ 2\ a, \quad 0\ x\ 2\ a\ 1\ y.$$

In general, the polynomials $R_n(x,y,z)$ can also be generated by the formal derivative D.

Theorem 3.6. *For $n \geq 1$, we have*

$$R_n(x,y,z) = D^{n-1}(ab)|_{a=1,b=1}. \tag{3.11}$$

The grammatical calculus shows that the generating function for the Roselle polynomials is essentially a product of the generating function of $L_n(x,y,z)$ and the generating function of the bivariate Eulerian polynomials.

Theorem 3.7. *We have*

$$\mathrm{Gen}(ab,t) = \frac{a(y-x)e^{zt}}{ye^{xt}-xe^{yt}}\left(\frac{x-y}{1-yx^{-1}e^{(x-y)t}} - x + b\right). \tag{3.12}$$

Proof. By the Leibniz rule, we get

$$\mathrm{Gen}(ab,t)=\sum_{n=0}^{\infty}D^n(ab)\frac{t^n}{n!}=\mathrm{Gen}(a,t)\mathrm{Gen}(b,t).$$

Since $D(b)=D(x)=xy$, it follows that

$$\mathrm{Gen}(b,t)=\mathrm{Gen}(x,t)-x+b=\frac{x-y}{1-yx^{-1}e^{(x-y)t}}-x+b,$$

which, together with Theorem 2.2, implies (3.12). ∎

Next we show that the generating function of $P_n(x,z)$ can be derived by using the grammatical calculus. Making substitutions in Theorem 3.6 gives the following relation.

Corollary 3.8. *For $n\geq 1$, we have*

$$P_n(x,z)=D^{n-1}(ab)|_{a=1,\,y=1,\,b=x,\,z=xz}\,. \tag{3.13}$$

Proof. Note that for any permutation σ of $[n]$, we have for $n\geq 1$,

$$1+\mathrm{jump}(\sigma)+\mathrm{suc}(\sigma)=\mathrm{asc}(\sigma). \tag{3.14}$$

By Theorem 3.6, we find that

$$\begin{aligned}D^{n-1}(ab)|_{a=1,\,y=1,\,b=x,\,z=xz} &= x\sum_{\sigma\in S_n}x^{\mathrm{jump}(\sigma)}(xz)^{\,\mathrm{suc}(\sigma)}\\ &= \sum_{\sigma\in S_n}x^{\mathrm{asc}(\sigma)}z^{\mathrm{suc}(\sigma)},\end{aligned}$$

as required. ∎

The above relation enables us to deduce the generating function of $P_n(x,z)$ from that of $R_n(x,y,z)$, that is,

$$\sum_{n=0}^{\infty}P_{n+1}(x,z)\frac{t^n}{n!}=\mathrm{Gen}(ab,t)|_{a=1,\,y=1,\,b=x,\,z=xz}=\frac{x(1-x)^2e^{(xz+1)t}}{(e^{xt}-xe^t)^2},$$

which is in accordance with (3.5).

We finish this section with a relation between $R_n(x,y,z)$ and $L_n(x,y,z)$, which can be readily verified by the grammatical calculus.

Theorem 3.9. *For $n\geq 0$, we have*

$$R_{n+1}(x,y,z)=L_n(x,y,z)+\sum_{k=1}^{n}\binom{n}{k}A_k(x,y)L_{n-k}(x,y,z), \tag{3.15}$$

where for $k\geq 1$, $A_k(x,y)$ are the bivariate Eulerian polynomials.

This relation also admits a combinatorial interpretation. Let T be a complete increasing binary tree on $[n+1]$. Suppose that we wish to interpret $R_{n+1}(x,y,z)$ in terms of complete increasing binary trees. We may adopt the following labeling for $L_{n+1}(x,y,z)$, except that if the root of T has a z-leaf, we should label it by 1 rather than z. If this is the case, then the right subtree of T can be viewed as a complete increasing tree on $[n]$ with a labeling, which contributes a term to $L_n(x,y,z)$. If the root of T has a nonempty left subtree, then this left subtree does not have any z-leaves, which can be reckoned as a labeling for the Eulerian polynomials, and so we are through.

4 An analogue of the Diaconis-Evans-Graham theorem

The main result of this paper is a left succession analogue of the Diaconis-Evan-Graham theorem. The grammar of Dumont can be utilized to produce a bijection from permutations with a given set of left successions to permutations with the same set of fixed points, which possesses an additional equidistribution property concerning $(\mathrm{jump}, \mathrm{des})$ and $(\mathrm{exc}, \mathrm{drop})$.

For $n \geq 1$ and a permutation $\sigma \in S_n$, define

$$\begin{aligned} M(\sigma) &= \{i \,|\, 1 \leq i \leq n-1,\ \sigma_i + 1 = \sigma_{i+1}\}, \\ G(\sigma) &= \{i \,|\, 1 \leq i \leq n-1,\ \sigma_i = i\}, \\ F(\sigma) &= \{i \,|\, 1 \leq i \leq n,\ \sigma_i = i\}. \end{aligned}$$

It should be noted that the index n is not taken into consideration in the definition of $G(\sigma)$. Given a subset $I \subseteq [n-1]$, denote by $M_n(I)$ the set of permutations of $[n]$ with I being the set of (interior) successions, and denote by $G_n(I)$ the set of permutations $\sigma \in S_n$ such that $G(\sigma) = I$. Similarly, $F_n(I)$ denotes the set of permutations σ of $[n]$ such that $F(\sigma) = I$.

Theorem 4.1 (Diaconis-Evans-Graham)**.** *Let $n \geq 1$ and $I \subseteq [n-1]$. Then there is a bijection between $M_n(I)$ and $G_n(I)$.*

For the special case $I = \emptyset$, a permutation without successions is called a relative derangement. Let D_n denote the number of derangements of $[n]$, and let Q_n denote the number of relative derangements of $[n]$. Roselle [9] and Brualdi [2] deduced that

$$Q_n = D_n + D_{n-1}. \tag{4.1}$$

A bijective proof of this relation was given in [3], appealing to the first fundamental transformation. Taking $I = \emptyset$, a permutation in $G_n(I)$ may or may not have n as a fixed point.

The permutations in these two cases are counted by D_{n-1} and D_n, respectively. Thus the $I = \emptyset$ case of the proof of the Diaconis-Evans-Graham theorem reduces to a combinatorial interpretation of (4.1).

Here comes the question of what happens for left successions. To fit in the picture of a grammar assisted bijection, we find it more convenient to work with a variant or a reformulation of the Diaconis-Evans-Graham theorem. Assume that $n \geq 1$ and $\sigma \in S_n$. Define

$$\overline{M}(\sigma) = \{\sigma_i \,|\, 1 \leq i \leq n-1,\ \sigma_i + 1 = \sigma_{i+1}\}.$$

It is readily seen that for any $\sigma \in S_n$,

$$\overline{M}(\sigma^{-1}) = M(\sigma), \tag{4.2}$$

$$G(\sigma^{-1}) = G(\sigma), \tag{4.3}$$

$$F(\sigma^{-1}) = F(\sigma). \tag{4.4}$$

where σ^{-1} stands for the inverse of σ.

Similar to the notation $\overline{M}(\sigma)$, for $n \geq 1$ and a subset $I \subseteq [n-1]$, we define $\overline{M}_n(I)$ to be the set of permutations $\sigma \in S_n$ such that $\overline{M}(\sigma) = I$. Then Theorem 4.1 can be reformulated as follows.

Theorem 4.2. *Let $n \geq 1$ and $I \subseteq [n-1]$. There is a bijection between $\overline{M}_n(I)$ and $G_n(I)$.*

As a left succession analogue of $\overline{M}(\sigma)$, for $n \geq 1$ and a permutation σ of $[n]$, we define

$$\overline{L}(\sigma) = \{\sigma_i \,|\, 1 \leq i \leq n,\ \sigma_{i-1} + 1 = \sigma_i\}.$$

For a subset I of $[n]$, define $\overline{L}_n(I)$ to be the set of permutations σ of S_n such that $\overline{L}(\sigma) = I$.

Theorem 4.3. *For $n \geq 1$ and any $I \subseteq [n]$, there is a bijection Φ from $\overline{L}_n(I)$ to $F_n(I)$ that maps* $(\mathrm{jump}, \mathrm{des})$ *to* $(\mathrm{exc}, \mathrm{drop})$.

Proof. Given a permutation $\sigma = \sigma_1 \cdots \sigma_n \in \overline{L}_n(I)$, we wish to construct a complete increasing binary T with the (a,x,y,z)-labeling such that $\sigma_i \in \overline{L}(\sigma)$ if and only if the vertex σ_i has a z-leaf in T. Once the correspondence is established, the equidistribution property can be deduced from the interpretations of the labelings.

The map can be described as a recursive procedure. For $n = 1$, the permutation $z1a$ is mapped to the complete increasing tree having one internal vertex 1 with a left z-leaf and a right a-leaf.

We now assume that $n \geq 1$ and that $\sigma = \sigma_1\sigma_2\cdots\sigma_n$ is a permutation of $[n]$. As the induction hypothesis, we assume that T is the tree corresponding to σ. For $1 \leq i \leq n$, the position i is referred to the position immediately before σ_i, whereas the position $n+1$ is meant to be the position after σ_n.

To keep the procedure running, we need to maintain additional properties of σ and T. Besides having the same weight, they should be synchronized in a certain sense. To be more specific, we say that the labeling of σ is coherent with the labeling of T provided that the following conditions are satisfied. In fact, these properties can be assured after each update.

- If the position i in σ is labeled by x, then the vertex σ_i in T has a x-leaf;
- If the position i in σ is labeled by y, then the vertex $\sigma_{i-1}+1$ in T has a y-leaf;
- If the position i in σ is labeled by z, then the vertex σ_i in T has a z-leaf.

Suppose that $* = n+1$ is to be inserted into σ. It is necessary to find out how to update the tree T accordingly. Now that there are $n+1$ (insertion) positions of σ and there are $n+1$ leaves of T, it suffices to define a map from the set of positions to the set of leaves of T with the understanding that when $*$ is inserted at position, say i, T will be updated to T' by turning the corresponding leaf of T into an internal vertex $*$. Denote by σ' the permutation produced from σ by inserting $*$ at the position i. There are four cases with regard to the four rules of the grammar.

1. If $*$ is inserted at a position labeled by a, we add $*$ to T at the position of the a-leaf. This operation is consistent with the rule $a \to az$.
2. For a label z at the position i, by the induction hypothesis, we know that the vertex σ_i in T has a z-leaf, so we can apply the rule $z \to xy$ to this z-leaf to update T. Notice that when $*$ is inserted, the label a on the right of n in σ will be switched to y. Observe that this y-label corresponds to the y-leaf of $*$ in T'. By inspection, we see that the labeling of σ' is coherent with the labeling of T'.
3. When the insertion occurs at position i labeled by x, by the induction hypothesis, we know that the vertex σ_i in T has a x-leaf. Then we apply the rule $x \to xy$ to this leaf. Notice that the y-leaf of $*$ in T' corresponds to the y-label on the right of n. Again, it can be seen that the labeling of σ' is coherent with the labeling of T'.
4. For a position i labeled by y, by the induction hypothesis, we know that the vertex $\sigma_{i-1}+1$ in T has a y-leaf. Then we can apply the rule $y \to xy$ to this leaf. In this case, the labeling of σ' remains coherent with the labeling of T'.

So far we have provided a procedure to update T depending on where the element $*$ is inserted into σ. Moreover, every stage of this procedure is reversible. The detailed examination is omitted. As the grammar ensures that the map is weight-preserving, that is, the weight of σ equals that of T.

It should be added that a left succession, the element σ_i for which $\sigma_{i-1}+1=\sigma_i$, to be precise, is created in σ whenever a vertex σ_i with a left z-leaf is created in T. Meanwhile, a left succession σ_i is destroyed in σ whenever a z-leaf with parent σ_i is destroyed.

It should also be noted that a jump σ_i for which $\sigma_{i-1}+2 \le \sigma_i$ is created in σ whenever a vertex σ_i with a left x-leaf is created in T. Meanwhile, a jump σ_i is destroyed in σ whenever a left x-leaf with parent σ_i is destroyed.

Since we have employed the cycle notation of a permutation, a vertex σ_i with a left z-leaf corresponds to a fixed point of a permutation, and an x-leaf corresponds to an excedance, that is, an element σ_i such that $\sigma_i > i$. This completes the proof. ∎

Figure 3 illustrates how to build the corresponding trees step by step, where an underlined label indicates where an insertion takes place.

For $n=3$, the correspondence is given in the table below. The cases when $\overline{L}_n(I)=\emptyset$ or $F_n(I)=\emptyset$ are not listed, such as $I=\{1,2\}$.

$I \subseteq [n]$	$\overline{L}_n(I)$	$F_n(I)$	(jump, des) of $\overline{L}_n(I) \leftrightarrow$ (exc, drop) of $F_n(I)$
$\emptyset$	2 1 3	$(1\,2\,3)$	$(2,1)$
	3 2 1	$(1\,3\,2)$	$(1,2)$
$\{1\}$	1 3 2	$(1)(2\,3)$	$(1,1)$
$\{2\}$	3 1 2	$(1\,3)(2)$	$(1,1)$
$\{3\}$	2 3 1	$(1\,2)(3)$	$(1,1)$
$\{1,2,3\}$	1 2 3	$(1)(2)(3)$	$(0,0)$

To conclude, we remark that the above grammar assisted bijection permits a refined equidistribution property in terms of set-valued statistics. As shown in [8], a grammar may be a helpful platform to deal with set-valued statistics. Roughly speaking, the above grammar assisted bijection maps elements associated with the x-labels in a permutation to elements associated with the x-labels in a complete increasing binary tree. More precisely, let

$$\overline{\mathrm{Jump}}(\sigma) = \{\sigma_i \,|\, 1 \le i \le n,\ \sigma_{i-1}+2 \le \sigma_i\},$$

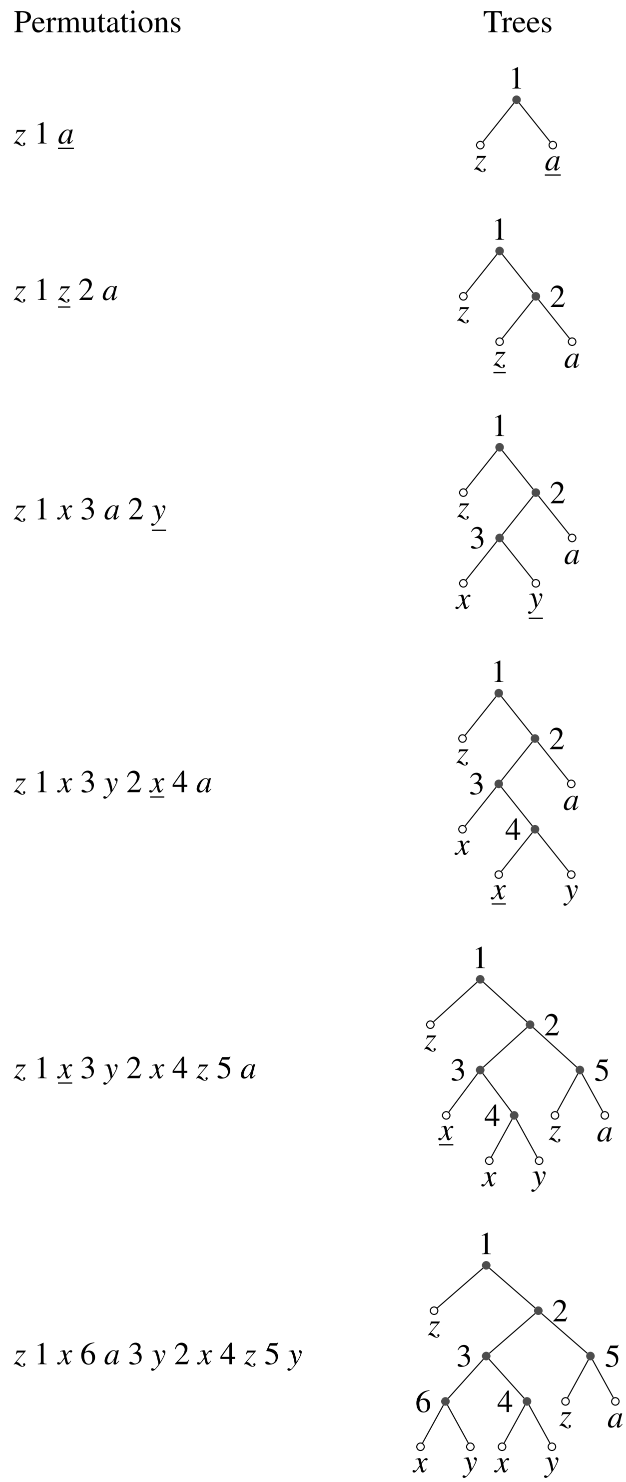

Figure 3: An example.

$$\overline{\mathrm{Exc}}(\sigma) \quad = \quad \{\sigma_i \,|\, 1 \leq i \leq n,\ \sigma_i > i\}.$$

In other words, the set $\overline{\mathrm{Jump}}(\sigma)$ consists of elements to the right of the x-labels of σ, whereas the elements in $\overline{\mathrm{Exc}}(\sigma)$ are exactly the vertices having an x-leaf in a complete increasing binary tree. Thus for the bijection Φ in the theorem and for any permutation σ of S_n, we have

$$\overline{\mathrm{Jump}}(\sigma) = \overline{\mathrm{Exc}}(\Phi(\sigma)). \tag{4.5}$$

For example, let $\sigma = 163245$. Then we have $\Phi(\sigma) = (1)(2634)(5)$. It is readily checked that

$$\overline{L}(\sigma) = F(\Phi(\sigma)) = \{1,5\}$$

and

$$\overline{\mathrm{Jump}}(\sigma) = \overline{\mathrm{Exc}}(\Phi(\sigma)) = \{4,6\}.$$

Similarly, the y-labels are related to the set-valued refinements of des and drop. So our grammar assisted bijection suits the purpose of producing a set-valued equidistribution.

Acknowledgments. We are grateful to the referee for insightful comments and substantial suggestions. This work was supported by the National Science Foundation of China.

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