The Wide Band Cayley Continuants

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Abstract

The Cayley continuants are referred to the determinants of tridiagonal matrices in connection with the Sylvester continuants. Munarini-Torri found a striking combinatorial interpretation of the Cayley continuants in terms of the joint distribution of the number of odd cycles and the number of even cycles of permutations of $[n] = \{1,2,\ldots,n\}$. In view of a general setting, r-regular cycles (with length not divisible by r) and r-singular cycles (with length divisible by r) have been extensively studied largely related to roots of permutations. We introduce the wide band Cayley continuants as an extension of the original Cayley continuants, and we show that they can be interpreted in terms of the joint distribution of the number of r-regular cycles and the number of r-singular cycles over permutations of $[n]$.

Keywords: r-regular cycles, r-singular cycles, roots of permutations, the Cayley continuants.

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1 Introduction

As a way to compute Sylvester's continuants, Cayley [4] introduced the tridiagonal determinants, as called the Cayley continuants by Munarini and Torri [7]. Cayley's approach may be perceived as a diagonalization technique in the ternomilogy of today, that is, generalizing first, then specializing. Such an understanding has been spelled out by in [7]. To be precise, for $n \geq 2$, the Cayley continuants are referred to the $n \times n$ tridiagonal determinants, where the zero entries are left blank,

$$U_n(x,y) = \begin{vmatrix} x & 1 & & & & \\ y & x & 2 & & & \\ & y-1 & x & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & x & n-1 \\ & & & & y-n+2 & x \end{vmatrix}_{n\times n}.$$

For $n = 0, 1$, the initial values are given by $U_0(x,y) = 1$ and $U_1(x,y) = x$. The determinants satisfy the three-term recurrence for $n \geq 2$,

$$U_n(x,y) = xU_{n-1}(x,y) - (n-1)(y-n+2)U_{n-2}(x,y).$$

From the above recursion, Cayley [4] derived the exponential generating function of $U_n(x,y)$ and he further computed the Sylvester's determinants. An exposition of Cayley's approach can be found in [7]. It is noteworthy that several classical polynomials can be derived from the Cayley continuants such as the Meixner polynomials of the first kind and the Mittag-Leffler polynomials, see also [7].

A striking combinatorial interpretation of the Cayley continuants has been found by Munarini and Torri [7] in terms of the joint distribution of the number of odd cycles and the number of even cycles. Munarini [6] obtained a representation of the Cayley continuants by means of the umbral operators.

For $n \geq 1$, let S_n be the set of permutations of $[n] = \{1, 2, \ldots, n\}$. For a permutation $\sigma \in S_n$, let $o(\sigma)$ and $e(\sigma)$ be the number of odd cycles and the number of even cycles of σ, respectively. Then for $n \geq 1$,

$$U_n(x,y) = \sum_{\sigma \in S_n} x^{o(\sigma)}(-y)^{e(\sigma)}.$$

To avoid the minus signs in the above combinatorial interpretation, one may replace y with $-y$. For $n \geq 2$, we come to the determinant

$$V_n(x,y) = \begin{vmatrix} x & 1 & & & & \\ -y & x & 2 & & & \\ & -(y+1) & x & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & x & n-1 \\ & & & & -(y+n-2) & x \end{vmatrix}_{n\times n}.$$

The initial values of $V_n(x,y)$ for $n = 0,1$ remain the same as those of $U_n(x,y)$. The first few values of $V_n(x,y)$ are given below:

$$\begin{aligned} V_0(x,y) &= 1, \\ V_1(x,y) &= x, \\ V_2(x,y) &= x^2+y, \\ V_3(x,y) &= x^3+3xy+2x, \\ V_4(x,y) &= x^4+6x^2y+8x^2+3y^2+6y, \\ V_5(x,y) &= x^5+10x^3y+20x^3+15xy^2+50xy+24x. \end{aligned}$$

Multiplying each odd row and each odd column of the determinant $V_n(x,y)$ by -1, it can be recast as

$$V_n(x,y) = \begin{vmatrix} x & -1 & & & & \\ y & x & -2 & & & \\ & y+1 & x & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & x & -(n-1) \\ & & & & y+n-2 & x \end{vmatrix}_{n\times n}. \tag{1.1}$$

Indeed, the above form of the Cayley continuants is the basis of our wide band extension.

The purpose of this paper is to make a connection between the wide band Cayley continuants and the joint distribution of the number of r-regular cycles and the number of r-singular cycles. A cycle is called r-regular if its length is not divisible by r, and is called r-singular if its length is divisible by r. We show that the wide band Cayley continuants can be interpreted as the polynomials for the joint distribution of the foregoing statistics.

2 The wide band Cayley continuants

We introduce the wide band Cayley continuants as an extension of the original Cayley continuants based on the expression of $V_n(x,y)$ in (1.1). Note that the term continuant is kept in a larger sense, since the wide band Cayley continuants are not restricted to be tridiagonal. For the sake of rigor, the wide band Cayley determinants may be considered as an alternative terminology.

Recall that an $n \times n$ matrix $A = (a_{i,j})_{n\times n}$ is called a (p,q)-band matrix if $a_{i,j} = 0$ for $j > i+p$ or $i > j+q$, where $0 \le p,q \le n-1$, see Börgers [3, Definition 5.1]. Like the case of $r = 2$, we use $V_n^{(r)}(x,y)$ to denote the wide band Cayley continuants, which generalize the Cayley continuants $U_n(x,y)$ with y replaced by $-y$.

Consider a specific kind of $n \times n$ $(1, r-1)$ band matrices $A_n^{(r)}(x,y)$, whose determinants are called the wide band Cayley continuants, denoted by $V_n^{(r)}(x,y)$. For $r \ge 2$ and $n \ge r$, the first superdiagonal (above the main diagonal) is $(-1,-2,\cdots,-(n-1))$, the main diagonal and the $(r-2)$ subdiagonals (below the main diagonal) have the same entry x, and the $(r-1)$-st subdiagonal is $(y, y-1, y+2, \cdots, y-n+r)$, as illustrated below:

$$V_n^{(r)}(x,y)=\begin{vmatrix} x & -1 & & & & & & \\ \vdots & x & -2 & & & & & \\ x & \vdots & x & \ddots & & & & \\ y & x & \vdots & \ddots & \ddots & & & \\ & y-1 & x & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & & \ddots & x & \cdots & x & -(n-1) \\ & & & & y-n+r & x & \cdots & x \end{vmatrix}_{n\times n}.$$

In particular, for $r=2$ and $n\geq 2$, $V_n^{(2)}(x,y)$ takes the form of $V_n(x,y)$ as in (1.1). For $r=3$ and $n=6$, the wide band Cayley continuant $V_6^{(3)}(x,y)$ is given by

$$V_6^{(3)}(x,y)=\begin{vmatrix} x & -1 & & & & \\ x & x & -2 & & & \\ y & x & x & -3 & & \\ & y-1 & x & x & -4 & \\ & & y-2 & x & x & -5 \\ & & & y-3 & x & x \end{vmatrix}.$$

As to the initial values, When $n=0$, set $A_0^{(r)}(x,y)$ to be the empty matrix with determinant $V_0^{(r)}(x,y)=1$. When $n=1$, set $A_0^{(r)}(x,y)=(x)$ and $V_1^{(r)}(x,y)=x$. For $2\leq n\leq r-1$, we define $V_n^{(r)}(x,y)$ to be a truncated form of the general case, that is,

$$V_n^{(r)}(x,y)=\begin{vmatrix} x & -1 & & & & \\ x & x & -2 & & & \\ x & x & x & \ddots & & \\ \vdots & \vdots & \vdots & \ddots & \ddots & \\ x & x & x & \cdots & x & -(n-1) \\ x & x & x & \cdots & x & x \end{vmatrix}_{n\times n}. \tag{2.1}$$

For example, we have

$$V_2^{(4)}(x,y) = \begin{vmatrix} x & -1 \\ x & x \end{vmatrix} \qquad \text{and} \qquad V_3^{(4)}(x,y) = \begin{vmatrix} x & -1 & \\ x & x & -2 \\ x & x & x \end{vmatrix}.$$

The initial values of $V_n^{(r)}(x,y)$ for $0 \le n \le r-1$ are given as follows, where $x^{(n)}$ stands for the rising factorial, that is, $x^{(0)} = 1$, and for $n \ge 1$,

$$x^{(n)} = x(x+1)\cdots(x+n-1).$$

Lemma 2.1. *For $r \ge 2$ and $0 \le n \le r-1$, we have*

$$V_n^{(r)}(x,y) = x^{(n)}.$$

Proof. For $n = 0, 1$, by definition there is nothing to be said. For $n \ge 2$, expanding the determinant in (2.1) along the last column, we see that $V_n^{(r)}(x,y)$ admits the same recurrence relation as $x^{(n)}$. ∎

Analogous to the recurrence relation for the Cayley continuants, the following relation holds for $V_n^{(r)}(x,y)$, where $(x)_n$ stands for the lower factorial, that is, $(x)_0 = 1$ and for $n \ge 1$,

$$(x)_n = x(x-1)\cdots(x-n+1).$$

Theorem 2.2. *For $r \ge 2$ and $n \ge r$, we have*

$$V_n^{(r)}(x,y) = x\sum_{i=1}^{r-1}(n-1)_{i-1}\, V_{n-i}^{(r)}(x,y) + (y+n-r)(n-1)_{r-1}\, V_{n-r}^{(r)}(x,y). \quad (2.2)$$

where $V_n^{(r)}(x,y) = x^{(n)}$ for $0 \le n \le r-1$.

Proof. Let $A_n^{(r)}(x,y) = (a_{i,j})_{n\times n}$ denote the matrix of the determinant $V_n^{(r)}(x,y)$. For $r \ge 2$ and $n \ge r$, expanding the determinant $V_n^{(r)}(x,y)$ along the last row, we get

$$V_n^{(r)}(x,y) = \sum_{i=1}^{r} a_{n,n-i+1}C_{n,n-i+1},$$

where $C_{n,n-i+1}$ is the cofactor of $a_{n,n-i+1}$. For $i=1$, we have

$$C_{n,n} = V_{n-1}^{(r)}(x,y) = (n-1)_{i-1}\, V_{n-i}^{(r)}(x,y).$$

Next, for $i=r$, consider the case when $a_{n,1} \neq 0$. It can happen only when $n=r$. In this case, we have

$$C_{n,1} = (n-1)_{n-1} = (n-1)_{i-1}\, V_{n-i}^{(r)}(x,y),$$

where we have made use of the fact that $V_0^{(r)}(x,y)=1$.

If $a_{n,1}=0$, that is, $n>r$, for $2 \le i \le r$, we have

$$C_{n,n-i+1} = (-1)^{i-1} \left| \begin{array}{c|cccc} A_{n-i}^{(r)}(x,y) & & & & \\ \hline & -(n-i+1) & & & \\ * & & -(n-i+2) & & \\ & & & \ddots & \\ & & * & & -(n-1) \end{array} \right| .$$

Taking the signs into account, we see that

$$C_{n,n-i+1} = (n-1)_{i-1} \left| A_{n-i}^{(r)}(x,y) \right| = (n-1)_{i-1} V_{n-i}^{(r)}(x,y).$$

Since $a_{n,n-i+1} = x$ for $1 \le i \le r-1$ and $a_{n,n-r+1} = y+n-r$, we obtain the required recurrence relation for $V_n^{(r)}(x,y)$. By Lemma 2.1, the initial values of $V_n^{(r)}(x,y)$ for $0 \le n \le r-1$ are given by $x^{(n)}$, as expected. ∎

We now turn to the exponential generating functions for the wide band Cayley continuants. Write

$$V_r(x,y;t) = \sum_{n\ge 0} V_n^{(r)}(x,y)\frac{t^n}{n!}.$$

Theorem 2.2 implies the differential equation

$$(1-t^r)V_r'(x,y;t) = \left(t^{r-1}y + x(1-t^{r-1})(1-t)^{-1}\right) V_r(x,y;t),$$

with the initial value $V_r(x,y;0)=1$. This gives the following formula for $V_r(x,y;t)$. As will be seen, the combinatorial interpretation in the next section along with the known generating functions also leads to the same expression for $V_r(x,y;t)$ without resorting to a differential equation.

Theorem 2.3. *For $r \geq 2$, we have*

$$V_r(x,y;t) = (1-t^r)^{\frac{x-y}{r}} (1-t)^{-x}. \tag{2.3}$$

For $r = 2$, the above formula reduces to the generating function of $V_n(x,y)$ as derived by Munarini and Torri [7], that is,

$$V_2(x,y;t) = (1-t^2)^{\frac{x-y}{2}} (1-t)^{-x} = \frac{(1+t)^{(x-y)/2}}{(1-t)^{(x+y)/2}}. \tag{2.4}$$

3 A combinatorial interpretation

In this section, we give a combinatorial interpretation of the above wide band Cayley continuants.

For $n \geq 0$, define $W_n^{(r)}(x,y)$ to be the polynomial for the joint distribution of the number of r-regular cycles and the number of r-singular cycles over permutations of $[n]$ with the assumption that $W_0^{(r)}(x,y) = 1$. To be more specific, define

$$W_n^{(r)}(x,y) = \sum_{\sigma \in S_n} x^{r(\sigma)} y^{s(\sigma)},$$

where $r(\sigma)$ and $s(\sigma)$ denote the number of r-regular cycles and the number of r-singular cycles of σ, respectively. For $r \geq 2$, by definition, we have $W_1^{(r)}(x,y) = x$.

For $r = 2$, Munarini and Torri [7] found a combinatorial interpretation of $V_n(x,y)$, that is,

$$V_n(x,y) = W_n^{(2)}(x,y).$$

In the following theorem, we give a recurrence relation for $W_n^{(r)}(x,y)$, which turns out to coincide with that for the wide band Cayley continuants. The argument is along the line of Munarini and Torri [7].

Theorem 3.1. *For $r \geq 2$ and $n \geq r$, we have*

$$W_n^{(r)}(x,y) = x \sum_{i=1}^{r-1} (n-1)_{i-1} W_{n-i}^{(r)}(x,y) + (y+n-r)(n-1)_{r-1} W_{n-r}^{(r)}(x,y), \tag{3.1}$$

where $W_n^{(r)}(x,y) = x^{(n)}$ for $0 \leq n \leq r-1$.

Proof. Let σ be a permutation of $[n]$. Assume σ is represented in the cycle notation, and each cycle begins with its minimum element. In particular, we call the cycle containing the element 1 the first cycle. There are three cases depending on the length of the first cycle.

Case 1: The first cycle length i is less than r. Then it is a r-regular cycle with weight x. Moreover, there are $(n-1)_{i-1}$ choices for the first cycle. Define the weight of σ to be $x^{r(\sigma)}y^{s(\sigma)}$. Then the sum of weights of all possible permutations in this case equals

$$x\sum_{i=1}^{r-1}(n-1)_{i-1}\,W_{n-i}^{(r)}(x,y).$$

Case 2: The first cycle length i equals r. Then it is a r-singular cycle of weight y. Moreover, there are $(n-1)_{r-1}$ choices for the first cycle. The sum of weights of all possible permutations in this case equals

$$y(n-1)_{r-1}\,W_{n-r}^{(r)}(x,y).$$

Case 3: The first cycle length i is greater than r. Let $C=(1\ j_2\ \cdots\ j_r\ j_{r+1}\ \cdots)$ be the first cycle, and let $D=j_{r+1}\ j_{r+2}\ \cdots$ be the permutation obtained from C by removing the first r elements. Therefore, σ can be recovered from the cycle $C'=(1\ j_2\ \cdots\ j_r)$, the permutation D, and the rest of the cycles of σ. Now, there are $(n-1)_{r-1}$ choices for C'.

Assume that C' is given. There are $n-r$ elements left. For any permutation π on these remaining elements, we need to specify an element to play the role of j_{r+1}. There are $n-r$ choices. Hence there are $(n-1)_r$ choices for $j_2\ j_3\ \cdots\ j_{r+1}$. At this point, we may reconstruct C from C', the specified element and the permutation π. Observe that C is r-regular if and only if the cycle $(j_{r+1}\cdots)$ in π is r-regular, and so σ and π have the same weight. Thus the sum of weights of all possible permutations σ in this case equals

$$(n-1)_r\,W_{n-r}^{(r)}(x,y).$$

Finally, when $1\le n\le r-1$, each cycle in the permutation of $[n]$ is r-regular and has weight x. By the combinatorial interpretation of the signless Stirling numbers of the fist kind, we see that $W_n^{(r)}(x,y)$ is precisely $x^{(n)}$. This completes the proof. ∎

In closing, we remark that in view of the combinatorial interpretation of $V_n^{(r)}(x,y)$, their generating function can be deduced from the generating functions for r-regular permutations and r-cycle permutations. A permutation is called r-regular if all its cycles are r-regular, whereas an r-cycle permutation is referred to a permutation in which every cycle is r-singular. For $n \geq 1$, let $\mathrm{Reg}_r(n)$ and $\mathrm{Cyc}_r(n)$ denote the set of all r-regular permutations and the set of all r-cycle permutations in S_n. As usual, we set $|\mathrm{Reg}_r(0)| = 1$ and $|\mathrm{Cyc}_r(0)| = 1$. Notice that the notations $NODIV_r(n)$ and $PERM_r(n)$, are used in Bóna-Mclennan-White [2] in lieu of $\mathrm{Reg}_r(n)$ and $\mathrm{Cyc}_r(n)$.

The following generating functions have long been known, see [1, 2, 5].

$$\sum_{n\geq 0} |\mathrm{Reg}_r(n)| \frac{t^n}{n!} = \exp\left(\sum_{n\neq 0 \bmod r} (n-1)!\, \frac{t^n}{n!} \right) = \frac{(1-t^r)^{1/r}}{1-t},$$

$$\sum_{n\geq 0} |\mathrm{Cyc}_r(n)| \frac{t^n}{n!} = \exp\left(\sum_{n\geq 1, n=0 \bmod r} (n-1)!\, \frac{t^n}{n!} \right) = (1-t^r)^{-1/r}.$$

It follows that

$$\begin{aligned} V_r(x,y;t) &= \exp\left(x \sum_{n\neq 0 \bmod r} (n-1)!\, \frac{t^n}{n!} \right) \exp\left(y \sum_{n\geq 1, n=0 \bmod r} (n-1)!\, \frac{t^n}{n!} \right) \\ &= \left(\sum_{n\geq 0} |\mathrm{Reg}_r(n)| \frac{t^n}{n!} \right)^x \left(\sum_{n\geq 0} |\mathrm{Cyc}_r(n)| \frac{t^n}{n!} \right)^y \\ &= (1-t^r)^{\frac{x-y}{r}} (1-t)^{-x}, \end{aligned}$$

which is in agreement with (2.3).

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