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Breaking Cycles, the Odd Versus the Even

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Abstract

In an award-winning expository article, V. Pozdnyakov and J.M. Steele gave a beautiful demonstration of the ramifications of a basic bijection for permutations. The aim of this note is to connect this correspondence to a seemingly unrelated problem concerning odd cycles and even cycles, arising in the combinatorial study of the Cayley continuants by E. Munarini and D. Torri. In extreme cases, one encounters two special classes of permutations of $2n$ elements with the same cardinality. A bijection of this appealing relation has been found by E. Sayag. A combinatorial study of permutations with only odd cycles has been carried out by M. Bóna, A. Mclennan and D. White. We find an intermediate structure which leads to a linkage between these two antipodal structures. A recursive setting reveals that everything boils down to only one trick – breaking the cycles.

Keywords: Permutations, cycles, bijection.

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1 Introduction

In an award-winning exposition, V. Pozdnyakov and J.M. Steele [6] elaborated on many a facet of a basic property of the cycle representation of permutations, viz., the number of permutations of $[n] = \{1, 2, \ldots, n\}$ $(n \geq 2)$ for which 1 and 2 occur in the same cycle equals the number of permutations of $[n]$ for which 1 and 2 do not occur in the same cycle. The heart of the plot lies in an operation of breaking a cycle into two cycles.

More precisely, given a cycle containing both 1 and 2, we can split it into two segments, one starting with with 1 and ending with the element preceding 2, whereas the other starting with 2 and ending with the element preceding 1. Keep in mind that a cycle can be expressed as a sequence starting with the minimum element.

The objective of this note is to supplement the showcase of Pozdnyakov-Steele with one more story. In a different scene, we meet up with two classes of permutations of $[2n]$ $(n \geq 1)$. Let A_n denote the set of permutations of $[n]$ consisting of odd cycles, let B_{2n} denote the set of permutations of $[2n]$ consisting of even cycles. A bijection between A_{2n} and B_{2n} can be found in [1, Section 6.2]. Let $a_n = |A_n|$ and $b_{2n} = |B_{2n}|$. As pointed out by Munarini and Torri [4], the generating function of the Cayley continuants specializes to the generating functions for a_{2n} and b_{2n}. In fact, we have $\mathrm{a}_{2n} = b_{2n} = ((2n-1)!!)^2$. The sequence $\{a_n\}$ is listed as $\#A000246$ in OEIS [5], and the sequence $\{b_{2n}\}$ is referred to as $\#A001818$. A further study of the sequence $\{a_n\}$ can be found in Bóna-Mclennan-White [2].

We take a different avenue to provide a combinatorial interpretation by employing the Pozdnyakov-Steele bijection with a twist of the roles of 1 and 2 in certain circumstances. As an intermediate step, we establish the following correspondence. Let P_{2n} be the set of permutations of $[2n]$ consisting of odd cycles except that the element 1 is in an even cycle.

Theorem 1.1 *There exists a bijection between A_{2n} and P_{2n}.*

2 A bijection

Before presenting the proof, let us consider how to apply the map in Theorem 1.1 to transform a permutation in A_{2n} to a permutation in B_{2n}. Starting with a permutation in A_{2n}, at the first step, we get a permutation with 1 appearing in an even cycle. Iterating this procedure for the remaining odd cycles, we are led to a permutation of even cycles. This proves that $a_{2n} = b_{2n}$.

The following inductive proof is essentially a description of a recursive algorithm.

Inductive Proof of Theorem 1.1. For $n = 1$, the required correspondence is merely the only way to break the even cycle (12) into two odd cycles (1) and (2).

Assume that $n > 1$ and that there is a one-to-one correspondence between A_{2m} and P_{2m} for $m < n$. We are going to put together a bijection between A_{2n} and P_{2n}. To this end, we define P_{2n}^{12} to be the set of permutations in P_{2n} such 1 and 2 belong to the same even cycle, and denote by P_{2n}^{1-2} the set of permutations in P_{2n} such that 1 appears in an even cycle but 2 appears in an odd cycle. Thus, $\mathrm{P}_{2n} = P_{2n}^{12} \cup P_{2n}^{1-2}$. For an even cycle containing both 1 and 2, we may break it into two cycles with one containing 1 and the other containing 2. Taking the parities into account, we find that $\mathrm{P}_{2n}^{12} \leftrightarrow A_{2n}^{1-2} \cup Q_{2n}^{1-2}$, where A_{2n}^{1-2} is the set of permutations of $[2n]$ consisting of odd cycles such that 1 and 2 do not appear in the same cycle, and Q_{2n}^{1-2} is the set of permutations of $[2n]$ such that 1 and 2 occur in different even cycles, whereas all other cycles are odd.

Thus, it suffices to justify the following one-to-one correspondence

$$P_{2n}^{1-2} \cup Q_{2n}^{1-2} \leftrightarrow A_{2n}^{12}, \tag{2.1}$$

where A_{2n}^{12} is the set of permutations of $[2n]$ consisting of odd cycles such that 1 and 2 appear in the same cycle. By splitting a permutation in A_{2n}^{12}, we see that $\mathrm{A}_{2n}^{12} = P_{2n}^{1-2} \cup U_{2n}^{1-2}$, where U_{2n}^{1-2} is the set of permutations of $[2n]$ such that 1 is in an odd cycle, 2 is in an even cycles and all other cycles are odd.

In order to justify (2.1), we only need to establish the following correspondence

$$Q_{2n}^{1-2} \leftrightarrow U_{2n}^{1-2}. \tag{2.2}$$

By exchanging the roles of 1 and 2, U_{2n}^{1-2} can be identified with the set of permutations such that 1 occurs in an even cycle and all other cycles are odd.

Notice that the relation (2.2) is nothing but a recursive statement of $A_{2n} \leftrightarrow P_{2n}$. To be more specific, let V_{2n}^{1-2} denote the set of permutations obtained from those U_{2n}^{1-2} by exchanging 1 and 2. Assume that σ is a permutation in V_{2n}^{1-2} and C is the even cycle of σ containing 1.

Invoking the induction hypothesis with respect to all the odd cycles in σ, we get an even cycle containing 2 along with all other odd cycles, which is precisely a permutation in Q_{2n}^{1-2}. This completes the proof. ∎

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References

[1] M. Bóna, *A Walk Through Combinatorics*, World Scientific, 2005.

[2] M. Bóna, A. Mclennan and D. White, *Permutations with roots*, Random Structures & Algorithms, 17 (2000), 157–167.

[3] MathOverflow, `https://mathoverflow.net/questions/207054`.

[4] E. Munarini and D. Torri, *Cayley continuants*, Theoret. Comput. Sci., 347 (2005), 353–369.

[5] The OEIS Foundation Inc., *The On-Line Encyclopedia of Integer Sequences*, `http://oeis.org`.

[6] V. Pozdnyakov and J.M. Steele, *Buses, bullies, and bijections*, Math. Mag., 89 (2016), 167–176.