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A Grammatical Calculus for Peaks and Runs of Permutations

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Abstract

We develop a nonstandard approach to exploring polynomials associated with peaks and runs of permutations. With the aid of a context-free grammar, or a set of substitution rules, one can perform a symbolic calculus, and the computation often becomes rather simple. From a grammar it follows at once a system of ordinary differential equations for the generating functions. Utilizing a certain constant property, it is even possible to deduce a single equation for each generating function. To bring the grammar to a combinatorial setting, we find a labeling scheme for up-down runs of a permutation, which can be regarded as a refined property, or the differentiability in a certain sense, in contrast to the usual counting argument for the recurrence relation. The labeling scheme also exhibits how the substitution rules arise in the construction of the combinatorial structures. Consequently, polynomials on peaks and runs can be dealt with in two ways, combinatorially or grammatically. The grammar also serves as a guideline to build a bijection between permutations and increasing trees that maps the number of up-down runs to the number of nonroot vertices of even degree. This correspondence can be adapted to left peaks and exterior peaks, and the key step of the construction is called the reflection principle.

Keywords: Context-free grammars, grammatical calculus, peak polynomials, up-down runs, alternating permutations, increasing trees.

AMS Classification: 05A15, 05A19

1 Introduction

We present a grammatical approach to studying peaks and runs of permutations. This is a classical topic in enumerative combinatorics, which has been extensively investigated for decades, see Bóna [1]. By injecting labels or variables into the conventional procedure of deriving recurrence relations, one obtains a set of substitution rules, or a context-free grammar. A grammar is more than just a recurrence relation, it offers a ground for a grammatical calculus, or a symbolic calculus in a rigorous sense, which can be quite efficient in acquiring generating functions. Besides, a grammar may shed light on the construction of a correspondence between two objects that have a grammar in common. Armed with the grammar, we find a bijection between permutations and increasing trees that maps the number of up-down runs and the number of peaks to the number of vertices subject to certain degree conditions.

To be more specific, for $n \geq 1$, let $\sigma = \sigma_1\sigma_2\cdots\sigma_n$ be a permutation of $[n] = \{1,2,\ldots,n\}$. A left peak of σ is an index i such that $1 \leq i \leq n-1$ and $\sigma_{i-1} < \sigma_i > \sigma_{i+1}$, under the convention that $\sigma_0 = 0$. An interior peak of σ is an index i such that $2 \leq i \leq n-1$ and $\sigma_{i-1} < \sigma_i > \sigma_{i+1}$. An exterior peak or an outer peak of σ is an index $1 \leq i \leq n$ such that $\sigma_{i-1} < \sigma_i > \sigma_{i+1}$ with the understanding that $\sigma_0 = \sigma_{n+1} = 0$.

For $n \geq 0$, the left peak polynomials, interior peak polynomials and the exterior peak polynomials are denoted by $L_n(x)$, $M_n(x)$ and $W_n(x)$, respectively. Let $L(x,t)$, $M(x,t)$ and $W(x,t)$ be the exponential generating functions of $L_n(x)$, $M_n(x)$ and $W_n(x)$, respectively. David-Barton [7] established partial differential equations for $L(x,t)$ and $M(x,t)$, and found a solution requiring one more step of integration. An explicit expression of $L(x,t)$ was given by Gessel, see [5, 6, 16]. A formula for $M(x,t)$ can also be deduced from a generating function of Carlitz-Scoville [3]. Note that $W(x,t)$ can be easily deduced from $M(x,t)$. Alternative proofs of the formulas for $L(x,t)$ and $M(x,t)$ have been given by Zhuang [16].

The grammar for the left peak polynomials $L_n(x)$ was discovered by Ma [11] via a recurrence relation and independently by Chen-Fu [5] with a grammatical labeling. Ma [11] further noticed that this grammar can be employed to generate the interior peak polynomials. Below is the concerned grammar

$$G = \{x \to xy,\ y \to x^2\}. \tag{1.1}$$

Let D denote the formal derivative with respect to G, or equivalently, D can be perceived as a differential operator

$$D = xy\frac{\partial}{\partial x} + x^2\frac{\partial}{\partial y}. \tag{1.2}$$

Notice that the operator D is a derivative so that the Leibniz formula holds, and this property makes it possible to perform the grammatical calculus with combinatorial motivations, see [4, 5, 8]. As remarked by Dumont [8], there are advantages to view G as a set of substitution rules. In fact, the setting of a grammar captures the combinatorial significance of a recursive procedure.

Despite previous attempts to explore the peak polynomials by means of a grammar, the story does not seem to be complete. For example, the authors were nearly there in obtaining a direct derivation of the generating functions solely by conducting the grammatical computation, but we missed the ultimate goal of reaching a sound understanding of the hyperbolic functions appearing in the formulas. This regret or oversight is now redeemed by considering the inverse of a generating function and by exploiting a constant property in a grammatical sense. It turns out that the constant property often facilitates the computation of a generating function.

One feature of the grammatical calculus is that we need to work with the bivariate versions of the peak polynomials. As far as the coefficients are concerned, the variable y is only a figurehead. But it is just the opposite that in the grammatical calculus the variable y in its own part is as equally important as x. In fact, with the company of the grammar, it is a pleasant journey to reach bivariate versions of the generating functions for the three kinds of peak polynomials. Meanwhile, this explains why the hyperbolic functions crop up with no need to mention differential equations.

As a bonus of the grammatical calculus, we get a simple relationship between the generating functions of the left peak polynomials and the exterior peak polynomials. In the context of peak polynomials, we illustrate how a grammar can be automatically translated into a system of ordinary differential equations of the generating functions. In light of a certain constant property, we may even deduce a single ordinary differential equation for each generating function.

Then we move on to the number of up-down runs of a permutation. The peaks and runs are closely related objects. For a permutation σ of $[n]$, we assume again that a zero is patched at the beginning, i.e., $\sigma_0 = 0$. An up-down run of σ is a maximal segment (a subsequence consisting of consecutive elements) that is either increasing or decreasing. If we drop the assumption $\sigma_0 = 0$, then a maximal increasing or decreasing segment is called an alternating run. For example, the permutation $3\,7\,5\,8\,6\,1\,4\,9\,2$ has six up-down runs:

$$0\,3\,7\;,\;7\,5,\;5\,8,\;8\,6\,1,\;1\,4\,9,\;9\,2$$

and it also has six alternating runs. However, the permutation $2\,1$ has two up-down runs and only one alternating run.

For $n \geq 0$, let $\Lambda_n(x)$ denote the polynomial associated with the number of permutations of $[n]$ with k up-down runs, and let $\Lambda(x,t)$ be exponential generating function of $\Lambda_n(x)$. Here we set our eyes on the Greek letter Λ because it bears an ideal resemblance to the up-down shape and we do not have to worry about interfering with the customary notation $A_n(x)$ for the Eulerian polynomials. As observed by Bóna, $\Lambda(x,t)$ can be deduced from a formula of David-Barton on the polynomials associated with the number of permutations of $[n]$ with k alternating runs. An equivalent formula for $\Lambda(x,t)$ was obtained by Stanley [13] in regard with permutations of $[n]$ having a given length of the longest alternating subsequences.

The grammar that governs the counting of up-down runs was found by Ma [12], that is,

$$G = \{a \rightarrow ax,\; x \rightarrow xy,\; y \rightarrow x^2\}, \tag{1.3}$$

which is equivalent to the operator

$$D = ax\frac{\partial}{\partial a} + xy\frac{\partial}{\partial x} + x^2\frac{\partial}{\partial y}. \tag{1.4}$$

In the framework of the grammatical calculus, we are led directly to a relationship between $\Lambda_n(x)$ and the peak polynomials. We notice that this connection also follows from two relations due to Ma [12]. Moreover, we exhibit that the grammar is informative for deriving exponential formulas on peaks and up-down runs of permutations.

If one asks why there exists a context-free grammar for the enumeration of a combinatorial structure, a vague answer would be that the involved quantities amount to local properties with respect to the growth of the structure. Indeed, this is what a context-free grammar is all about. For example, when generating permutations of $[n+1]$ out of permutations of $[n]$, the operation of inserting $n+1$ has rather a local impact on the number of various peaks. Thus in a certain sense, a grammar is a way of showing locality or differentiability.

For up-down runs, we give an explicit labeling of permutations, called the up-down labeling, which reveals how the insertion operation affects the number of up-down runs during the procedure of generating a permutation of $[n+1]$ from a permutation of $[n]$. With this labeling scheme in hand, we see that the grammar is more than just a recurrence relation, and we can make use of it in two ways, either as an apparatus of a grammatical calculus, or as a bridge to facilitate finding bijections between two objects that have a grammar in common.

This line of thinking yields a bijection between permutations and increasing trees that maps the number of up-down runs of a permutation to the number of nonroot vertices of even degree of an increasing tree. The grammar plays a vital role in the justification of the bijection. For this reason, we call the bijection a grammar assisted bijection. In particular, when restricted to alternating permutations, we arrive at a grammar assisted correspondence between alternating permutations and increasing even trees. A construction in this regard has been given by Kuznetsov, Pak and Postnikov [10].

As for exterior peaks, it is known that the classical bijection between permutations and increasing binary trees maps the number of the exterior peaks to the number of certain type of vertices, see Stanley [15]. Stemming from two labeling schemes relative to the same grammar, we obtain a bijection between permutations and increasing trees connecting the number of exterior peaks of a permutation to the number of vertices of even degree of an increasing tree.

It is our hope that this grammatical proposal could be applicable to more occasions.

2 A grammatical calculus for peaks

The objective of this section is to demonstrate the efficiency of the grammatical calculus in the study of the three kinds of peak polynomials. All we need is a grammar and the Leibniz rule relative to the grammar.

For $n \geq 1$ and $0 \leq k \leq \lfloor n/2 \rfloor$, let $L(n,k)$ denote the number of permutations of $[n]$ with k left peaks. Analogously, $M(n,k)$ and $W(n,k)$ are defined in regard with interior peaks and exterior peaks, respectively, with k in the valid range. For $n = 0$, we set $L(0,0) =$

$M(0,0)=W(0,0)=1$. Define

$$L_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} L(n,k)x^k, \tag{2.1}$$

$$M_n(x) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} M(n,k)x^k, \tag{2.2}$$

$$W_n(x) = \sum_{k=1}^{\lfloor (n+1)/2 \rfloor} W(n,k)x^k. \tag{2.3}$$

In connection with the grammar

$$G = \{x \to xy,\ \ y \to x^2\}, \tag{2.4}$$

for $n \geq 1$, the bivariate peak polynomials are defined by

$$L_n(x,y) = \sum_{k=0}^{\lfloor n/2 \rfloor} L(n,k)x^{2k+1}y^{n-2k}, \tag{2.5}$$

$$M_n(x,y) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} M(n,k)x^{2k+2}y^{n-2k-1}, \tag{2.6}$$

$$W_n(x,y) = \sum_{k=1}^{\lfloor (n+1)/2 \rfloor} W(n,k)x^{2k}y^{n-2k+1}. \tag{2.7}$$

For $n=0$, we define $L_0(x,y)=x$, $M_0(x,y)=1$ and $W_0(x,y)=y$. Let D be the formal derivative with respect to G given in (2.4). We have the grammatical interpretations of the three bivariate peak polynomials, for $n \geq 0$,

$$D^n(x) = L_n(x,y), \quad D^n(y) = W_n(x,y)$$

and, for $n \geq 1$, $D^n(y) = M_n(x,y)$, respectively.

To perform the grammatical calculus, we need to consider the generating function of a Laurent polynomial f in x and y with respect to the operator D, as defined by

$$\mathrm{Gen}(f,t) = \sum_{n=0}^{\infty} D^n(f)\frac{t^n}{n!}. \tag{2.8}$$

The formal derivative D is related to the derivative with respect to the variable t of the generating function $\mathrm{Gen}(f,t)$ via the following relation:

$$\mathrm{Gen}(D(f),t) = \mathrm{Gen}'(f,t), \tag{2.9}$$

where the prime signifies the derivative relative to t.

To see how the grammatical approach works, we give a simple relation for the generating functions $L(x,y,t)$ and $W(x,y,t)$, that is,

$$L(x,y,t) = \mathrm{Gen}(x,t), \quad W(x,y,t) = \mathrm{Gen}(y,t).$$

Then we present a combinatorial interpretation of this fact.

Theorem 2.1. *We have*

$$\mathrm{Gen}^2(x,t) = \mathrm{Gen}^2(y,t) + x^2 - y^2. \tag{2.10}$$

Proof. It is easily checked that $x^2 - y^2$ is a constant relative to D, that is, $D(x^2) = D(y^2)$, see [5]. Thus we have $D^n(x^2) = D^n(y^2)$ for $n \geq 1$, and hence

$$\mathrm{Gen}(x^2,t) = x^2 + \sum_{n=1}^{\infty} D^n(x^2)\frac{t^n}{n!} = x^2 + \sum_{n=1}^{\infty} D^n(y^2)\frac{t^n}{n!}, \tag{2.11}$$

which is the right-hand side of (2.10), as claimed. ∎

Unlike the approach of David-Barton by means of partial differential equations, in virtue of the above relation (2.10), we may aim at ordinary differential equations for $L(x,y,t)$ and $W(x,y,t)$, which are within the reach of Maple. Note that we need the constant property in deriving the equations. This way of deriving ordinary differential equations may suit other instances. For example, for the grammar $G = \{x \to xy,\ y \to xy\}$, which generates the Eulerian polynomials, $x - y$ is a constant. For the grammar $G = \{x \to xy,\ y \to x\}$, which generates the André polynomials, $y^2 - 2x$ is a constant. In each case, one can deduce an ordinary differential equation from the grammar. In fact, once an equation for $\mathrm{Gen}(x,t)$ is obtained, we may treat y merely as a parameter and set $y = 1$ for the purpose of computing the generating function only involving x. We also remark that the relation (2.9) enables us to read off a system of ordinary differential equations on the generating functions for all the variables. As for the peak polynomials, we come to the following system of equations reminiscent of the grammar,

$$\begin{cases} L'(t) = L(t)W(t), \\ W'(t) = L^2(t), \end{cases} \tag{2.12}$$

with boundary conditions $L(0)=x$ and $W(0)=y$, where the parameters x and y in $L(x,y,t)$ and $W(x,y,t)$ are suppressed to emphasize that the derivative is taken with respect to t.

Theorem 2.2. *The following ordinary differential equations hold with boundary conditions $L(x,y,0)=x$ and $W(x,y,0)=y$:*

$$L'(t) = L(t)\sqrt{L^2(t)-x^2+y^2}, \tag{2.13}$$

$$W'(t) = W^2(t)+x^2-y^2. \tag{2.14}$$

There is a combinatorial explanation of the fact that $D^n(x^2)=D^n(y^2)$ for $n\geq 1$. Since $D(y)=x^2$, $D^n(x^2)$ can be rewritten as $D^{n+1}(y)$. Thus, the combinatorial reason behind the relation (2.10) lies in the following two convolution formulas for $n\geq 1$,

$$W_{n+1}(x,y) = \sum_{k=0}^{n}\binom{n}{k}W_k(x,y)W_{n-k}(x,y), \tag{2.15}$$

$$W_{n+1}(x,y) = \sum_{k=0}^{n}\binom{n}{k}L_k(x,y)L_{n-k}(x,y). \tag{2.16}$$

For a permutation σ, we use $L(\sigma)$, $M(\sigma)$ and $W(\sigma)$ to denote the number of left peaks, interior peaks and exterior peaks of σ, respectively. A bijective argument for (2.15) and (2.16) goes as follows. For $n\geq 1$, let σ be a permutation of $[n+1]$. Write $\sigma=\pi 1\tau$. It is readily seen

$$W(\sigma)=W(\pi)+W(\tau).$$

Thus we arrive at (2.15).

On the other hand, let us write $\sigma=\pi(n+1)\tau$, and let τ' denote the reverse of τ. We find that

$$W(\sigma)=L(\pi)+L(\tau')+1,$$

since $n+1$ is always an exterior peak. This special exterior peak is counted by the exponent of x in the definition of $L_n(x,y)$, with each contributing a factor x.

With the above two interpretations of $W_{n+1}(x,y)$, we may burn the bridge after having crossed the river. In doing so, we get a direct correspondence between the right-hand sides of (2.15) and (2.16).

The next theorem gives the generating function of x^{-1} with respect to D, which is the inverse of the bivariate form of Gessel's formula.

Theorem 2.3. *We have*

$$\mathrm{Gen}(x^{-1},t) = \frac{\sqrt{y^2-x^2}\cosh(t\sqrt{y^2-x^2}) - y\sinh(t\sqrt{y^2-x^2})}{x\sqrt{y^2-x^2}}. \tag{2.17}$$

Proof. As noted in [5],

$$D(x^{-1}) = -x^{-1}y, \quad D^2(x^{-1}) = x^{-1}(y^2-x^2).$$

Now the following pattern emerges. For $n \geq 0$,

$$D^{2n}(x^{-1}) = x^{-1}(y^2-x^2)^n, \tag{2.18}$$

$$D^{2n+1}(x^{-1}) = -x^{-1}y(y^2-x^2)^n. \tag{2.19}$$

Taking the parity into account, we find that

$$\sum_{n=0}^{\infty} D^{2n}(x^{-1})\frac{t^{2n}}{(2n)!} = x^{-1}\cosh(t\sqrt{y^2-x^2}), \tag{2.20}$$

$$\sum_{n=0}^{\infty} D^{2n+1}(x^{-1})\frac{t^{2n+1}}{(2n+1)!} = -\frac{x^{-1}y}{\sqrt{y^2-x^2}}\sinh(t\sqrt{y^2-x^2}). \tag{2.21}$$

Putting the above sums together completes the proof. ∎

Corollary 2.4. *We have*

$$\mathrm{Gen}(x,t) = \frac{x\sqrt{y^2-x^2}}{\sqrt{y^2-x^2}\cosh(t\sqrt{y^2-x^2}) - y\sinh(t\sqrt{y^2-x^2})}. \tag{2.22}$$

Let $L(x,t)$ be the exponential generating function of $L_n(x)$. Dividing both sides by x, replacing x^2 by x and setting $y=1$ yields Gessel's formula:

$$L(x,t) = \frac{\sqrt{1-x}}{\sqrt{1-x}\cosh(t\sqrt{1-x}) - \sinh(t\sqrt{1-x})}. \tag{2.23}$$

Given the above formula for $\mathrm{Gen}(x,t)$, the generating function $\mathrm{Gen}(y,t)$ for exterior peaks can be deduced from the relation

$$\mathrm{Gen}'(x,t) = \mathrm{Gen}(x,t)\mathrm{Gen}(y,t),$$

or equivalently,

$$\mathrm{Gen}(y,t) = \frac{\mathrm{Gen}'(x,t)}{\mathrm{Gen}(x,t)}. \tag{2.24}$$

Corollary 2.5. *We have*

$$\mathrm{Gen}(y,t)=\frac{y\sqrt{y^2-x^2}\cosh(t\sqrt{y^2-x^2})-(y^2-x^2)\sinh(t\sqrt{y^2-x^2})}{\sqrt{y^2-x^2}\cosh(t\sqrt{y^2-x^2})-y\sinh(t\sqrt{y^2-x^2})}. \tag{2.25}$$

There are alternative ways to derive the above formula for $\mathrm{Gen}(y,t)$. Analogous to $\mathrm{Gen}(x^{-1},t)$, it is easy to compute $\mathrm{Gen}(x^{-1}y,t)$.

Theorem 2.6. *We have*

$$\mathrm{Gen}(x^{-1}y,t)=x^{-1}y\cosh(t\sqrt{y^2-x^2})-x^{-1}\sqrt{y^2-x^2}\sinh(t\sqrt{y^2-x^2}). \tag{2.26}$$

Notice that the above relation also implies the formula (2.25) for $\mathrm{Gen}(y,t)$, since

$$\mathrm{Gen}(y,t)=\mathrm{Gen}(x,t)\mathrm{Gen}(x^{-1}y,t). \tag{2.27}$$

One more way to relate $\mathrm{Gen}(y,t)$ to $\mathrm{Gen}(x,t)$ is to utilize the relation (2.10).

Let $M(x,y,t)$ be the exponential generating function of $M_n(x,y)$. Owing to the fact

$$D^n(y)=M_n(x,y),$$

for $n\geq 1$, and the initial value $M_0(x,y)=1$, we see that

$$M(x,y,t)=1-y+\mathrm{Gen}(y,t), \tag{2.28}$$

which implies the David-Barton formula in the bivariate form.

Corollary 2.7. *We have*

$$\sum_{n=0}^{\infty}M_n(x,y)\frac{t^n}{n!}=\frac{\sqrt{y^2-x^2}\cosh(t\sqrt{y^2-x^2})+(1-y)\sinh(t\sqrt{y^2-x^2})}{\sqrt{y^2-x^2}\cosh(t\sqrt{y^2-x^2})-y\sinh(t\sqrt{y^2-x^2})}. \tag{2.29}$$

Let $M(x,t)$ be the exponential generating function of $M_n(x)$. Setting $y=1$ and replacing x^2 by x, we get

$$M(x,t)=\frac{\sqrt{1-x}\cosh(t\sqrt{1-x})}{\sqrt{1-x}\cosh(t\sqrt{1-x})-\sinh(t\sqrt{1-x})}. \tag{2.30}$$

Considering a grammar

$$H=\{a\rightarrow ay,\ x\rightarrow xy,\ y\rightarrow x^2\}, \tag{2.31}$$

we find that $L_n(x)$ and $W_n(x)$ satisfy an exponential relation. It is not hard to see that $D^n(a)$ relative to H coincides with $D^n(x)$ relative to $G=\{x\rightarrow xy,\ y\rightarrow x^2\}$.

Theorem 2.8. *We have*

$$\sum_{n=0}^{\infty} L_n(x)\frac{t^n}{n!} = \exp\left(\sum_{n=0}^{\infty} W_n(x)\frac{t^{n+1}}{(n+1)!}\right). \tag{2.32}$$

Next, we seek a combinatorial interpretation of (2.32). To this end, we introduce a decomposition of a permutation, called the *LW*-decomposition, which is essentially the cycle decomposition, or Foata's first fundamental transformation of a permutation. Assume that σ is a permutation of $[n]$. If 1 appears at the end of σ, then nothing needs to be done. Otherwise, write $\sigma = \pi\tau$, where π ends with 1. Now we make π the first block of the decomposition and repeat the process for τ. If τ ends with the minimum element, then we are done. Otherwise, we continue to decompose τ in the same manner. For example, the permutation $2\,6\,1\,3\,8\,4\,7\,9\,5$ is decomposed into four segments:

$$2\;6\;1 \mid 3 \mid 8\;4 \mid 7\;9\;5.$$

Notice that the blocks are displayed in the increasing order of their minimum elements.

Proof of Theorem 2.8. Let σ be a permutation of $[n]$, where $n \geq 1$, and let $\sigma = \pi_1|\pi_2|\cdots|\pi_k$ be the *LW*-decomposition of σ. For $1 \leq i \leq k$, let π_i' be the permutation obtained from π_i by removing the minimum element at the end. We proceed to show that

$$L(\sigma) = W(\pi_1') + W(\pi_2') + \cdots + W(\pi_k'). \tag{2.33}$$

Asssume that the minimum element of π_i is the element σ_j. We claim that j cannot be a left peak in σ. Otherwise, the element σ_{j+1} would be smaller than σ_j, which is contradictory to the construction of the W-decomposition.

On the other hand, assume that σ_j is not at the end of any segment π_i of the *LW*-decomposition of σ, say σ_j is an element of π_i'. It is evident that j is a left peak of σ if and only if the corresponding position is an exterior peak of π_i'. Thus we arrive at (2.33). This completes the proof. ∎

3 A grammatical calculus for up-down runs

Up-down runs of a permutation are intuitively related to peaks. It is also known that the number of up-down runs of a permutation of $[n]$ equals the length of the longest alternating subsequences, see Stanley [13]. From the viewpoint of grammars, the structure of up-down runs can be understood as an exponential structure built on left peaks. This connection will be made precise in Theorem 3.2. This section is devoted to a grammatical calculus for the number of permutations of $[n]$ with k up-down runs.

For $n \geq 2$, let $\Lambda(n,k)$ denote the number of permutations of $[n]$ with k up-down runs. In particular, define $\Lambda(0,0) = \Lambda(1,1) = 1$ and $\Lambda(0,k) = \Lambda(n,0) = 0$ whenever $n,k \geq 1$. For $n \geq 0$, write

$$\Lambda_n(x) = \sum_{k=1}^{n} \Lambda(n,k)x^k. \tag{3.1}$$

The following grammar was discovered by Ma [12]:

$$G = \{a \to ax,\ x \to xy,\ y \to x^2\}. \tag{3.2}$$

For $n \geq 0$, the bivariate form of $\Lambda_n(x)$ is defined by

$$\Lambda_n(x,y) = \sum_{k=1}^{n} \Lambda(n,k)x^k y^{n-k}. \tag{3.3}$$

In view of the recurrence relation

$$\Lambda(n,k) = k\Lambda(n-1,k) + \Lambda(n-1,k-1) + (n-k+1)\Lambda(n-1,k-2), \tag{3.4}$$

for $n,k \geq 1$, Ma obtained the following expression for $\Lambda_n(x,y)$ in terms of the above grammar G.

Theorem 3.1 (Ma [12])**.** *For $n \geq 0$, we have*

$$D^n(a) = a\Lambda_n(x,y). \tag{3.5}$$

The first few values of $D^n(a)$ are given below:

$$\begin{aligned}
D(a) &= ax,\\
D^2(a) &= axy + ax^2,\\
D^3(a) &= axy^2 + 3ax^2y + 2ax^3,\\
D^4(a) &= axy^3 + 7ax^2y^2 + 11ax^3y + 5ax^4,\\
D^5(a) &= axy^4 + 15ax^2y^3 + 43ax^3y^2 + 45ax^4y + 16ax^5,
\end{aligned}$$

$$D^6(a) \quad = \quad axy^5 + 31ax^2y^4 + 148ax^3y^3 + 268ax^4y^2 + 211ax^5y + 61ax^6.$$

As observed by Bóna [1, 2], see also Stanley [14], $\Lambda_n(x)$ can be expressed in terms of the polynomials $R_n(x)$ associated with alternating runs, and in turn $R_n(x)$ can be expressed in terms of the Eulerian polynomials as shown by David-Barton [7], see also Knuth [9]. More precisely, for $n \geq 1$ and $1 \leq k \leq n$, let $R(n,k)$ denote the number of permutations of $[n]$ with k alternating runs. For $n \geq 0$, let

$$R_n(x,y) = \sum_{k=1}^{n} R(n+1,k)x^k y^{n-k}. \tag{3.6}$$

Ma [12] showed that for $n \geq 0$,

$$D^n(a^2) = a^2 R_n(x,y). \tag{3.7}$$

Let $R(x,t)$ denote the exponential generating function of $R_n(x,y)$ with y set to 1, and let $\Lambda(x,t)$ denote the exponential generating function of $\Lambda_n(x)$. With respect to the grammar G, the relation

$$\mathrm{Gen}(a^2,t) = \mathrm{Gen}^2(a,t)$$

takes the form of the following identity on generating functions, which seems to have been unnoticed before, at least not explicitly,

$$R(x,t) = \Lambda^2(x,t). \tag{3.8}$$

The grammatical labeling to be given in the next section can be regarded as a combinatorial interpretation of Theorem 3.1.

To explore the connections between $\Lambda_n(x)$ and the peak polynomials from the aspect of the grammar G, we present an exponential formula for $\Lambda_n(x)$ and $L_n(x)$. Then we provide a grammatical derivation of a relation on $\Lambda_n(x,y)$, $L_n(x,y)$ and $W_n(x,y)$. Moreover, we find a transformation of grammars leading to a relation on $\Lambda_n(x)$ and $M_n(x)$ due to Ma [12].

Theorem 3.2. *We have*

$$\sum_{n=0}^{\infty} \Lambda_n(x)\frac{t^n}{n!} = \exp\left(\sum_{n=0}^{\infty} xL_n(x^2)\frac{t^{n+1}}{(n+1)!}\right). \tag{3.9}$$

To give a combinatorial interpretation of the above theorem, we present a decomposition of a permutation σ such that the number of up-down runs of σ can be computed from the numbers of left peaks of its components. This decomposition is called the *AL*-decomposition. Let $n \geq 0$ and let σ be a permutation of $[n]$. First, we observe that the number of up-down runs of the permutation $\pi = \sigma(n+1)$ equals $2L(\sigma)+1$. Therefore, $xL_n(x^2)$ equals the generating function of permutations of $[n+1]$ ending with the maximum element $n+1$ with respect to the number of up-down runs.

Given a permutation σ of $[n]$, we may write $\sigma = \pi_1\pi_2\cdots\pi_k$ such that π_1 ends with the maximum element of σ, π_2 ends with the maximum element of $\pi_2\cdots\pi_k$, and so forth. This decomposition can be considered as a combinatorial interpretation of (3.9).

The *LW*-decomposition and the *AL*-decomposition can be reckoned as the dual of each other. The following identity is a bivariate form of a relation observed by Ma [12].

Theorem 3.3. *We have*

$$(x+y)\mathrm{Gen}(a,t) = a\left(\mathrm{Gen}(x,t)+\mathrm{Gen}(y,t)\right), \tag{3.10}$$

or equivalently, for $n \geq 0$,

$$(x+y)\Lambda_n(x,y) = L_n(x,y)+W_n(x,y). \tag{3.11}$$

As a matter of fact, it is a direct consequence of following relations also given by Ma [12]. For $n \geq 1$ and $1 \leq k \leq \lfloor (n-1)/2 \rfloor$, we have

$$L(n,k) = \Lambda(n,2k)+\Lambda(n,2k+1), \tag{3.12}$$

and for $n \geq 1$ and $0 \leq k \leq \lfloor (n-2)/2 \rfloor$,

$$M(n,k) = \Lambda(n,2k+1)+\Lambda(n,2k+2). \tag{3.13}$$

We now proceed to compute $\mathrm{Gen}(a,t)$ using the grammatical calculus.

Proof. Observe that $\frac{x+y}{a}$ is a constant with respect to grammar, that is,

$$D\left(\frac{x+y}{a}\right) = 0.$$

Consequently,

$$\mathrm{Gen}(x,t)+\mathrm{Gen}(y,t) = \mathrm{Gen}(x+y,t) = \mathrm{Gen}\left(a\cdot\frac{x+y}{a},t\right) = \frac{x+y}{a}\mathrm{Gen}(a,t),$$

as claimed. ∎

Substituting (2.22) and (2.25) into (3.10) and using the exponential forms for sinh and cosh, we obtain the following formula.

Corollary 3.4. *We have*

$$\mathrm{Gen}(a,t)=a(y-x)\frac{y+\sqrt{y^2-x^2}+2xe^{\sqrt{y^2-x^2}t}+(y-\sqrt{y^2-x^2})e^{2\sqrt{y^2-x^2}t}}{y^2-x^2+y\sqrt{y^2-x^2}+(y^2-x^2-y\sqrt{y^2-x^2})e^{2\sqrt{y^2-x^2}t}}. \tag{3.14}$$

Setting $a=y=1$ yields Stanley's formula [14]:

$$\Lambda(x,t)=(1-x)\frac{1+\rho+2xe^{\rho t}+(1-\rho)e^{2\rho t}}{1+\rho-x^2+(1-\rho-x^2)e^{2\rho t}}, \tag{3.15}$$

where $\rho=\sqrt{1-x^2}$.

We conclude this section with a grammatical derivation of the following relation on $\Lambda_n(x)$ and $M_n(x)$ due to Ma [11]. For $n\geq 0$,

$$\Lambda_n(x)=\frac{x(1+x)^{n-1}}{2^{n-1}}M_n\left(\frac{2x}{1+x}\right). \tag{3.16}$$

Grammatical Proof of (3.16). Set

$$u=x+y,\ v=\sqrt{x(x+y)}.$$

The grammar $G=\{x\rightarrow xy,\ y\rightarrow x^2\}$ is transformed into

$$G=\left\{u\rightarrow v^2,\ v\rightarrow\frac{uv}{2}\right\}.$$

Using the grammatical interpretation of the interior peak polynomials $M_n(x,y)$, we find that

$$D^n(u)=v^2\sum_{k=0}^{\lfloor(n-1)/2\rfloor}\frac{1}{2^{n-1-k}}M(n,k)u^{n-1-2k}v^{2k}. \tag{3.17}$$

By Theorem 3.3, we see that

$$D^n(a)=\frac{a}{x+y}D^n(x+y), \tag{3.18}$$

which yields

$$D^n(a)\ =\ ax\sum_{k=0}^{\lfloor(n-1)/2\rfloor}\frac{1}{2^{n-1-k}}M(n,k)(x+y)^{n-1-2k}(x(x+y))^k$$

$$= \quad ax\frac{(x+y)^{n-1}}{2^{n-1}}\sum_{k=0}^{\lfloor (n-1)/2\rfloor} M(n,k)\frac{(2x)^k}{(x+y)^k}.$$

Setting $a = y = 1$ completes the proof. ∎

4 A grammatical labeling for up-down runs

For a permutation σ of $[n]$, the up-down labeling of σ can be described as follows. Like the L-labeling given in [6], the labels are assigned to the positions next to each element of σ. For $1 \le i \le n+1$, by a position i we mean the position immediately before σ_i, whereas the position $n+1$ is meant to be the position after σ_n. Associated with a labeling, the weight of a permutation is referred to the product of the labels.

To define the labeling for up-down runs of a permutation σ, which we call the A-labeling, we first patch a zero at the beginning of σ, or equivalently, set $\sigma_0 = 0$. Then the labels are given by the following procedure.

Case 1. For each up run $\sigma_i\sigma_{i+1}\cdots\sigma_j$ possibly with $i = 0$, where $i < j \le n$, two possibilities arise. If $j < n$, that is, σ_j is not the last element of σ, we label the positions $i+1,\ldots,j-1$ by y and label the position j by x.

If $j = n$, that is, σ_j is the last element of σ, then label the position n by a and position $n+1$ by x, and the other positions by y. This case looks a little peculiar, but that is perhaps the way it is.

Case 2. For each down run $\sigma_i\sigma_{i+1}\cdots\sigma_j$, where $i < j$, we always label the position $i+1$ by x, and label the other positions $i+2,\ldots,j$ by y. Moreover, if $j = n$, we label the position $n+1$ by a.

For example, below is the A-labeling of a permutation ending with a down run:

$$0\; y\; 3\; x\; 7\; x\; 5\; x\; 8\; x\; 6\; y\; 1\; y\;\; 4\; x\; 9\; x\; 2\; a.$$

The weight of the above permutation equals ax^6y^3. For an example of σ ending with an up run, let $n = 9$ and $\sigma = 375861249$. Below is the A-labeling,

$$0\; y\; 3\; x\; 7\; x\; 5\; x\; 8\; x\; 6\; y\; 1\; y\;\; 2\; y\; 4\; a\; 9\; x.$$

In this case the weight of σ equals ax^5y^4.

A labeling of a permutation is called a grammatical labeling with respect to G, we mean that when generating the permutations of $[n+1]$ by inserting the element $n+1$ into a permutaiton σ of $[n]$, the substitution rules are applied to each label exactly once. This property makes it possible to compute the total weight (sum of the products of labels) of permutations of $[n+1]$ from the total weight of permutations of $[n]$ by taking the formal derivative with respect to the grammar. Moreover, the labels can be treated as ingredients of the grammatical calculus. The notion of a grammatical labeling was introduced in [5], which says that the combinatorial structure is differentiable in a certain sense. The following theorem justifies the grammatical labeling for the purpose of updating the weights of permutations upon the insertion operations.

Theorem 4.1. *Let $n \geq 1$ and let σ be a permutation of $[n]$. Assume that $1 \leq i \leq n+1$. Let π be the permutation obtained from σ by inserting $n+1$ in σ at position i. Then the weight of π can be derived from that of σ by applying the substitution rule to the label of the element of σ at position i.*

Before presenting the proof, let us give an example. Let $n = 5$ and $\sigma = 2\,5\,4\,1\,3$. In the table below, an underlined label signifies where the element 6 is inserted which is also the label to which the substitution rule is applied.

σ	π	Substitution
$0\,\underline{y}\,2\,x\,5\,x\,4\,y\,1\,a\,3\,x$	$0\,x\,6\,x\,2\,x\,5\,x\,4\,y\,1\,a\,3\,x$	$y \to x^2$
$0\,y\,2\,\underline{x}\,5\,x\,4\,y\,1\,a\,3\,x$	$0\,y\,2\,x\,6\,x\,5\,y\,4\,y\,1\,a\,3\,x$	$x \to xy$
$0\,y\,2\,x\,5\,\underline{x}\,4\,y\,1\,a\,3\,x$	$0\,y\,2\,y\,5\,x\,6\,x\,4\,y\,1\,a\,3\,x$	$x \to xy$
$0\,y\,2\,x\,5\,x\,4\,\underline{y}\,1\,a\,3\,x$	$0\,y\,2\,x\,5\,x\,4\,x\,6\,x\,1\,a\,3\,x$	$y \to x^2$
$0\,y\,2\,x\,5\,x\,4\,y\,1\,\underline{a}\,3\,x$	$0\,y\,2\,x\,5\,x\,4\,y\,1\,x\,6\,x\,3\,a$	$a \to ax$
$0\,y\,2\,x\,5\,x\,4\,y\,1\,a\,3\,\underline{x}$	$0\,y\,2\,x\,5\,x\,4\,y\,1\,y\,3\,a\,6\,x$	$x \to xy$

Proof of Theorem 4.1. It is readily seen that the Theorem is valid for $n = 1, 2$. Now assume that $n \geq 2$ and σ is a permutation of $[n]$. Suppose that π is a permutation of $[n+1]$ created from σ by inserting the element $n+1$ at the position before σ_i, where $1 \leq i \leq n$, or at the position $n+1$, that is, at the end of σ.

First, we consider the case when $i < n$. If the position is labeled by x, it can be seen that if $n+1$ is inserted in σ at position i, the change of weights is connected with the substitution rule $x \to xy$. The two possibilities are illustrated in Figure 1, where $*$ stands for the element $n+1$ and a dotted line indicates the position of insertion.

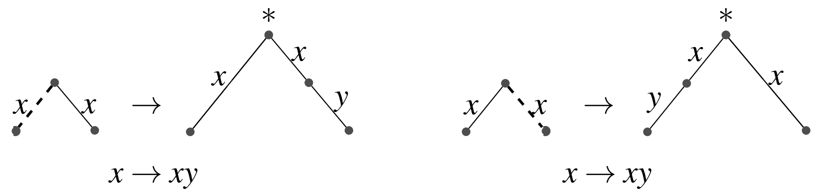

Figure 1: Insertion at a position labeled by x.

If the position i is labeled by y, no matter whether it is in an up run or a down run, the change of weights is always consistent with the substitution rule $y \to x^2$, as illustrated in Figure 2.

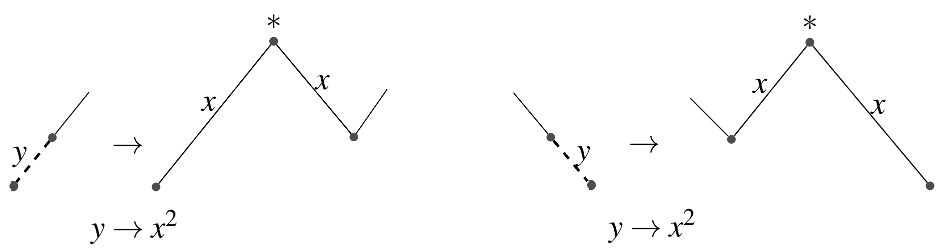

Figure 2: Insertion at a position labeled by y.

We are left with three cases regarding the last two elements of σ. Keep in mind that if σ ends with an up run, the last two labels must be ax.

Case 1. σ ends with a down run and the last two labels are x and a. As discussed before, the insertion at the position labeled by x is in accordance with the rule $x \to xy$ and the insertion at the position of a is reflected by the rule $a \to ax$, see Figure 3.

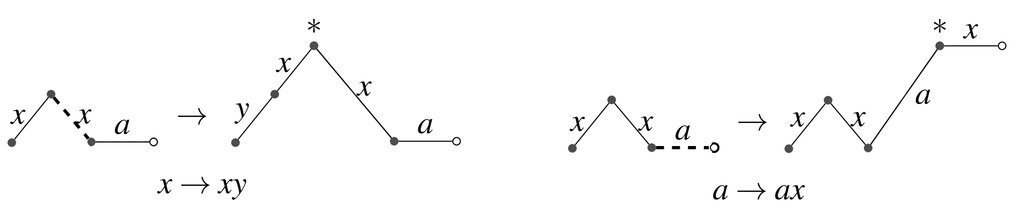

Figure 3: The labeling of σ ends with xa.

Case 2. The labeling of σ ends with ya. As shown in Figure 4, the changes of weights caused by the insertions are coded by the corresponding substitutions.

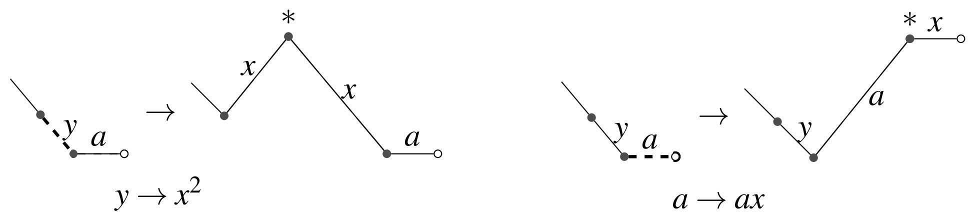

Figure 4: The labeling of σ ends with ya.

Case 3. σ ends with an up run, that is, the last two labels are a and x. As depicted in Figure 5, in either case, the change of weights is in compliance with the grammar, namely, the rules $a \rightarrow ax$ and $x \rightarrow xy$.

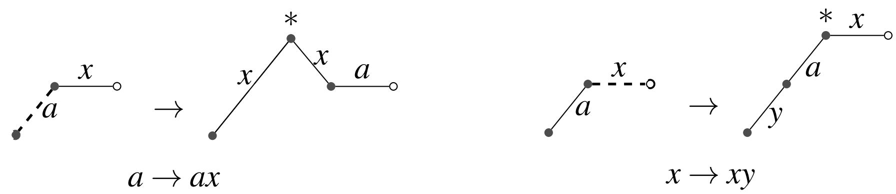

Figure 5: The labeling of σ ends with ax.

Summing up all the cases completes the proof. ∎

Once we have the above up-down labeling of a permutation, we see that $D^n(a) = a\Lambda_n(x,y)$ for all n. The up-down labeling also gives rise to a labeling for alternating runs. Let $n \geq 1$, and let σ be a permutation of $[n]$. Assume that $\sigma = \pi 1 \tau$. Then the alternating labeling of σ consists of the up-down labelings of π' and τ, where π' is the reverse of π. For example, the alternating labeling of the permutation $7\,3\,5\,8\,6\,1\,4\,9\,2$ is given by

$$(x\ 7\ a\ 3\ y\ 5\ x\ 8\ x\ 6\ y\ 1)\ (1\ y\ 4\ x\ 9\ x\ 2\ a).$$

However, we shall not dwell further in this respect.

5 Grammar assisted bijections

In this section, we present a bijection between permutations and increasing trees that maps the number of up-down runs to the number of even degree nonroot vertices of the corresponding increasing tree. While the existence of such a bijection is assured by the two grammatical labelings of the same grammar, a direct construction is less obvious. We shall work out an explicit correspondence.

We begin with a labeling scheme for increasing trees, which we call the parity labeling. Let $n \geq 1$, and let T be an increasing tree on $\{0, 1, \ldots, n\}$. The degree of a vertex of T is understood to be the number of its children. Then the root of T is labeled by a. If v is not the root, then it is labeled by x if it is of even degree, and it is labeled by y if it is of odd degree. For example, an increasing tree along with the parity labeling is shown in Figure 6.

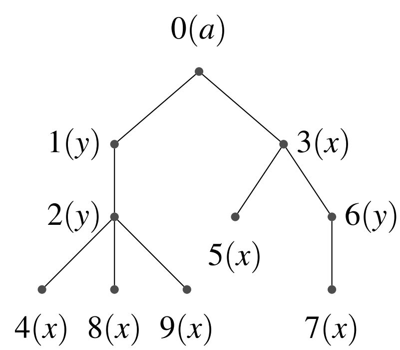

Figure 6: An increasing tree with the parity labeling.

The grammar implies the following bijection.

Theorem 5.1. *For $n \geq 1$, there is a bijection ϕ between the set of permutations σ of $[n]$ and the set of increasing trees T on $\{0, 1, \ldots, n\}$ such that the number of up-down runs of σ equals the number of even degree nonroot vertices of $\phi(\sigma)$.*

Proof. We proceed to describe the construction of the map ϕ. For $n = 1$, there is nothing to be said. For $n = 2$, the correspondence is unique subject to the weight preserving requirement, and it is shown in Figure 7.

We now assume that $n \geq 3$. Let σ be a permutation of $[n]$, and let $\sigma^{(i)}$ be the permutation obtained from σ by removing all the elements that are greater than i. In order to construct an increasing tree T on $\{0, 1, \ldots, n\}$ from σ with the weight preserving property, we devise a procedure to generate a sequence of increasing trees $T^{(2)}$, $T^{(3)}$, …, $T^{(n)}$, where $T^{(2)}$ is the increasing tree corresponding to $\sigma^{(2)}$. As will be seen, for any $2 \leq i \leq n$, $\sigma^{(i)}$ and $T^{(i)}$ not only have the same weight, but also share the same labeling that will

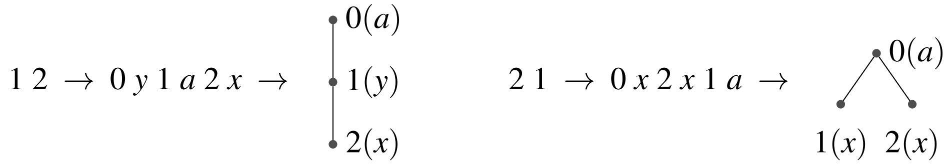

Figure 7: The correspondence for $n = 2$.

be made clear later in the course of the construction. We say that the labeling of $\sigma^{(i)}$ is coherent with the labeling of $T^{(i)}$ provided that the following conditions are satisfied. Set $\pi = \sigma^{(i)}$.

- If $\pi_{i-1} > \pi_i$, then for $1 \leq k \leq i$, the position k of π has the same label as the vertex π_k in $T^{(i)}$ and the position $i+1$ has the same label a as the root of $T^{(i)}$.
- If $\pi_{i-1} < \pi_i$, then for $1 \leq k \leq i-1$, the position k of π has the same label as the vertex π_k in $T^{(i)}$, the position i has the same label a as the root of $T^{(i)}$ and the position $i+1$ of π has the same label x as the vertex π_i in $T^{(i)}$.

Let us now describe the construction of $T^{(i+1)}$ from $T^{(i)}$. Let $\pi = \sigma^{(i)}$. Assume that $\sigma^{(i+1)}$ is obtained from π by inserting $i+1$ at position k, where $1 \leq k \leq i+1$. We need to distinguish two main cases. First, we assume that π ends with a down run, that is, π has an even number of up-down runs, and so the number of x labels of π is even. Then the increasing tree $T^{(i+1)}$ is generated via the following procedure.

1. Assume that the position k of π is labeled by x and k is on the rise, then the next position must be a down step labeled by x, which implies that $k \leq i-1$. We call the position $k+1$ the dual position of k and adjoin the vertex $i+1$ to the vertex π_{k+1} of $T^{(i)}$ as a child to obtain $T^{(i+1)}$ for which the label of π_{k+1} changes from x to y, see Figure 8, where the square vertex signifies where to attach the vertex $i+1$ to $T^{(i)}$. Intuitively, for a position k labeled by x that is on the rise, we need to look at the next position for corresponding operation on the increasing tree. Fortunately, one finds that the assumption that the position $k+1$ is labeled by x is the very property required for the procedure to work. Once it has been noticed, it is not hard to see that it is valid all along.

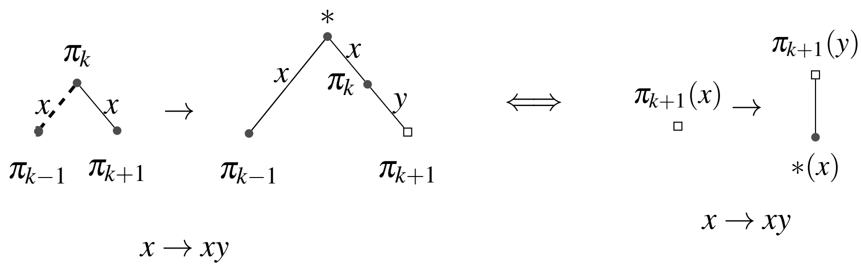

Figure 8: x is on the rise.

2. Assume that the position k of π is labeled by x and k is on the fall. We call the position $k-1$ the dual position of k, and adjoin the vertex $i+1$ to the vertex π_{k-1} of $T^{(i)}$ as a child to obtain $T^{(i+1)}$, see Figure 9, where and throughout the proof the symbol $*$ stands for the element $i+1$ to be added into $\sigma^{(i)}$ and $T^{(i)}$. As before, there is no danger to assume that the vertex π_{k-1} is labeled by x.

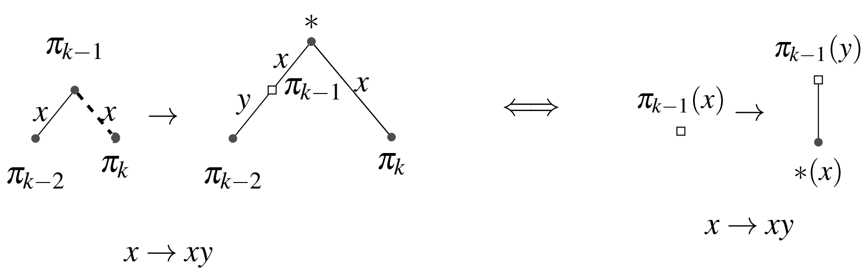

Figure 9: x is on the fall.

3. Assume that the position k of π is labeled by y. Then adjoin $i+1$ to the vertex π_k in $T^{(i)}$ as a child, see Figures 10 and 11.

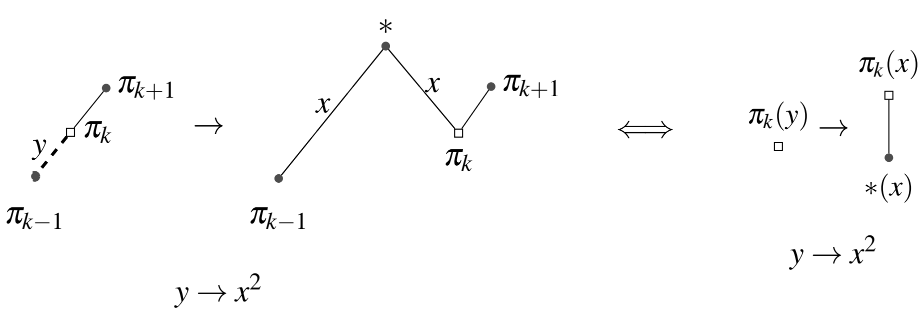

Figure 10: y is on the rise.

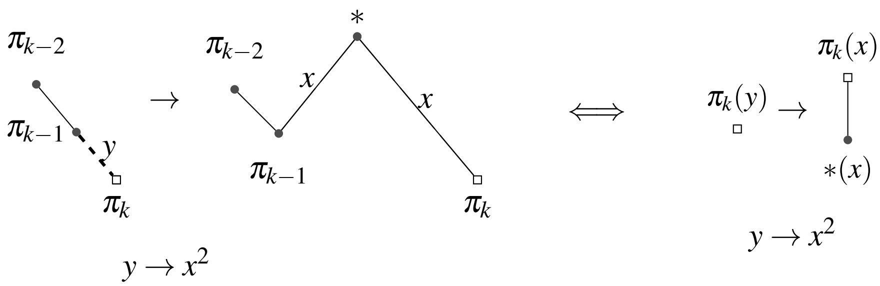

Figure 11: y is on the fall.

4. If $k=i+1$, then adjoin $i+1$ to the root of $T^{(i)}$. In this case, the involved substitution is $a \to ax$.

In any of the above cases, it is observable that after the update of $T^{(i)}$, the labeling of $\sigma^{(i+1)}$ is consistent with the labeling of $T^{(i+1)}$. In other words, all the assumptions in the above argument are well-grounded.

We are now left with the case that π has an odd number of up-down runs, that is, the labels of π end with ax. Here we encounter three possibilities.

1. If $i+1$ is inserted at position i, namely, at the position before π_i, then adjoin the vertex $i+1$ to the root 0. In this case, the vertex $i+1$ is labeled by x and the root remains to be labeled by a, see Fig. 12.

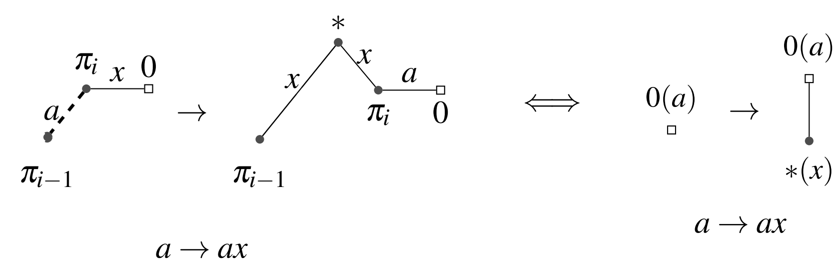

Figure 12: The case $k=i$ and $\pi_{i-1} < \pi_i$.

2. If $i+1$ inserted into π at position $i+1$, that is, $i+1$ is inserted at the end of π, then adjoin $i+1$ to the vertex π_i of $T^{(i)}$. In this case, $i+1$ is labeled by x in $T^{(i+1)}$ and the the label of π_i is switched from x to y, see Fig. 13.

3. If $i+1$ is inserted at position k with $k \le i-1$, then we may follow the above procedure for the case when π has an even number of up-down runs to generate an

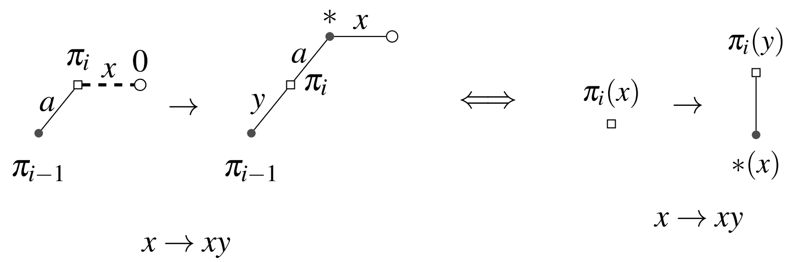

Figure 13: The case $k = i+1$ and $\pi_{i-1} < \pi_i$.

increasing tree $T^{(i+1)}$. It should be pointed out that there are no worries even when the position k is on the rise and labeled by y, in which case $\pi_1\pi_2\cdots\pi_k$ has an odd number of up-down runs, since under this circumstance the x label still appear in pairs in π before the position k.

Up till now, we have provided a procedure to build an increasing tree T from a permutation σ. To see that the process is reversible, we may extract from T a sequence of increasing trees $T^{(2)}$, $T^{(3)}$, ..., $T^{(n)}$, where $T^{(i)}$ is obtained from T by removing all the vertices that are greater than i.

As $T^{(2)}$ is in one-to-one correspondence with a permutation $\sigma^{(2)}$, we may carry out the following steps by starting with $i = 2$. Using $\sigma^{(i)}$ and $T^{(i+1)}$ along with the labeling assumption, one sees that $\sigma^{(i+1)}$ can be retrieved, however the detailed reasoning is omitted. So we reach the conclusion that the procedure leads to a desired bijection. ∎

The operation of locating the vertex in an increasing tree when an insertion into a permutation takes place at a position labeled by x is called the reflection principle. For example, for the increasing tree in Figure 6, the intermediate permutations are given in the table below, where an underlined label indicates where the element $i+1$ is inserted into $\sigma^{(i)}$. Notice that σ has six up-down runs, whereas T has six nonroot vertices of even degree.

i	$\sigma^{(i)}$ with labeling	Weight	Substitution
2	$0\ y\ 1\ \underline{a}\ 2\ x$	axy	$a \to ax$
3	$0\ y\ 1\ x\ 3\ \underline{x}\ 2\ a$	ax^2y	$x \to xy$
4	$0\ y\ 1\ \underline{x}\ 4\ x\ 3\ y\ 2\ a$	ax^2y^2	$x \to xy$
5	$0\ y\ 1\ x\ 5\ x\ 4\ \underline{y}\ 3\ y\ 2\ a$	ax^2y^3	$y \to x^2$
6	$0\ y\ 1\ x\ 5\ x\ 4\ x\ 6\ \underline{x}\ 3\ y\ 2\ a$	ax^4y^2	$x \to xy$
7	$0\ y\ 1\ x\ 5\ x\ 4\ y\ 6\ x\ 7\ x\ 3\ \underline{y}\ 2\ a$	ax^4y^3	$y \to x^2$
8	$0\ y\ 1\ x\ 5\ x\ 4\ y\ 6\ x\ 7\ x\ 3\ \underline{x}\ 8\ x\ 2\ a$	ax^6y^2	$x \to xy$
9	$0\ y\ 1\ x\ 5\ x\ 4\ y\ 6\ x\ 7\ x\ 3\ x\ 9\ x\ 8\ y\ 2\ a$	ax^6y^3	

It should be noted that the above bijection reduces to a correspondence between alternating (down-up) permutations of $[n]$ and even increasing trees on $\{0,1,\ldots,n\}$. Recall that an alternating permutation $[n]$ is referred to a permutation $\sigma_1\sigma_2\cdots\sigma_n$ such that $\sigma_1 > \sigma_2 < \sigma_3 > \cdots$, see Stanley [14], whereas an increasing tree is called even if every vertex possibly except the root is of even degree. Notice that by the alternating condition some authors mean the up-down condition $\sigma_1 < \sigma_2 > \sigma_3 < \cdots$ instead. Clearly, a permutation is alternating if and only if it has no y labels in the up-down labeling, and down-up permutations are in one-to-one correspondence with up-down permutations via complementation. A bijection between up-down permutations and even increasing trees has been given by Kuznetsov, Pak and Postnikov [10].

When applied to a down-up permutation, the procedure in the above theorem produces an even increasing tree. For example, below is a down-up permutation together with the up-down labeling:

$$0\ x\ 3\ x\ 2\ x\ 9\ x\ 6\ x\ 7\ x\ 1\ x\ 8\ x\ 4\ a\ 5\ x.$$

The corresponding even increasing tree is displayed in Figure 14.

We remark that a variant of the construction in the above proof gives a grammar assisted bijection linking the number of left peaks of a permutation to the number of vertices of even degree of an increasing tree, as given in [5].

Theorem 5.2 ([5]). *For $n \geq 1$ and $0 \leq m \leq \lfloor n/2 \rfloor$, there is a one-to-one correspondence between the set of permutations of $[n]$ with m left peaks and the set of increasing trees on*

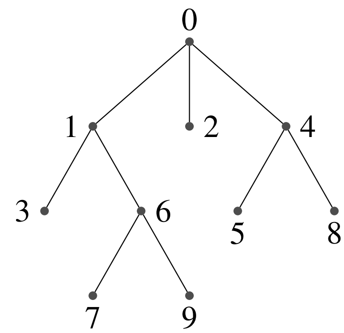

Figure 14: An even increasing tree.

$\{0,1,\ldots,n\}$ *with* $2m+1$ *nonroot vertices of even degree.*

The grammar assisted bijection can be described as follows. For $n \geq 2$, given a permutation σ of $[n]$, let $\sigma^{(2)}$, $\sigma^{(3)}$, ..., $\sigma^{(n)} = \sigma$ be the sequences of permutations such that $\sigma^{(i)}$ is obtained from σ by removing all the elements that are greater than i. We wish to generate a sequence of increasing trees $T^{(2)}$, $T^{(3)}$, ..., $T^{(n)}$ such that $\sigma^{(i)}$ and $T^{(i)}$ have the same weight, as clarified below.

Recall that the L-labeling of σ is defined to assign the label x to the two positions that form a left peak as well as to the last position, and the rest of the positions are endowed with the label y. For example, below is the L-labeling of a permutation of $[9]$:

$$0\; y\; 1\; x\; 9\; x\; 8\; y\; 3\; x\; 6\; x\; 5\; y\; 4\; y\; 2\; y\; 7\; x.$$

On the other hand, for an increasing tree T on $\{0,1,\ldots,n\}$, a vertex is labeled by x is of even degree, otherwise it is labeled by y. We call this labeling scheme the L-labeling of T, see Figure 15.

Suppose that $\sigma^{(i+1)}$ is obtained from $\sigma^{(i)}$ by inserting $i+1$ at position k, where $1 \leq k \leq i+1$. Let $\pi = \sigma^{(i)}$. Observe that the last label of $\sigma^{(i)}$ is always x and the rest of the x labels always appear in pairs. To produce $T^{(i+1)}$ from $T^{(i)}$, we stand by the following rules:

1. Assume that $k \leq i-1$. Depending upon whether the position k is labeled by x or y, we proceed as in the case for up-down runs.

2. Assume that $k = i$. In this case, the position k is labeled by y. If $\pi_{i-1} < \pi_i$, then

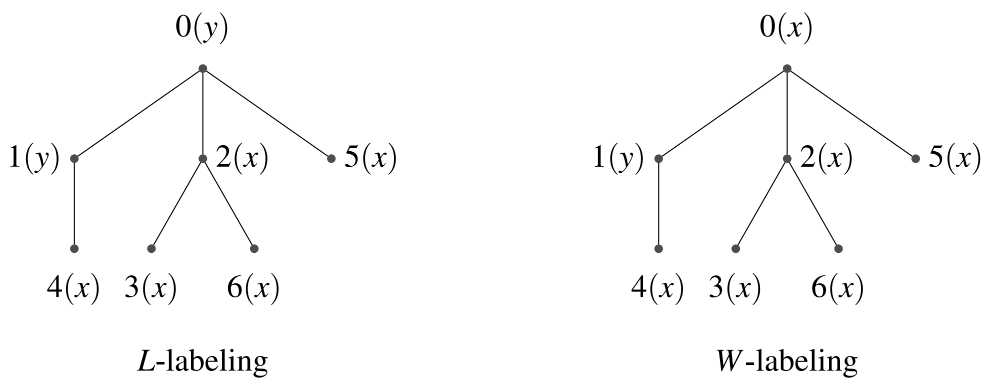

Figure 15: Two labelings of an increasing tree.

adjoin $i+1$ to the root in $T^{(i)}$ to generate $T^{(i+1)}$; otherwise adjoin $i+1$ to the vertex π_i.

3. Assume that $k=i+1$. If $\pi_{i-1}<\pi_i$, then adjoin $i+1$ to the vertex π_i of $T^{(i)}$; otherwise adjoin $i+1$ to the root to generate $T^{(i+1)}$.

For instance, let us consider the permutation taken from the example for Theorem 5.1,

$$\sigma = 1\,5\,4\,6\,7\,3\,9\,8\,2,$$

it can be checked that σ corresponds to the same increasing tree as in Figure 6 with the label a of the root replaced by x. This is no coincidence, as will be seen.

Naturally, one may wonder how to map the number of exterior peaks to a statistic of an increasing tree. Here is a key observation about the W-labeling, that is, the x labels always appear in pairs. This property allows us to present the above grammar-assisted bijections in a much more concise manner.

Imagine that an increasing tree with a single vertex 0 is labeled by y. Then we comply with the parity rules as before to produce an increasing tree by successively adding vertices. The associated labeling is called the W-labeling of an increasing tree, see Figure 15.

Theorem 5.3. *For $n\geq 1$, there is a bijection mapping a permutation σ of $[n]$ with k exterior peaks to an increasing tree T on $\{0,1,\ldots,n\}$ with j vertices of even degree such that $k=\lfloor (j+1)/2\rfloor$.*

Note that for a permutation σ with k exterior peaks, the corresponding increasing tree has $2k$ vertices labeled by x. This parity property becomes transparent if we define the degree of a vertex of an increasing tree as the number of adjacent vertices as if the tree is a graph in the usual sense. Obviously such a concern is superficial, and we had better stick to the usual terminology for rooted trees especially when we have a clear picture in mind.

We conclude with a remark that all the aforementioned bijections can be presented in a unified way solely in terms of the reflection principle. That is to say, the bijection in Theorem 5.3 does the same job as the other grammar assisted bijections. For example, the increasing tree in Figure 15 invariably corresponds to the permutation 6 2 4 3 1 5 for any of the three labeling schemes.

We now return to the key step of the reflection operation. Let $n \geq 1$, assume that $\sigma = \sigma_1\sigma_2\cdots\sigma_n$ is a permutation of $[n]$. For $1 \leq k \leq n+1$, the position before σ_k is called position k, whereas the last position is referred to position 0. Now, the exterior peaks gather the corresponding positions in pairs, so that we do not have to take special care of the last two elements concerning their relative order. Moreover, we may index the positions of σ by $0, 1, \ldots, n$, and this yields a bijection that is seemingly different, but is of the same nature.

Needless to say, this bijection provides a correspondence between down-up permutations and even increasing trees, without using the language of the up-down labelings.

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