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# The Dumont Ansatz for the Eulerian Polynomials, Peak Polynomials and Derivative Polynomials

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**Abstract**

We observe that three context-free grammars of Dumont can be brought to a common ground, via the idea of transformations of grammars, proposed by Ma-Ma-Yeh. Then we develop a unified perspective to investigate several combinatorial objects in connection with the bivariate Eulerian polynomials. We call this approach the Dumont ansatz. As applications, we provide grammatical treatments, in the spirit of the symbolic method, of relations on the Springer numbers, the Euler numbers, the three kinds of peak polynomials, an identity of Petersen, and the two kinds of derivative polynomials, introduced by Knuth-Buckholtz and Carlitz-Scoville, and later by Hoffman in a broader context. We obtain a convolution formula on the left peak polynomials, leading to the Gessel formula. In this framework, we come to the combinatorial interpretations of the derivative polynomials due to Josuat-Vergès.

**Keywords:** Dumont ansatz, Eulerian polynomials, peak polynomials, derivative polynomials, context-free grammars

**AMS Classification:** 05A15, 05A19

## 1 Introduction

The theme of this work is to present a unified approach to several enumeration problems in connection with the classical Eulerian polynomials via a formal calculus based on context-free grammars. We call this approach the Dumont ansatz because it is largely built on the three grammars of Dumont related to the Eulerian polynomials. In some sense, we may say that the Dumont ansatz is a handful of the grammars of Dumont, reinforced by the idea of Ma-Ma-Yeh [23] concerning transformations of grammars.

The grammar of Dumont for the Eulerian polynomials reads

$$G = \{x \to xy, \;\; y \to xy\}. \tag{1.1}$$

Let $D$ be the formal derivative with respect to the above grammar $G$. For $n \geq 0$, the bivariate Eulerian polynomial $A_n(x,y)$ is defined by

$$A_n(x,y) = D^n(y).$$

Recall that a grammar $G$ on a set $V = \{x_1, x_2, \ldots\}$ of variables is defined to be a set of substitution rules mapping each variable $x_i$ to a Laurent polynomial $F_i(x_1, x_2, \ldots)$ on $V$, and the formal derivative $D$ with respect to $G$ can be expressed as a differential operator

$$D = \sum_i F_i(x_1, x_2, \ldots) \frac{\partial}{\partial x_i}.$$

The generating function of a Laurent polynomial $f$ on $V$ is defined by

$$\mathrm{Gen}(f,t) = \sum_{n=0}^{\infty} D^n(f) \frac{t^n}{n!}. \tag{1.2}$$

If $g$ is also a Laurent polynomial on $V$, then $D$ satisfies the product rule

$$D(fg) = fD(g) + D(f)g. \tag{1.3}$$

In general, $D$ obeys the Leibniz rule, i.e., for $n \geq 0$,

$$D^n(fg) = \sum_{k=0}^{n} \binom{n}{k} D^k(f) D^{n-k}(g), \tag{1.4}$$

or equivalently, the following multiplicative property holds,

$$\mathrm{Gen}(fg,t) = \mathrm{Gen}(f,t)\,\mathrm{Gen}(g,t). \tag{1.5}$$

For the above grammar $G$ in (1.1), we have

$$D = xy\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right).$$

Setting $y = 1$, the bivariate Eulerian polynomials $A_n(x,y)$ reduce to the Eulerian polynomials $A_n(x)$. The generating function of $A_n(x)$ is given by

$$\sum_{n \geq 0} A_n(x) \frac{t^n}{n!} = \frac{1-x}{1-xe^{(1-x)t}}. \tag{1.6}$$

See, for example, [28]. The above relation is equivalent to the bivariate version

$$\sum_{n=0}^{\infty} A_n(x,y) \frac{t^n}{n!} = \frac{y-x}{1-xy^{-1}e^{(y-x)t}}. \tag{1.7}$$

A grammatical derivation of (1.7) is given in [5]. The following form of the generating function of $A_n(x,y)$ for $n \geq 1$ is due to Carlitz-Scoville [2], and it is equivalent to the expression in the univariate case,

$$\sum_{n=1}^{\infty} A_n(x,y)\frac{t^n}{n!} = xy\,\frac{e^{xt}-e^{yt}}{xe^{yt}-ye^{xt}}. \tag{1.8}$$

It is a well-known fact due to Foata-Schützenberger [13] that the Eulerian polynomials $A_n(x)$ have the $\gamma$-expansion with nonnegative coefficients,

$$A_n(x) = \sum_{k=1}^{\lfloor (n+1)/2 \rfloor} \gamma_{n,k}\, x^k (1+x)^{n+1-2k},$$

where $\gamma_{n,k}$ are nonnegative.

Let $u = xy$ and $v = x+y$. Then the grammar for the Eulerian polynomials takes the form

$$G = \{u \to uv, \;\; v \to 2u\}$$

and $A_n(x,y)$ can be expressed as $D^n(u)$. For $1 \leq n \leq 6$, the $\gamma$-expansions of $A_n(x,y)$ are as follows,

$$\begin{aligned}
A_1(x,y) &= u,\\
A_2(x,y) &= uv,\\
A_3(x,y) &= uv^2+2u^2,\\
A_4(x,y) &= uv^3+8u^2v,\\
A_5(x,y) &= uv^4+22u^2v^2+16u^3,\\
A_6(x,y) &= uv^5+52u^2v^3+136u^3v.
\end{aligned}$$

While the above grammar serves the purpose for the computation of the $\gamma$-coefficients of the Eulerian polynomials, for the reason that will be seen later, there is an advantage to make the substitutions

$$u = xy, \;\; 2v = x+y. \tag{1.9}$$

In this way, the transformed grammar becomes

$$G = \{u \to 2uv, \;\; v \to u\}, \tag{1.10}$$

which is exactly the grammar given by Dumont for increasing binary trees. Then we define the bivariate Dumont polynomials, denoted $D_n(u,v)$, in terms of the formal derivative of the grammar (1.10), or equivalently, in terms of increasing binary trees.

Under the substitutions in (1.9), we can express $x$ and $y$ in terms of $u$ and $v$, to wit,

$$x = v+\sqrt{v^2-u}, \;\; y = v-\sqrt{v^2-u}. \tag{1.11}$$

It is remarkable that Dumont also discovered an analogous grammar for 0-1-2 increasing trees, that is,

$$G = \{u \to uv,\ \ v \to u\}.$$

Despite the striking resemblance of the aforementioned three grammars, they have been playing their own roles without supporting each other. Thanks to the idea of transformations of grammars, due to Ma-Ma-Yeh [23], we recognize that these three grammars can be brought to a common ground in the name of the Dumont ansatz.

Applying the strategies of the Dumont ansatz to the peak polynomials and the derivative polynomials, we illustrate how to make connections to the Eulerian polynomials. A left peak of a permutation is also called an exterior peak or a peak. There are other two relevant statistics, the number of interior peaks and the number of left-right peaks (or outer peaks). We give grammatical labelings, called the $M$-labeling and the $W$-labeling, so that we can bring the three kinds of peak polynomials to the test ground of the Dumont ansatz.

It should be stressed that once we have a grammar on file, it is often not hard to find a combinatorial structure of a recursive nature such as increasing plane trees or increasing binary trees, as an interpretation of the corresponding polynomials. This means that a grammar can be instrumental in search for appropriate combinatorial structures.

The derivative polynomials $P_n(x)$ and $Q_n(x)$ for the tangent and the secant were introduced by Knuth-Buckholtz [19] in their studies of the tangent, Euler and Bernoulli numbers. They were studied later by Carlitz-Scoville [2] and Hoffman [15, 16] in broader contexts, see also [20]. When evaluated at $x = 1$, the derivative polynomials $Q_n(x)$ turn out to be the Springer numbers. Using the Dumont ansatz, we quickly get the combinatorial interpretations of the derivative polynomials established by Josuat-Vergès [17]. Moreover, we see how the the derivative polynomials $P_n(x)$ are related to the Eulerian polynomial $A_n(x)$.

The Dumont ansatz not only provides a mechanism to unify many known results, but also offers a rigorous platform to exploit the grammatical calculus, in the spirit of the symbolic method, in the course of proving and discovering combinatorial identities. For example, we found it possible to give a derivation of an identity of Petersen by using the grammatical calculus. We also obtain a convolution identity on the left peak polynomials, which can be used to derive the formula of Gessel on the generating function of the left peak polynomials [25, Sequence A008971], see also [33].

## 2 The Dumont ansatz

In this section, we first give an overview of three grammars of Dumont, for the Eulerian polynomials, increasing binary trees, and the André polynomials. We see that these three grammars share the same nature and the corresponding generating functions can be deduced from each other via a change of variables. We use an intermediate structure as a unified model to deal with the relations among the generating functions in the family, and we call this approach the Dumont ansatz. Roughly speaking, the idea behind the Dumont ansatz is to establish connections among combinatorial polynomials by means of transformations of grammars.

## 2.1 The grammar for the Eulerian polynomials

As described in the Introduction, the bivariate Eulerian polynomials $A_n(x,y)$ can be expressed as $D^n(y)$, where $D$ is the formal derivative with respect to the grammar

$$G = \{x \to xy,\ \ y \to xy\}. \tag{2.1}$$

The bivariate polynomials $A_n(x,y)$ can be expressed in terms of the numbers of descents and ascents of permutations, or in terms of complete increasing binary trees, which are called the Gessel trees in [6].

For $n = 0$, we define $A_0(x,y) = y$, the first few values of $A_n(x,y)$ are given below,

$$\begin{aligned}
A_1(x,y) &= xy,\\
A_2(x,y) &= xy^2 + x^2y,\\
A_3(x,y) &= xy^3 + 4x^2y^2 + x^3y,\\
A_4(x,y) &= xy^4 + 11x^2y^3 + 11x^3y^2 + x^4y,\\
A_5(x,y) &= xy^5 + 26x^2y^4 + 66x^3y^3 + 26x^4y^2 + x^5y,\\
A_6(x,y) &= xy^6 + 57x^2y^5 + 302x^3y^4 + 302x^4y^3 + 57x^5y^2 + x^6y.
\end{aligned}$$

For more information about context-free grammars for combinatorial polynomials and Eulerian polynomials, see [4, 5, 11]. The journey of the Dumont ansatz starts with the above grammar $G$ and the Eulerian polynomials $A_n(x,y)$.

## 2.2 The grammar for increasing binary trees

The second grammar of Dumont we will be concerned with is

$$G = \{u \to 2uv,\ \ v \to u\}. \tag{2.2}$$

Here we deliberately choose to use the variables $u$ and $v$ rather than $x$ and $y$ as in [11], because it is related to the grammar (2.1) for the Eulerian polynomials via the following substitutions

$$u = xy,\ \ 2v = x + y. \tag{2.3}$$

Since

$$D(u) = D(xy) = xy(x+y) = 2uv$$

and

$$D(v) = D(x+y)/2 = xy = u,$$

the grammar $G$ in (2.1) is transformed into the grammar $G$ in (2.2). This transformation implies that the Eulerian polynomials $A_n(x)$ are $\gamma$-positive, as observed by Ma-Ma-Yeh [23].

Dumont [11] showed that the grammar $G$ in (2.2) gives a weighted counting of increasing binary trees $T$ on $[n]$ if we label a leaf by $u$ and a degree one vertex by $v$, where the degree of a vertex in a binary tree is referred to the number of its children. Then we define the weight of $T$ as the product of the labels associated with $T$. The labels of the increasing binary tree in Figure 1 are shown in parentheses. This way of labeling an increasing binary tree is called the $(u,v)$-labeling.

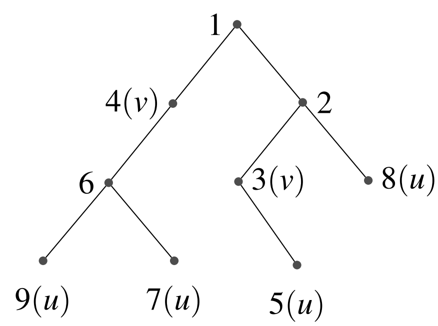


Figure 1: An increasing binary tree with the $(u,v)$-labeling.

The initiative of the Dumont ansatz grew out of the realization that the underlying combinatorial structures of the original grammar and the transformed grammar are essentially the same. The substitutions of variables are reflected by different labeling schemes. It should be mentioned that the idea of using a change of variables to compute the $\gamma$-coefficients of the Eulerian polynomials has appeared in the work of Chow [7].

**Definition 2.1.** *For $n \geq 0$, the Dumont polynomial $D_n(u,v)$ is referred to the polynomial $D^n(v)$, where $D$ is the formal derivative with respect to the grammar $G$ in (2.2).*

The following theorem is due to Dumont [11].

**Theorem 2.2** (Dumont)**.** *For $n \geq 1$, $D_n(u,v)$ equals the sum of weights of all increasing binary trees on $[n]$ endowed with the $(u,v)$-labeling.*

The first few values of $D_n(u,v)$ are listed below,

$$\begin{aligned}
D_0(u,v) &= v,\\
D_1(u,v) &= u,\\
D_2(u,v) &= 2uv,\\
D_3(u,v) &= 4uv^2+2u^2,\\
D_4(u,v) &= 8uv^3+16u^2v,\\
D_5(u,v) &= 16uv^4+88u^2v^2+16u^3,\\
D_6(u,v) &= 32uv^5+416u^2v^3+272u^3v.
\end{aligned}$$

## 2.3 0-1-2 increasing plane trees

In the study of the $\gamma$-coefficients of the Eulerian polynomials, 0-1-2 increasing plane trees arise as an underlying structure for the combinatorial interpretation. In accordance with the grammar $G$ in (2.2), we need an alternative labeling scheme.

Let $n \geq 1$, and let $T$ be a 0-1-2 increasing plane tree on $[n]$. A $(u, 2v)$-labeling of $T$ is referred to labeling every leaf by $u$ and labeling every degree one vertex by $2v$. Given a $(u, 2v)$-labeling, the weight of $T$ is defined to be the product of labels associated with $T$. Figure 2 depicts a 0-1-2 increasing plane tree with the $(u, 2v)$-labeling.

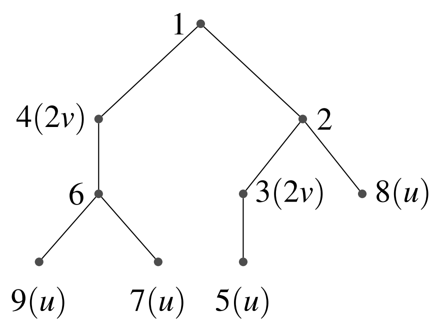


Figure 2: A 0-1-2 increasing plane tree with the $(u, 2v)$-labeling.

Using the $(u, 2v)$-labeling of 0-1-2 increasing plane trees, the theorem of Dumont concerning the grammar (2.2) can be reformulated as follows.

**Theorem 2.3** (Dumont)**.** *For $n \geq 1$, the Dumont polynomial $D_n(u, v)$ equals the sum of weights of all 0-1-2 increasing plane trees on $[n]$ with the $(u, 2v)$-labeling.*

It is evident that the $(u, 2v)$-labeling for 0-1-2 increasing plane trees is equivalent to the $(u, v)$-labeling for increasing binary trees. Thus, for $n \geq 1$,

$$D_n(u, v) = A_n(x, y), \tag{2.4}$$

where $(u, v)$ and $(x, y)$ are related by (2.3), i.e., $u = xy$ and $2v = x + y$.

## 2.4 The André polynomials

Dumont [11] showed that the following grammar

$$G = \{u \to uv, \ \ v \to u\}$$

can be used to generate the bivariate André polynomials $E_n(x, y)$, which are defined in terms of 0-1-2 increasing trees as follows,

$$E_n(u, v) = \sum_T u^{f_0(T)} v^{f_1(T)},$$

where the sum ranges over all 0-1-2 increasing trees $T$ on $[n]$, and $f_i(T)$ denotes the numbers of degree $i$ vertices of $T$ for $i = 0,1,2$. For $n \geq 1$ and $x = y = 1$, $E_n(x,y)$ reduces to the Euler number $E_n$, that is, the number of alternating permutations on $[n]$.

Define $E_0(u,v) = 1$. The first few values of $E_n(u,v)$ are given below,

$$\begin{aligned}
E_1(u,v) &= u,\\
E_2(u,v) &= uv,\\
E_3(u,v) &= uv^2+u^2,\\
E_4(u,v) &= uv^3+4u^2v,\\
E_5(u,v) &= uv^4+11u^2v^2+4u^3,\\
E_6(u,v) &= uv^5+26u^2v^3+34u^3v.
\end{aligned}$$

Notice that the bivariate version of the André polynomials can be recovered from the one variable version. Let $E_n(u) = E_n(u,1)$. Then for $n \geq 1$,

$$E_n(u,v) = v^{n+1}E_n\left(\frac{u}{v^2}\right). \tag{2.5}$$

The generating function of the André polynomials $E_n(x)$ was obtained by Foata-Schützenberger [14]. An alternative proof can be found in Foata-Han [12]. A derivation utilizing the grammar of Dumont was given in [4]. The following is manifest since both are generated by essentially the same grammar.

**Theorem 2.4.** *For $n \geq 0$, we have*

$$D_n(2u,v) = 2^nE_n(u,v). \tag{2.6}$$

*Proof.* Clearly, ordering the two children of a degree two vertex in a 0-1-2 increasing tree is equivalent to assigning the number two to this vertex as a label, so that

$$D_n(u,v) = \sum_L 2^{f_2(T)}u^{f_0(T)}(2v)^{f_1(T)}, \tag{2.7}$$

where the sum ranges over 0-1-2 increasing trees on $[n]$. It follows that

$$D_n(2u,v) = \sum_T 2^{f_0(T)+f_1(T)+f_2(T)}u^{f_0(T)}v^{f_1(T)} = 2^n\sum_T u^{f_0(T)}v^{f_1(T)},$$

where $T$ has the same range as in (2.7), whereupon the theorem is proved. ∎

Now we see that the André polynomials can be expressed in terms of the Eulerian polynomials, see [25, A094503]. For $n \geq 0$, we have

$$2^nE_n(u,v) = A_n(x,y), \tag{2.8}$$

where $x$ and $y$ are determined by $xy = 2u$ and $x+y = 2v$.

For $u = 1$ and $v = 1$, (2.8) becomes the known identity on the Euler numbers,

$$E_n = \frac{A_n(i)}{(1+i)^{n-1}}, \tag{2.9}$$

where $n \geq 1$ and $i = \sqrt{-1}$, see [25, Sequence A000111].

# 3 The peak polynomials

The objective of this section is to demonstrate that the peak polynomials, all of the three kinds, fall into the framework of the Dumont ansatz. First of all, let us now get the notation straight. Our proposal is to employ symbols that are meaningful and yet easy to remember. It turns out that the $LMW$-notation seems to be a sensible choice. As for left peaks (exterior peaks), the letter $L$ looks like having a peak on the left, and so we use $L(n,k)$ to denote the number of permutations of $[n]$ with $k$ left peaks. Accordingly, we use $L_n(x)$ and $L_n(x,y)$ to denote the one variable and bivariate left peak polynomials, respectively, that is,

$$L_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} L(n,k)x^k, \tag{3.1}$$

and

$$L_n(x,y) = \sum_{k=0}^{\lfloor n/2 \rfloor} L(n,k)x^{2k+1}y^{n-2k}. \tag{3.2}$$

The first few values of $L_n(x,y)$ are given below,

$$\begin{aligned}
L_0(x,y) &= x, \\
L_1(x,y) &= xy, \\
L_2(x,y) &= xy^2 + x^3, \\
L_3(x,y) &= xy^3 + 5x^3y, \\
L_4(x,y) &= xy^4 + 18x^3y^2 + 5x^5, \\
L_5(x,y) &= xy^5 + 58x^3y^3 + 61x^5y, \\
L_6(x,y) &= xy^6 + 179x^3y^4 + 479x^5y^2 + 61x^7.
\end{aligned}$$

To be more specific, let $n \geq 1$ and let $\sigma = \sigma_1\sigma_2\cdots\sigma_n$ be a permutation of $[n]$. We assume that $\sigma_0 = \sigma_{n+1} = 0$. Then an index $i$ is said to be a left peak if $1 \leq i < n$ and $\sigma_{i-1} < \sigma_i > \sigma_{i+1}$, or an interior peak if $1 < i < n$ and $\sigma_{i-1} < \sigma_i > \sigma_{i+1}$, or a left-right peak if $1 \leq i \leq n$ and $\sigma_{i-1} < \sigma_i > \sigma_{i+1}$,

Next, we choose the letter $M$ for the case of interior peaks, because the two peaks in $M$ bear a striking resemblance to interior peaks. Therefore, we shall use $M(n,k)$ to denote the number of permutations of $[n]$ with $k$ interior peaks. For $n \geq 1$, the interior peak polynomials are defined by

$$M_n(x) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} M(n,k)x^k. \tag{3.3}$$

In the case of left-right peaks or outer peaks, the letter $W$ signifies three left-right peaks including the two at both ends, and so we use $W(n,k)$ to denote the number of permutations of $[n]$ with $k$ left-right peaks. We move on to define

$$W_n(x) = \sum_{k=1}^{\lfloor (n+1)/2 \rfloor} W(n,k)x^k. \tag{3.4}$$

Note that various notations for the peak polynomials and their coefficients have appeared in the literature, see, for example, [1, 8, 18, 20, 22, 27], while they do not necessarily mean the same as in here. The bivariate versions of $M_n(x)$ and $W_n(x)$ are crucial as far as the grammars are concerned, which are defined by

$$M_n(x,y) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} M(n,k) x^{2k+2} y^{n-2k-1}, \tag{3.5}$$

$$W_n(x,y) = \sum_{k=1}^{\lfloor (n+1)/2 \rfloor} W(n,k) x^{2k} y^{n-2k+1}. \tag{3.6}$$

In fact, there is a reason to express $W_n(x,y)$ as

$$W_n(x,y) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} W(n,k+1) x^{2k+2} y^{n-2k-1}. \tag{3.7}$$

The first few values of $W_n(x)$ are given below,

$$\begin{aligned}
W_0(x,y) &= y, \\
W_1(x,y) &= x^2, \\
W_2(x,y) &= 2x^2y, \\
W_3(x,y) &= 4x^2y^2 + 2x^4, \\
W_4(x,y) &= 8x^2y^3 + 16x^4y, \\
W_5(x,y) &= 16x^2y^4 + 88x^4y^2 + 16x^6, \\
W_6(x,y) &= 32x^2y^5 + 416x^4y^3 + 272x^6y.
\end{aligned}$$

It can be seen from the above table that the polynomials $W_n(x)$ have the same coefficients as the Dumont polynomials $D_n(u,v)$.

## 3.1 Connection to the Gessel formula

Adopting the Dumont ansatz, we obtain a convolution formula connecting the left peak polynomials with the Dumont polynomials, which yields the Gessel formula on the generating function of $L_n(x)$, i.e.,

$$L(x,t) = \sum_{n=0}^{\infty} L_n(x) \frac{t^n}{n!}.$$

Similarly, let

$$M(x,t) = \sum_{n=0}^{\infty} M_n(x) \frac{t^n}{n!}.$$

Note that by way of recurrence relations, David-Barton [9] established partial differential equations on $L(x,t)$ and $M(x,t)$ and found solutions in differential forms.

**Theorem 3.1** (David-Barton)**.** *We have*

$$2x(1-x)\frac{\partial L(x,t)}{\partial x}+(xt-1)\frac{\partial L(x,t)}{\partial t}+L(x,t)+1 \quad = \quad 0, \tag{3.8}$$

$$2x(1-x)\frac{\partial M(x,t)}{\partial x}+(xt-1)\frac{\partial M(x,t)}{\partial t}+M(x,t)+\sqrt{x} \quad = \quad 0. \tag{3.9}$$

*Put*

$$z=t\sqrt{1-x}+\log\left(\frac{\sqrt{x}}{1+\sqrt{1-x}}\right).$$

*Then we have*

$$\frac{\partial L(x,t)}{\partial x} \quad = \quad \frac{1}{2}\left(\frac{1}{\cosh(z)-1}+\frac{1}{\cosh(z)+1}\right), \tag{3.10}$$

$$\frac{\partial M(x,t)}{\partial t} \quad = \quad \frac{1}{2}\left(\frac{1}{\cosh(z)-1}-\frac{1}{\cosh(z)+1}\right). \tag{3.11}$$

Gessel [25, Sequence A008971] obtained the following explicit formula for $L(x,t)$.

**Theorem 3.2** (Gessel)**.** *We have*

$$L(x,t)=\frac{\sqrt{1-x}}{\sqrt{1-x}\cosh\left(t\sqrt{1-x}\right)-\sinh t\sqrt{1-x}}. \tag{3.12}$$

As brought up by Stanley [29], an explicit expression for $M(x,t)$ can be deduced from Equation (3.11) of David-Barton. An extension to a more general enumeration problem was given by Carlitz-Scoville [2].

Our point of departure is the following grammar

$$G=\{x\rightarrow xy, \quad y\rightarrow x^2\}, \tag{3.13}$$

independently found by Chen-Fu [4] and Ma [20]. Let $D$ be the formal derivative of the grammar $G$. It has been shown that the left peak polynomials $L_n(x,y)$ can be generated by the grammar $G$, i.e., for $n\geq 0$,

$$L_n(x,y)=D^n(x). \tag{3.14}$$

Setting $u=x^2$ and $v=y$, the grammar in (3.13) takes the form

$$G=\{u\rightarrow 2uv, \ v\rightarrow u\}, \tag{3.15}$$

which turns out to be exactly a grammar of the Dumont ansatz. This transformation enables us to establish a convolution identity on $L_n(x,y)$.

Let $D$ be the formal derivative with respect to the grammars in (3.13) and (3.15). Bear in mind that there is no ambiguity because the substitution rules act on distinct variables. Let us consider the polynomials $D^n(v)$. There are two ways to look at $D^n(v)$. On one hand, $D_n(u,v)$ can be considered as a polynomial in $u,v$, which equals the Dumont polynomial $D_n(u,v)$. On the other hand, $D^n(v)$ can be treated as a polynomial in $x,y$.

Since

$$D^{n+1}(y) = D^n(x^2),$$

we obtain the following convolution identity.

**Theorem 3.3.** *For $n \geq 0$,*

$$D_{n+1}(x^2, y) = x^2 \sum_{k=0}^{n} \binom{n}{k} L_k(x,y) L_{n-k}(x,y). \tag{3.16}$$

The above formula gives rise to a relation on the generating function of $L_n(x,y)$. Let $L(x,y,t)$ be the generating function of $L_n(x,y)$.

**Corollary 3.4.** *Let $A(\bar{x},\bar{y},t)$ be the generating function of the Eulerian polynomials $A_n(\bar{x},\bar{y})$, and let $A'(\bar{x},\bar{y},t)$ denote the differentiation with respect to $t$. Then we have*

$$A'(\bar{x},\bar{y},t) = L^2(x,y,t), \tag{3.17}$$

*where $\bar{x} = y + \sqrt{y^2 - x^2}$ and $\bar{y} = y - \sqrt{y^2 - x^2}$.*

Let us illustrate how to compute $L(x,y,t)$ from $A(\bar{x},\bar{y},t)$. Invoking (1.8), that is,

$$A(\bar{x},\bar{y},t) = \bar{x}\bar{y} \frac{e^{\bar{x}t} - e^{\bar{y}t}}{\bar{x}e^{\bar{y}t} - \bar{y}e^{\bar{x}t}}.$$

we find that

$$A'(\bar{x},\bar{y},t) = \bar{x}\bar{y}e^{(\bar{x}+\bar{y})t} \frac{(\bar{x}-\bar{y})^2}{(\bar{x}e^{\bar{y}t} - \bar{y}e^{\bar{x}t})^2}. \tag{3.18}$$

In order to connect $A'(\bar{x},\bar{y},t)$ with $L(x,y,t)$, it is necessary to express the Dumont polynomial $D_n(u,v)$ in terms of the Eulerian polynomial $A_n(x,y)$. Substituting $u = x^2$ and $v = y$ into (1.11), we find that

$$A_n(y + \sqrt{y^2 - x^2}, y - \sqrt{y^2 - x^2}) = D_n(x^2, y). \tag{3.19}$$

Plugging

$$\bar{x} = y + \sqrt{y^2 - x^2}, \quad \bar{y} = y - \sqrt{y^2 - x^2}$$

into (3.18), a routine calculation shows that

$$A'(y + \sqrt{y^2 - x^2}, y - \sqrt{y^2 - x^2}, t) = \left( \frac{xy\sqrt{y^2 - x^2}}{\sqrt{y^2 - x^2}\cosh(t\sqrt{y^2 - x^2}) - y\sinh t\sqrt{y^2 - x^2}} \right)^2,$$

which, together with (3.17), gives

$$L(x,y,t) = \frac{xy\sqrt{y^2 - x^2}}{\sqrt{y^2 - x^2}\cosh(t\sqrt{y^2 - x}) - y\sinh t\sqrt{y^2 - x^2}}. \tag{3.20}$$

Noting that $\sqrt{x}L_n(x) = L_n(\sqrt{x}, 1)$, so that $L(x,t) = \sqrt{x}L(\sqrt{x}, 1, t)$. By (3.20), we obtain (3.12).

Petersen [26] established a relation between $L_n(x)$ and the Eulerian polynomials, see [20], to wit,

$$L_n\left(\frac{4x}{(1+x)^2}\right) = \frac{1}{(1+x)^n}\sum_{k=0}^{n}\binom{n}{k}(1-x)^{n-k}2^kA_k(x). \tag{3.21}$$

To seek a grammatical understanding of the above identity, we find a rather simple transformation forging a bridge between $L_n(x,y)$ and $A_n(x,y)$.

**Theorem 3.5.** *Setting $x=\sqrt{\bar{x}\bar{y}}$ and $y=(\bar{x}+\bar{y})/2$, the grammar*

$$G=\{\bar{x}\to\bar{x}\bar{y},\ \ \bar{y}\to\bar{x}\bar{y}\}$$

*for the Eulerian polynomials is transformed into the grammar*

$$G=\{x\to xy, \quad y\to x^2\}$$

*for the left peak polynomials.*

*Proof.* By definition, we have

$$D(x)=D(\sqrt{\bar{x}\bar{y}})=\frac{D(\bar{x}\bar{y})}{2\sqrt{\bar{x}\bar{y}}}=\sqrt{\bar{x}\bar{y}}\,\frac{\bar{x}+\bar{y}}{2}=xy$$

and

$$D(y)=D\left(\frac{\bar{x}+\bar{y}}{2}\right)=\bar{x}\bar{y}=y^2,$$

as requested. ∎

It should be mentioned that while we usually consider Laurent polynomials for the action of a formal derivative. But as noted in [3], we are not confined to Laurent polynomials. Indeed, taking square root is not an issue at all. To help understand the formal derivative of $\sqrt{\bar{x}\bar{y}}$, we may set $z=\sqrt{\bar{x}\bar{y}}$ and then apply the product rule

$$D(z^2)=D(\bar{x}\bar{y})=\bar{x}\bar{y}(\bar{x}+\bar{y})=2zD(z)$$

to obtain $D(z)$.

*Grammatical Proof of Petersen's Identity (3.21).* Recall that

$$D^n(x)=L_n(x,y)=\sum_{k=0}^{[n/2]}L(n,k)x^{2k+1}y^{n-2k}.$$

In view of the grammar transformation given in Theorem 3.5, we see that

$$D^n(\sqrt{\bar{x}\bar{y}})=L_n\left(\sqrt{\bar{x}\bar{y}},\frac{\bar{x}+\bar{y}}{2}\right).$$

On the other hand, by the Leibniz rule, we have

$$D^n(\sqrt{\bar{x}\bar{y}}) \quad = \quad D^n\left(\frac{y}{\sqrt{\bar{x}^{-1}\bar{y}}}\right)$$

$$
\begin{aligned}
&= \sum_{k=0}^{n} \binom{n}{k} D^k(\bar{x}) D^{n-k}(\sqrt{\bar{x}\bar{y}^{-1}}) \\
&= \sqrt{\bar{x}\bar{y}^{-1}} \sum_{k=0}^{n} \binom{n}{k} A_k(\bar{x},\bar{y}) \frac{(\bar{y}-\bar{x})^{n-k}}{2^{n-k}},
\end{aligned}
$$

where we have made use of the fact that for $k \geq 0$,

$$
D^k(\bar{x}\bar{y}^{-1}) = \bar{x}\bar{y}^{-1}(\bar{y}-\bar{x})^k, \tag{3.22}
$$

as observed in [4]. It follows that

$$
L_n\left(\sqrt{\bar{x}\bar{y}}, \frac{\bar{x}+\bar{y}}{2}\right) = \sqrt{\bar{x}\bar{y}^{-1}} \sum_{k=0}^{n} \binom{n}{k} A_k(\bar{x},\bar{y}) \frac{(\bar{y}-\bar{x})^{n-k}}{2^{n-k}}. \tag{3.23}
$$

Setting $\bar{y} = 1$ and replacing $\bar{x}$ by $x$ yields

$$
\begin{aligned}
L_n\left(\sqrt{x}, \frac{1+x}{2}\right) &= \sqrt{x} \sum_{k=0}^{n} \binom{n}{k} A_k(x) \frac{(1-x)^{n-k}}{2^{n-k}} \\
&= \frac{\sqrt{x}}{2^n} \sum_{k=0}^{n} \binom{n}{k} 2^k A_k(x)(1-x)^{n-k}. \qquad (3.24)
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
L_n\left(\sqrt{x}, \frac{1+x}{2}\right) &= \sum_{k=0}^{\lfloor n/2 \rfloor} L(n,k)(\sqrt{x})^{2k+1} \frac{(1+x)^{n-2k}}{2^{n-2k}} \\
&= \frac{\sqrt{x}(1+x)^n}{2^n} \sum_{k=0}^{\lfloor n/2 \rfloor} L(n,k) \frac{(4x)^k}{(1+x)^{2k}}. \qquad (3.25)
\end{aligned}
$$

Comparing (3.24) and (3.25), we arrive at (3.21). ∎

## 3.2 The grammatical labelings

A grammatical labeling of permutations was given in [4] to produce the left peak polynomials. Similar tactics can be applied to the other two kinds of peak polynomials.

The labeling for $L_n(x,y)$, called the $L$-labeling, can be described as follows. Let $\sigma$ be a permutation of $[n]$. We patch a zero to $\sigma$ at both ends so that there are $n+1$ positions between two adjacent elements, and these are the possible positions to insert $n+1$ in $\sigma$ to generate a permutation of $[n+1]$. The $L$-labeling of $\sigma$ is meant to label the last position by $x$, label the two positions next to any left peak by $x$, and label the remaining positions by $y$. Notice that the $L$-labeling is an equivalent representation of the labeling given in [4]. Below is an example,

$$
3\,1\,4\,5\,6\,2 \stackrel{L}{\longrightarrow} 0\,x\,3\,x\,1\,y\,4\,y\,5\,x\,6\,x\,2\,x\,0.
$$

The procedure of generating the above permutation, along with the $L$-labelings and the corresponding substitution rules, is displayed in the table below,

| $n$ | The $L$-labeling | Weight | Rule |
|---|---|---|---|
| 1 | $0\,y\,1\,x\,0$ | $xy$ | $x \to xy$ |
| 2 | $0\,y\,1\,y\,2\,x\,0$ | $xy^2$ | $x \to xy$ |
| 3 | $0\,x\,3\,x\,1\,y\,2\,x\,0$ | $x^3y$ | $y \to x^2$ |
| 4 | $0\,x\,3\,x\,1\,x\;\;4\,x\,2\,x\,0$ | $x^5$ | $y \to x^2$ |
| 5 | $0\,x\,3\,x\,1\,y\,4\,x\,5\,x\,2\,x\,0$ | $x^5y$ | $x \to xy$ |
| 6 | $0\,x\,3\,x\,1\,y\,4\,y\,5\,x\,6\,x\,2\,x\,0$ | $x^5y^2$ | $x \to xy$ |

The $M$-labeling of $\sigma$ is supposed to label the two positions at both ends by $x$, label the two positions next to any interior peak by $x$, and label the remaining positions by $y$. Below is an example,

$$3\,1\,4\,5\,6\,2 \xrightarrow{M} 0\,x\,3\,y\,1\,y\,4\,y\,5\,x\,6\,x\,2\,x\,0.$$

The procedure of generating the above permutation along with the $M$-labelings is displayed in the table below,

| $n$ | The $M$-labeling | Weight | Rule |
|---|---|---|---|
| 1 | $0\,x\,1\,x\,0$ | $x^2$ | $y \to x^2$ |
| 2 | $0\,x\,1\,y\,2\,x\,0$ | $x^2y$ | $x \to xy$ |
| 3 | $0\,x\,3\,y\,1\,y\,2\,x\,0$ | $x^2y^2$ | $x \to xy$ |
| 4 | $0\,x\,3\,y\,1\,x\;\;4\,x\,2\,x\,0$ | $x^4y$ | $y \to x^2$ |
| 5 | $0\,x\,3\,y\,1\,y\,4\,x\,5\,x\,2\,x\,0$ | $x^4y^2$ | $x \to xy$ |
| 6 | $0\,x\,3\,y\,1\,y\,4\,y\,5\,x\,6\,x\,2\,x\,0$ | $x^4y^3$ | $x \to xy$ |

The following relation was derived by Ma [20] via recurrence relations.

**Theorem 3.6.** *For $n \geq 1$, we have*

$$D^n(y) = M_n(x,y). \tag{3.26}$$

Indeed, it is only a matter of formality to check that the $M$-labeling justifies the above conclusion. Let us now turn to the $W$-labeling. For a permutation $\sigma$ of $[n]$, a zero is patched at both ends. Consider the positions between two adjacent elements of $\sigma$, where $\sigma_0 = \sigma_{n+1} = 0$. For any element $1 \leq i \leq n$, if $\sigma_i$ is a left-right peak, then label the positions next to $i$ by $x$, and label the rest of the positions by $y$.

For example, the $W$-labeling of the permutation 1 is $0\,x\,1\,x\,0$, and below is a permutation accompanied by its $W$-labeling,

$$3\,1\,4\,5\,6\,2 \xrightarrow{W} 0\,x\,3\,x\,1\,y\,4\,y\,5\,x\,6\,x\,2\,y\,0.$$

By examining the change of labels when inserting the element $n+1$ to a permutation on $[n]$, we are led to the following interpretation of the grammatical expansion of $D^n(y)$. The same reasoning as for the grammatical generation of the polynomials $L_n(x,y)$ given in [4] applies to $W_n(x,y)$. Thus the justification with full rigor will not be repeated here. Instead, we shall give an example. The process of generating the preceding permutation is described in the following table,

| $n$ | The $W$-labeling | Weight | Rule |
|---|---|---|---|
| 1 | $0\,x\,1\,x\,0$ | $x^2$ | $y \to x^2$ |
| 2 | $0\,y\,1\,x\,2\,x\,0$ | $x^2y$ | $x \to xy$ |
| 3 | $0\,x\,3\,x\,1\,x\,2\,x\,0$ | $x^4$ | $y \to x^2$ |
| 4 | $0\,x\,3\,x\,1\,x\,4\,x\,2\,y\,0$ | $x^4y$ | $x \to xy$ |
| 5 | $0\,x\,3\,x\,1\,y\,4\,x\,5\,x\,2\,y\,0$ | $x^4y^2$ | $x \to xy$ |
| 6 | $0\,x\,3\,x\,1\,y\,4\,y\,5\,x\,6\,x\,2\,y\,0$ | $x^4y^3$ | $x \to xy$ |

**Theorem 3.7.** *For $n \geq 1$, we have*

$$D^n(y) = W_n(x,y). \tag{3.27}$$

Now that $W_n(x) = xM_n(x)$, by comparing (3.5) with (3.7), we retrieve the following combinatorial property, see Zhuang [33].

**Theorem 3.8.** *For $n \geq 1$ and $0 \leq k \leq \lfloor (n-1)/2 \rfloor$, the number of permutations of $[n]$ with $k+1$ left-right peaks equals the number of permutations of $[n]$ with $k$ interior peaks, that is,*

$$W(n,k+1) = M(n,k). \tag{3.28}$$

Applying the Dumont ansatz, we get the following relation, where $D_n(u,v)$ are the Dumont polynomials.

**Theorem 3.9.** *For $n \geq 0$,*

$$D_n(u,v) = M_n(x,y), \tag{3.29}$$

*where $u = x^2$ and $v = y$.*

Using the theory of enriched $P$-partitions, Stembridge [30] showed that for $n \geq 1$,

$$xM_n\left(\frac{4x}{(1+x)^2}\right) = \frac{2^{n-1}}{(1+x)^{n-1}} A_n(x), \tag{3.30}$$

see also, [20, 28]. In fact, Theorem 3.9 spells out how the these polynomials (in the bivariate forms) are related via a change of variables.

## 3.3 Convolution formulas

From the point of view of grammars, one can convert the Leibniz formulas into convolution formulas for combinatorial polynomials, and we shall exemplify how this is the case for peak polynomials.

Returning to $M_n(x)$ and $W_n(x)$, we see that for $n \geq 1$,

$$xM_n(x) = W_n(x), \tag{3.31}$$

since $x^2M_n(x^2) = M_n(x,1)$ and $W_n(x^2) = W_n(x,1)$.

In consideration of the initial values, we have no choice but to impose that

$$M_0(x) = x^{-1} \quad \text{and} \quad W_0(x) = 1. \tag{3.32}$$

Since $D(x) = xy$, we have for $n \geq 0$,

$$D^{n+1}(x) = D^n(xy).$$

and so the following convolution formula is immediate.

**Theorem 3.10.** *For $n \geq 0$, we have*

$$L_{n+1}(x,y) = \sum_{k=0}^{n} \binom{n}{k} L_k(x,y) M_{n-k}(x,y), \tag{3.33}$$

*or equivalently,*

$$L_{n+1}(x) = x \sum_{k=0}^{n} \binom{n}{k} L_k(x) M_{n-k}(x). \tag{3.34}$$

Since $D(x^2) = D(y^2) = 2x^2y$, we deduce that for $n \geq 1$,

$$D^n(x^2) = D^n(y^2),$$

which can be translated into the following convolution identity.

**Theorem 3.11.** *For $n \geq 1$, we have*

$$\sum_{k=0}^{n} \binom{n}{k} L_k(x,y) L_{n-k}(x,y) = \sum_{k=0}^{n} \binom{n}{k} M_k(x,y) M_{n-k}(x,y). \tag{3.35}$$

Given that $D(x^2) = D(y^2)$, we find that for $n \geq 1$,

$$D^{n+1}(y) = D^n(x^2) = D^n(y^2).$$

Therefore, we reach a convolution formula for $M_n(x,y)$, which is reminiscent of that for the derivative polynomials $P_n(x)$. This is no surprise because the polynomials $M_n(x,y)$ and $P_n(x)$ admit the same grammatical structure.

**Theorem 3.12.** *For $n \geq 1$, we have*

$$M_{n+1}(x,y) = \sum_{k=0}^{n} \binom{n}{k} M_k(x,y) M_{n-k}(x,y). \tag{3.36}$$

With the understanding that $M_0(x) = x^{-1}$ and $M_1(x) = 1$, for $n \geq 1$, (3.36) may be replaced by

$$M_{n+1}(x) = x \sum_{k=0}^{n} \binom{n}{k} M_k(x) M_{n-k}(x). \tag{3.37}$$

In the same vein, for $n \geq 1$, the convolution formula (3.35) can be recast as

$$\sum_{k=0}^{n} \binom{n}{k} L_k(x) L_{n-k}(x) = \sum_{k=0}^{n} \binom{n}{k} M_k(x) M_{n-k}(x). \tag{3.38}$$

Ma [20] considered the polynomials $R_n(x,y) = D^n(x+y)$, where $n \geq 0$. Since

$$D^{n+1}(x+y) = D^n(x(x+y)),$$

we get the following convolution formula at once.

**Theorem 3.13.** *For $n \geq 0$, we have*

$$R_{n+1}(x,y) = \sum_{k=0}^{n} \binom{n}{k} L_k(x,y) R_{n-k}(x,y). \tag{3.39}$$

The convolution of $L_n(x)$ appeared in Ma-Yeh [24]. However, the above convolution identity for $L_n(x)$ went unnoticed, even though there is no barrier to deduce it in the context.

# 4 The derivative polynomials

The derivative polynomials $P_n(x)$ and $Q_n(x)$ for the tangent and the secant were introduced by Knuth-Buckholtz [19] for the computation of tangent, Euler and Bernoulli numbers. They were later studied by Carlitz-Scoville [2]. Hoffman [15] defined the derivative polynomials in a more general context. Recall that

$$(\tan(x))' = \tan^2(x) + 1.$$

Define $P_n(x)$ by

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n} \tan(x) = P_n(\tan(x)). \tag{4.1}$$

The derivative polynomials $P_n(x)$ are listed as the Sequence A008293 in OEIS, and $P_n(0)$ are the tangent numbers. As pointed out by Dumont [10], the numbers $P_n(1)$ go back to Euler, and they are listed as Sequence A000831 in OEIS [25] with initial values

$$1, 2, 4, 16, 80, 512, 3904, 34816, 354560, 4063232, \ldots.$$

In the same vein, the derivative polynomials $Q_n(x)$ for the secant are defined by

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n} \sec(x) = Q_n(\tan(x)) \sec(x). \tag{4.2}$$

For $x = 0$, $Q_n(x)$ are the secent numbers, that is, $Q_n(0) = 0$ for $n$ odd, and $Q_n(0) = E_n$ for $n$ even. The numbers $Q_n(1)$ are called the Springer numbers or the generalized Euler numbers, denoted by $S_n$.

The derivative polynomials satisfy the recurrence relations for $n \geq 0$,

$$P_{n+1}(x) = (1+x^2)\frac{\mathrm{d}}{\mathrm{d}x}P_n(x), \tag{4.3}$$

$$Q_{n+1}(x) = (1+x^2)\frac{\mathrm{d}}{\mathrm{d}x}Q_n(x) + xQ_n(x). \tag{4.4}$$

Note that the generating functions of $P_n(x)$ and $Q_n(x)$ were given by Hoffman [15].

**Theorem 4.1** (Hoffman)**.** *We have*

$$\sum_{n=0}^{\infty} P_n(x)\frac{t^n}{n!} = \frac{x+\tan(t)}{1-x\tan(t)}, \tag{4.5}$$

$$\sum_{n=0}^{\infty} Q_n(x)\frac{t^n}{n!} = \frac{1}{\cos(t)-x\sin(t)}. \tag{4.6}$$

Notice that the generating function of $Q_n(x)$ tells that $Q_n(1)$ coincides with the Springer number $S_n$ for any $n$.

## 4.1 The grammar

Whereas the following theorem is merely a paraphrase of the recursive formulas for $P_n(x)$ and $Q_n(x)$, it lends a different perspective to look at the derivative polynomials, as well as a channel to the Dumont ansatz.

**Theorem 4.2.** *Let*

$$G = \{a \to ax,\ \ x \to 1+x^2\}, \tag{4.7}$$

*and let $D$ be the formal derivative with respect to $G$. Then for $n \geq 0$,*

$$D^n(x) = P_n(x), \tag{4.8}$$

$$D^n(a) = aQ_n(x). \tag{4.9}$$

For $n=0$, we have $P_0(x) = x$ and $Q_0(x) = 1$. The first few values of $P_n(x)$ and $Q_n(x)$ are given below,

$$\begin{aligned}
P_1(x) &= 1+x^2,\\
P_2(x) &= 2x+2x^3,\\
P_3(x) &= 2+8x^2+6x^4,\\
P_4(x) &= 16x+40x^3+24x^5,\\
P_5(x) &= 16+136x^2+240x^4+120x^6,\\
P_6(x) &= 272x+1232x^3+1680x^5+720x^7,
\end{aligned}$$

and

$$\begin{aligned}
Q_1(x) &= x, \\
Q_2(x) &= 1+x^2, \\
Q_3(x) &= 5x+6x^3, \\
Q_4(x) &= 5+28x^2+24x^4, \\
Q_5(x) &= 61x+180x^3+120x^5, \\
Q_6(x) &= 61+662x^2+1320x^4+720x^6.
\end{aligned}$$

## 4.2 The generating functions

We present a proof of the generating functions of $P_n(x)$ and $Q_n(x)$ to demonstrate the rigor and the efficiency of the grammatical calculus. In the formal setting, the generating functions of $P_n(x)$ and $Q_n(x)$ can be expressed as follows.

**Theorem 4.3.** *We have*

$$\mathrm{Gen}(a,t) = \frac{a}{\cos(t)-x\sin(t)}, \tag{4.10}$$

$$\mathrm{Gen}(x,t) = \frac{x\cos(t)+\sin(t)}{\cos(t)-x\sin(t)}. \tag{4.11}$$

*Grammatical Proof.* Since

$$\mathrm{Gen}(a,t) = \frac{1}{\mathrm{Gen}(a^{-1},t)} \tag{4.12}$$

and

$$\mathrm{Gen}(x,t) = \frac{\mathrm{Gen}(a^{-1}x,t)}{\mathrm{Gen}(a^{-1},t)}, \tag{4.13}$$

we proceed to compute $D^n(a^{-1})$ and $D^n(a^{-1}x)$. Clearly,

$$D(a^{-1}) = -a^{-2}D(a) = -a^{-2}(ax) = -a^{-1}x,$$

so we get

$$D^2(a^{-1}) = D(-a^{-1}x) = a^{-1}x^2 - a^{-1}(1+x^2) = -a^{-1},$$

and hence

$$D^3(a^{-1}) = a^{-1}x.$$

It follows that for $n \geq 0$,

$$D^{2n}(a^{-1}) = (-1)^n a^{-1}, \tag{4.14}$$

$$D^{2n+1}(a^{-1}) = (-1)^{n+1} a^{-1}x. \tag{4.15}$$

Consequently,

$$\begin{aligned}
\mathrm{Gen}(a^{-1},t) &= \sum_{n=0}^{\infty} D^n(a^{-1})\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty}(-1)^n a^{-1}\frac{t^{2n}}{(2n)!}+\sum_{n=0}^{\infty}(-1)^{n+1}a^{-1}x\frac{t^{2n+1}}{(2n+1)!} \\
&= a^{-1}\cos(t)-a^{-1}x\sin(t).
\end{aligned}$$

This proves (4.10).

In order to compute $\mathrm{Gen}(a^{-1}x,t)$, observe that (4.14) and (4.15) can be alternatively expressed as

$$D^{2n}(a^{-1}x) = (-1)^n a^{-1}x, \tag{4.16}$$

$$D^{2n+1}(a^{-1}x) = (-1)^n a^{-1}, \tag{4.17}$$

from which we infer that

$$\mathrm{Gen}(a^{-1}x,t) = a^{-1}x\cos(t)+a^{-1}\sin(t). \tag{4.18}$$

According to (4.13), we reach (4.11). This completes the proof. ∎

## 4.3 Convolution formulas

To show how the grammatical calculus can be performed, we consider the following convolution identities of Hoffman [15]. For $n \geq 1$,

$$P_{n+1}(x) = \sum_{k=0}^{n}\binom{n}{k}P_k(x)P_{n-k}(x), \tag{4.19}$$

$$Q_{n+1}(x) = \sum_{k=0}^{n}\binom{n}{k}P_k(x)Q_{n-k}(x). \tag{4.20}$$

Since $D^{n+1}(x)=D^n(1+x^2)=D^n(x^2)$, (4.19) immediately follows from the Leibniz rule. To verify (4.20), we only need the condition $n \geq 0$. Note that $D^{n+1}(a)=D^n(ax)$. Thus (4.20) is a consequence of the Leibniz rule.

The following convolution formula found by Ma-Fang-Mansour-Yeh [22] also follows from the same line of reasoning. For $n \geq 0$,

$$P_{n+2}(x) = 2\sum_{k=0}^{n}\binom{n}{k}P_k(x)P_{n+1-k}(x). \tag{4.21}$$

Noting that

$$D^2(x)=D(1+x^2)=D(x^2)=2xD(x),$$

we have

$$D^{n+2}(x) = 2D^n(xD(x)).$$

Applying the Leibniz rule yields (4.21).

The following convolution identity is due to Hoffman [15], which can be used to evaluate $Q_n(1)$.

**Theorem 4.4** (Hoffman)**.** *For $n \geq 0$,*

$$P_{n+1}(x) = (1+x^2)\sum_{k=0}^{n}\binom{n}{k}Q_k(x)Q_{n-k}(x). \tag{4.22}$$

There is a grammatical derivation of (4.22). We need the following property.

**Proposition 4.5.** *We have*

$$D\left(\frac{1+x^2}{a^2}\right) = 0. \tag{4.23}$$

*Proof.* Since

$$D(a^{-2}) = -2a^{-3}D(a) = -2a^{-2}x$$

and

$$D(1+x^2) = 2xD(x) = 2x(1+x^2),$$

we see that

$$D\left(\frac{1+x^2}{a^2}\right) = \frac{-2x(1+x^2)}{a^2} + \frac{2x(1+x^2)}{a^2} = 0,$$

as claimed. ∎

*Grammatical Proof of (4.22).* It suffices to show that

$$a^2D^{n+1}(x) = (1+x^2)\sum_{k=0}^{n}\binom{n}{k}D^k(a)D^{n-k}(a). \tag{4.24}$$

In view of the Leibniz rule, the above relation takes the form

$$a^2D^{n+1}(x) = (1+x^2)D^n(a^2). \tag{4.25}$$

But (4.23) says that $\frac{1+x^2}{a^2}$ is a constant with respect to $D$, this implies that

$$\frac{1+x^2}{a^2}D^n(a^2) = D^n(1+x^2) = D^{n+1}(x),$$

as required. ∎

## 4.4 The Josuat-Vergès trees

As will be seen, the grammar $G$ is a reflection of the recursive construction of a structure introduced by Josuat-Vergès which serves as the basis for combinatorial interpretations of the derivative polynomials.

Let us recall the definition. Let $n \geq 1$. A Josuat-Vergès tree is a complete increasing binary tree possibly with unlabeled leaves, called empty leaves. More precisely, the vertices from the root to any leaf are in the increasing order regardless of the empty leaf, if any. For $n = 0$, the only Josuat-Vergès tree is the empty leaf. The labeling stipulation of a Josuat-Vergès tree is simple: each empty leaf is labeled by $x$. For $n = 2$, the four Josuat-Vergès trees are listed in Figure 3, with the total weight amounting to $P_2(x) = 2x + 2x^3$.

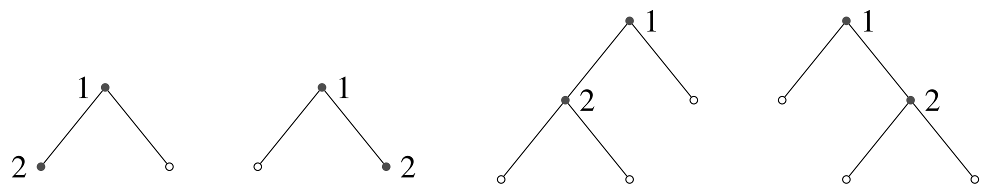


Figure 3: The four Josuat-Vergès trees on $\{1,2\}$.

Now, it can be seen that the substitution rule $x \to 1 + x^2$ corresponds to adding the element $n+1$ to a Josuat-Vergès tree on $[n]$ by turning an empty leaf into a new vertex with no children or an internal vertex with two empty children. This operation would result in a Josuat-Vergès tree on $[n+1]$, and it offers a recipe to understand the following statement.

**Theorem 4.6.** *For $n \geq 1$, $P_n(x)$ equals the sum of weights of all Josuat-Vergès trees on $[n]$.*

## 4.5 Exponentiation of the Dumont grammar

Examining the partition argument for the Faà de Bruno formula in regard with higher order derivatives of the composition of two functions, or equivalently, the combinatorial interpretation for successive applications of the formal derivative of the grammar

$$G = \{f_i \to f_{i+1} g_1,\ \ g_j \to g_{j+1} \,|\, i = 0, 1, 2, \ldots,\ j = 1, 2, \ldots\},$$

see [3], we are led to a combinatorial interpretation of $Q_n(x)$ based on that of $P_n(x)$. The proof is solely a repetition of the aforementioned testimony, and hence is omitted.

To be more specific, we consider the grammar

$$G = \{a \to av,\ \ v \to u,\ \ u \to 2uv\}. \tag{4.26}$$

Let $D$ denote the formal derivative with respect to $G$. The partition argument shows that $D^n(a)$ corresponds to the exponential structure built on the structure of the original grammar of Dumont. This makes it possible to lift the Dumont ansatz to the exponential level, involving forests of planted increasing binary trees.

Let us take up Theorem 2.2 as an example. It is obvious that for $n \geq 1$, $D^n(v)$ equals the sum of weights of planted increasing binary trees on $[n]$. A forest of planted increasing binary trees on $[n]$ can be naturally defined as a forest on $[n]$ in the usual sense, with each component being a planted increasing binary tree. The labeling schemes for planted binary trees can be carried over to forests. However, we should pay special attention to a tree with a single vertex. If a leaf differs from the root, we label it with $u$. If a degree one vertex is not the root, we label it with $v$. If the root is the only vertex in the tree, then we label it with $v$ as it is the starting label, otherwise the root is not labeled, bearing in mind that there is at most one child of the root. For example, Figure 4 depicts a planted increasing binary tree with labels in parentheses.

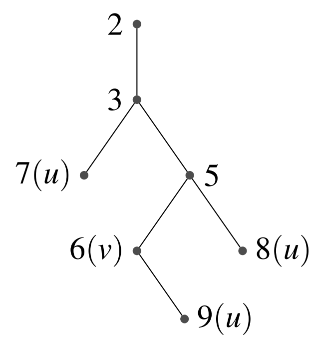


Figure 4: A planted increasing binary tree.

The exponential counterpart of Theorem 2.2 can be described as follows.

**Theorem 4.7.** *For $n \geq 1$, $D^n(a)$ equals the sum of weights of all forests of planted increasing binary trees on $[n]$ with the $(u,v)$-labeling.*

In the framework of the Dumont ansatz, we are ready to make a connection to the Josuat-Vergès forests on $[n]$, which are referred to as plane rooted forests in [17]. First, a planted Josuat-Vergès tree on a nonempty subset $S$ of $[n]$ is defined as an increasing rooted tree possibly with empty leaves such that the root has only one child, which is allowed to be an empty leaf, and the subtree of the root, if not empty, is a Josuat-Vergès tree on the rest of the elements in $S$. For the special case when $S$ contains only one element, it is necessary to note that the unique planted Josuat-Vergès tree on $S$ consists of the root along with an empty leaf.

Then a Josuat-Vergès forest is defined as a forest on $[n]$ consisting of planted Josuat-Vergès trees. The weight of a Josuat-Vergès forest is defined to be the product of the weights of all trees in the forest. For example, Figure 5 demonstrates a Josuat-Vergès forest on $[9]$.

**Theorem 4.8.** *[Josuat-Vergès] For $n \geq 1$, $Q_n(x)$ equals the sum of weights of all Josuat-Vergès forests on $[n]$.*

By the resemblance between the above combinatorial interpretation of $Q_n(x)$ and the exponentiation of the Dumont grammar, one sees that the Josuat-Vergès forests can be regarded exactly as an expanded version of the forests for the Dumont grammar. In our labeling

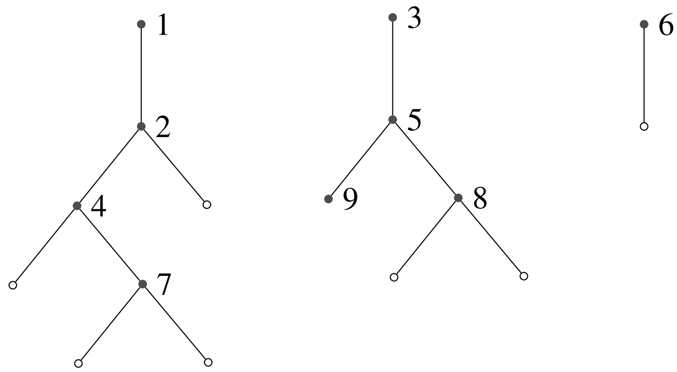


Figure 5: A Josuat-Vergès forest.

scheme, a leaf is labeled with $u = 1 + x^2$. This means that the leaf can be an endpoint (a real leaf) or a vertex having two empty leaves as its children, and this explains why we should label an empty leaf by $x$.

We remark that the exponentiation property concerning the grammars implies the following relation on the generating functions of $P_n(x)$ and $Q_n(x)$, as derived combinatorially with the aid of cycle alternating permutations by Josuat-Vergès [17],

$$\sum_{n=0} P_n(x) \frac{t^{n+1}}{(n+1)!} = \log\left( \sum_{n=0}^{\infty} Q_n(x) \frac{t^n}{n!} \right) = \log\left( \frac{1}{\cos(t) - x\sin(t)} \right), \tag{4.27}$$

which yields the generating function of $P_n(x)$ after differentiation. For more details about the composition of two grammars, see [3].

### 4.6 The $\beta$-expansions

Ma-Ma-Yeh [23] came up with a beautiful idea to deduce the $\gamma$-positivity of a polynomial by making use of a transformation of grammars. We will demonstrate that this idea can be adapted to the derivative polynomials.

Ma [20] obtained expansions of $P_n(x)$ and $Q_n(x)$ based on the grammar definitions and a trigonometric identity, which we call the $\beta$-expansions. The coefficients in the expansions turn out to be the coefficients of the peak polynomials. It has been shown that

$$P_n(x) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} M(n,k) x^{n-2k-1} (1+x^2)^{k+1}, \tag{4.28}$$

$$Q_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} L(n,k) x^{n-2k} (1+x^2)^k. \tag{4.29}$$

As noted by Zhu-Yeh-Lu [32], the Springer number $S_n$ equals $L_n(2)$. Similarly, we have $P_n(1) = M_n(2)$. Appealing to the Dumont ansatz for 0-1-2 increasing plane trees, we see that the $\beta$-expansion of $P_n(x)$ can be viewed as the $\gamma$-expansion of the Eulerian polynomials $A_n(x,y)$.

**Theorem 4.9.** *For $n \geq 1$, we have*

$$P_n(x) = \sum_{k=1}^{\lfloor (n+1)/2 \rfloor} \beta_{n,k} (2x)^{n+1-2k} (1+x^2)^k, \tag{4.30}$$

*where $\beta_{n,k}$ equals the number of 0-1-2 increasing plane trees on $[n]$ with $k$ leaves.*

Given $u = 1+x^2$ and $v = x$, the parameters $(x,y)$ in (2.4) for the Eulerian polynomials $A_n(x,y)$ are substituted by

$$(x+i, x-i), \tag{4.31}$$

where $i = \sqrt{-1}$. Hence the following relation is valid.

**Theorem 4.10.** *For $n \geq 0$, we have*

$$P_n(x) = (x-i)^{n+1} A_n\left(\frac{x+i}{x-i}\right). \tag{4.32}$$

Again, resorting to the Dumont ansatz, we may associate $P_n(x)$ with the André polynomials. For $n \geq 1$, we have

$$P_n(x) = 2^n E_n\left(\frac{1+x^2}{2}, x\right). \tag{4.33}$$

By virtue of the relation (2.5), we may rewrite (4.33) as

$$P_n(x) = 2^n x^{n+1} E_n\left(\frac{1+x^2}{2x^2}\right). \tag{4.34}$$

For $x = 1$, (4.34) reduces to

$$P_n(1) = 2^n E_n, \tag{4.35}$$

which is due to Knuth-Buckholtz [19]. Notice that the identity (4.35) also follows from (2.9) and (4.32). The proof of (4.33) can be regarded as a combinatorial argument. It is worth mentioning that, as conjectured by Sun [31] and resolved by Zhu-Yeh-Lu [32], the sequence of Springer numbers is log-convex.

The following relation is due to Ma [21], which specializes to (4.35) when $x = 0$. Let $E(x)$ be the generating function of the Euler numbers,

$$E(x) = \tan(x) + \sec(x).$$

Then

$$2^n \frac{\mathrm{d}^n}{\mathrm{d}x^n} E(x) = P_n(E(x)). \tag{4.36}$$

As for the $\beta$-expansion (4.29) of $Q_n(x)$, in light of Theorem 4.7 and Theorem 4.8, one sees that the coefficient $L(n,k)$ can also be interpreted as the number of forests of planted increasing binary trees on $[n]$ with $k$ leaves.

Next, we give a grammatical proof of (4.29). Recall the grammar for $Q_n(x)$ as given in (4.7), i.e.,

$$G = \{a \to ax, \ \ x \to 1 + x^2\}. \tag{4.37}$$

Under the substitutions $v = x$ and $u = 1 + x^2$, we get the grammar

$$G = \{a \to av, \ \ v \to u, \ \ u \to 2uv\}. \tag{4.38}$$

Let $D$ be the formal derivative of the above grammar $G$. Then we have, for $n \geq 0$,

$$D^n(a) = aQ_n(x).$$

The first few values of $D^n(a)$ are given below,

$$\begin{aligned} D(a) &= av, \\ D^2(a) &= a(v^2 + vu), \\ D^3(a) &= a(v^3 + 5vu), \\ D^4(a) &= a(v^4 + 18v^2u + 5u^2), \\ D^5(a) &= a(v^5 + 58v^3u + 61vu^2), \\ D^6(a) &= a(v^6 + 179v^4u + 479v^2u^2 + 61u^3). \end{aligned}$$

*Grammatical Proof of* (4.29). Setting $z = x^2$, the grammar

$$G = \{x \to xy, \ \ y \to x^2\} \tag{4.39}$$

is transformed into

$$G = \{x \to xy, \ \ y \to z, \ \ z \to 2yz\}. \tag{4.40}$$

Now, we may take a comparative look at (4.38) and (4.40) to understand what is going on. Recall that for the grammar $G$ in (4.39),

$$D^n(x) = x \sum_{k=0}^{\lfloor n/2 \rfloor} L(n,k) x^{2k} y^{n-2k}. \tag{4.41}$$

Observe that the above coefficient $L(n,k)$ of $x^{2k+1}y^{n-2k}$ in $D^n(x)$ equals the coefficient of $xy^{n-2k}z^k$ in $D^n(x)$ linked with the grammar in (4.40), as claimed. ∎

For example, for the grammar $G$ in (4.40), we have

$$D^5(x) = x(y^5 + 58y^3z + 61yz^2).$$

Renaming $x$ by $a$, $y$ by $x$ and $z$ by $1 + x^2$, the above expression is in accordance with the $\beta$-expansion of $Q_5(x)$:

$$Q_5(x) = 61x + 180x^3 + 120x^5 = x^5 + 58x^3(1 + x^2) + 61x(1 + x^2)^2.$$

To conclude, we remark that the relation (4.29) opens a new avenue for the Gessel formula for the generating function of $L_n(x,y)$. Yet another possibility would be to explore the connection between $L_n(x,y)$ and $P_n(x)$ along the line of the exponential formula based on (4.27).

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