European J. Combin.
109 (2023) Paper No. 103655.

The Gessel Correspondence and the Partial γ-Positivity of the Eulerian Polynomials on Multiset Stirling Permutations

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Abstract

Pondering upon the grammatical labeling of 0-1-2 increasing plane trees, we come to the realization that the grammatical labels play a role as records of chopped off leaves of the original increasing binary trees. While such an understanding is purely psychological, it does give rise to an efficient apparatus to tackle the partial γ-positivity of the Eulerian polynomials on multiset Stirling permutations, as long as we bear in mind the combinatorial meanings of the labels x and y in the Gessel representation of a k-Stirling permutation by means of an increasing $(k+1)$-ary tree. More precisely, we introduce a Foata-Strehl action on the Gessel trees resulting in an interpretation of the partial γ-coefficients of the aforementioned Eulerian polynomials, different from the ones found by Lin-Ma-Zhang and Yan-Huang-Yang. In particular, our strategy can be adapted to deal with the partial γ-coefficients of the second order Eulerian polynomials, which in turn can be readily converted to the combinatorial formulation due to Ma-Ma-Yeh in connection with certain statistics of Stirling permutations.

Keywords: Eulerian polynomials, Stirling permutations on a multiset, γ-positivity, increasing plane trees

AMS MSC: 05A15, 05A19

1 Introduction

This work is concerned with the partial γ-coefficients of the Eulerian polynomials on multiset Stirling permutations, which are also called the Stirling polynomials.

For $n \geq 1$, let S_n denote the set of permutations of $[n] = \{1, 2, \ldots, n\}$. For a permutation $\sigma = \sigma_1\sigma_2 \cdots \sigma_n$ in S_n, we adopt the convention that a zero is patched both at the beginning and at the end of σ, that is, $\sigma_0 = \sigma_{n+1} = 0$. An ascent of σ is defined to be an index i $(1 \leq i \leq n)$ such that $\sigma_{i-1} < \sigma_i$, whereas a descent is defined to be an index i $(1 \leq i \leq n)$ such that $\sigma_i > \sigma_{i+1}$. The numbers of ascents and descents of σ are denoted by $\mathrm{asc}(\sigma)$ and $\mathrm{des}(\sigma)$, respectively. The bivariate Eulerian polynomials $A_n(x, y)$ are defined by

$$A_n(x, y) = \sum_{\sigma \in S_n} x^{\mathrm{asc}(\sigma)} y^{\mathrm{des}(\sigma)}. \tag{1.1}$$

Setting $y = 1$, $A_n(x, y)$ takes the form of the usual Eulerian polynomials, or the descent polynomials of S_n.

One of the most remarkable facts about the Eulerian polynomials is the γ-positivity discovered by Foata and Schüzenberger [9], which has been extensively studied ever since, see, for example, [1, 3, 6, 8].

We shall choose to work with the bivariate version $A_n(x, y)$. For $n \geq 1$, the following expression of $A_n(x, y)$ is called γ-expansion:

$$A_n(x, y) = \sum_{k=1}^{\lfloor (n+1)/2 \rfloor} \gamma_{n,k} (xy)^k (x + y)^{n+1-2k}. \tag{1.2}$$

The coefficients $\gamma_{n,k}$ are called the γ-coefficients. Foata and Schüzenberger discovered a combinatorial interpretation of the γ-coefficients implying the positivity. More precisely, it has been shown that

$$\gamma_{n,k} = |\{\sigma \in S_n \mid \mathrm{des}(\sigma) = k, \mathrm{ddes}(\sigma) = 0\}|, \tag{1.3}$$

where $\mathrm{ddes}(\sigma)$ means the number of double descents of σ, that is, the number of indices i such that $\sigma_{i-1} > \sigma_i > \sigma_{i+1}$.

As a notable extension of the Eulerian polynomials, Gessel and Stanley [12] introduced the notion of Stirling permutations whose descent polynomials have been called the second order Eulerian polynomials.

For $n \geq 1$, let $[n]_2$ denote the multiset $\{1^2, 2^2, \ldots, n^2\}$, where i^2 signifies two occurrences of i. A permutation σ on $[n]_2$ is said to be a Stirling permutation if for any i, the elements between the two occurrences of i in σ, if any, are greater than i. For $n \geq 1$, the set of Stirling permutations of $[n]_2$ is usually denoted by Q_n. As before, we assume that a Stirling permutation is patched a zero both at the beginning and at the end. The statistics asc and des can be analogously defined for Stirling permutations.

The number of Stirling permutations in Q_n with $k+1$ descents, often denoted by $C(n, k)$, is called the second order Eulerian polynomial. For Stirling permutations, one more statistic naturally comes on the scene, that is, the number of plateaux. It appears that the notion of a plateau was first introduced by Dumont [7] under the name of a repetition. Let $\sigma = \sigma_1\sigma_2\cdots\sigma_{2n} \in Q_n$. An index i $(1 \leq i \leq 2n)$ is called a plateau if $\sigma_i = \sigma_{i+1}$. The number of plateaux of σ is denoted by $\mathrm{plat}(\sigma)$.

Bóna [2] proved that the three statistics asc, plat and des are equidistributed over Q_n. Janson [14] constructed an urn model to prove the symmetry of the joint distribution of the three statistics.

In fact, Dumont [7] defined the trivariate second order Eulerian polynomials

$$C_n(x,y,z) = \sum_{\sigma\in Q_n} x^{\mathrm{asc}(\sigma)} y^{\mathrm{des}(\sigma)} z^{\mathrm{plat}(\sigma)}, \tag{1.4}$$

which can be regarded as an extension of the second order Eulerian polynomials of Gessel-Stanley and the bivariate Eulerian polynomials. Apparently, when a Stirling permutation $\sigma \in Q_n$ has n plateaux, it can be considered as a permutation on $[n]$ with each element i replaced by ii. It was noticed by Dumont that $C_n(x,y,z)$ are symmetric in x, y, z.

The question of γ-positivity for $C_n(x,y,z)$ has been studied by Ma-Ma-Yeh [17]. Write

$$C_n(x,y,z) = \sum_{i=1}^{n} z^i \sum_{j=0}^{\lfloor(2n+1-i)/2\rfloor} \gamma_{n,i,j}(xy)^j(x+y)^{2n+1-i-2j}, \tag{1.5}$$

which is called the partial γ-expansion of $C_n(x,y,z)$. The coefficients $\gamma_{n,i,j}$ are called the partial γ-coefficients. Making use of a context-free grammar argument, Ma-Ma-Yeh showed that $C_n(x,y,z)$ are partial γ-positive in the sense that the coefficients $\gamma_{n,i,j}$ are nonnegative. Moreover, they obtained a combinatorial interpretation of $\gamma_{n,i,j}$ resorting to certain statistics on Stirling permutations.

The structure of a Stirling permutation can be further extended to a multiset. Throughout this paper, we assume that $n \geq 1$. Unless specified otherwise, we always assume that

$$M = \{1^{k_1}, 2^{k_2}, \ldots, n^{k_n}\}, \tag{1.6}$$

where $k_i \geq 1$ for all i and i^{k_i} stands for k_i occurrences of i. Moreover, we always designate K to denote $k_1 + k_2 + \cdots + k_n$.

A permutation $\sigma = \sigma_1\sigma_2\ldots\sigma_K$ of M is said to be a Stirling permutation if $\sigma_i = \sigma_j$ with $i < j$, then $\sigma_k \geq \sigma_i$ for any $i < k < j$. The set of Stirling permutations of M will be denoted by Q_M. For $M = [n]_k = \{1^k, 2^k, \ldots, n^k\}$, a Stirling permutation on M is called a k-Stirling permutation.

The statistics asc, des and plat for Stirling permutations in Q_n can be literally carried over to Q_M. Then the Eulerian polynomials on Stirling permutations of a multiset M can be defined by

$$C_M(x,y,z) = \sum_{\sigma\in Q_M} x^{\mathrm{asc}(\sigma)} y^{\mathrm{des}(\sigma)} z^{\mathrm{plat}(\sigma)}, \tag{1.7}$$

see also [16]. In particular, the descent polynomial over Q_M is often denoted by

$$Q_M(x) = \sum_{\sigma\in Q_M} x^{\mathrm{des}(\sigma)}, \tag{1.8}$$

While $C_M(x,y,z)$ are no longer symmetric in general, they are symmetric in x and y. This means that $C_M(x,y,z)$ can be expressed as

$$C_M(x,y,z) = \sum_{i=0}^{K-n} z^i \sum_{j=1}^{\lfloor (K+1-i)/2 \rfloor} \gamma_{M,i,j} (xy)^j (x+y)^{K+1-i-2j}. \tag{1.9}$$

The above relation (1.9) is called the partial γ-expansion of $C_M(x,y,z)$. The coefficients $\gamma_{M,i,j}$ are called the partial γ-coefficients of $C_M(x,y,z)$. The nonnegativity of the coefficients $\gamma_{M,i,j}$ is referred to as the partial γ-positivity.

The partial γ-positivity for several multivariate polynomials associated various classes of permutations has recently been studied in Ma-Ma-Yeh [17], Lin-Ma-Zhang [16] and Yan-Huang-Yang [18].

The objective of this paper is to present a combinatorial treatment of the partial γ-coefficients of $C_M(x,y,z)$. In particular, our strategy can be adapted to deal with the partial γ-coefficients of the second order Eulerian polynomials, which in turn can be readily converted to the combinatorial formulation obtained by Ma-Ma-Yeh [17]. Our combinatorial interpretation of $\gamma_{M,i,j}$ is built on the Gessel trees which are increasing plane trees in which the internal vertices are represented by distinct numbers, whereas in the combinatorial framework of Yan, Huang and Yang [18], two vertices are allowed to be represented by the same number.

The underlying combinatorial structure employed in this work is that of a Gessel tree for a Stirling permutation of a multiset. Our main result (Theorem 3.1) is a combinatorial interpretation in light of canonical Gessel trees. A closely related approach is to utilize context-free grammars, which leads to the notion of pruned Gessel trees. A careful study of the Gessel correspondence reveals the properties needed to turn Theorem 3.1 into an equivalent statement on multiset Stirling permutations (Theorem 5.2). Specializing to the set Q_n of Stirling permutations, we are led to the combinatorial interpretation of the partial γ-coefficients (Theorem 6.1) due to Ma-Ma-Yeh [17].

The rest of this paper is organized as follows. In Section 2, we give an overview of the Gessel correspondence between increasing trees and multiset Stirling permutations. We also address a refined property of the Gessel correspondence. Section 3 is devoted to a classification of Gessel trees based on the structure of canonical Gessel trees. An operation in the spirit of the Foata-Strehl action [10] is introduced to serve the purpose. As a main result of this work, we present a combinatorial interpretation of the partial γ-coefficients of $C_M(x,y,z)$. In Section 4, we present a context-free grammar approach. In fact, such a consideration has spurred the notion of pruned Gessel trees and has provided a motivation for the Foata-Strehl action in Section 3. The aim of Section 5 is to present a combinatorial explanation of the partial γ-coefficients in terms of multiset Stirling permutations. In Section 6, we explain how to get to the result of Ma-Ma-Yeh. It turns out that the symmetry property of $C_n(x,y,z)$ is required to fulfill the task.

2 The Gessel correspondence

The Gessel correspondence [11] is an extension of the classical bijection between permutations and

It is the aim of this paper to utilize this correspondence to study the γ-positivity and the partial γ-positivity of the Eulerian polynomials on multiset Stirling permutations. Gessel [11] established a bijection ϕ between k-Stirling permutations and $(k+1)$-ary increasing trees in which only the internal vertices are labeled and represented by solid dots, whereas the external vertices (leaves) are not labeled and represented by circles.

For a Stirling permutation σ on M. If $\sigma = \emptyset$, let $\phi(\sigma)$ be the tree with only one (unlabeled) vertex. If $\sigma \neq \emptyset$, let i be the smallest element of σ. Then σ can be uniquely decomposed as $w_0\, i\, w_1\, i \cdots w_{k-1}\, i\, w_k$, where w_j is either empty or a Stirling permutation for all $0 \leq j \leq k$. Set i to be the root of $\phi(T)$, and set $\phi(T_j)$ to be the j-th subtree (counting from left to right) of i. This procedure yields a recursive construction of $\phi(\sigma)$.

For example, let $\sigma = 33552217714664 \in Q_7$, the corresponding ternary increasing tree is illustrated in Figure 1.

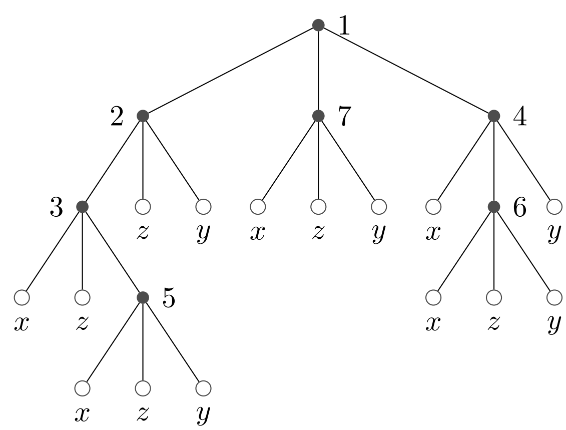


Figure 1: A ternary increasing tree.

Recall that for the classical representation of permutations by increasing binary trees, every vertex (regardless of an internal vertex or a leaf) is labeled, whereas in the Gessel representation, only the internal vertices are labeled, and external vertices (leaves) are not labeled, where a leaf is drawn as a circle.

In the Gessel representation, a leaf is called an x-leaf if it is the first child, and is called a y-leaf if it is the last child; otherwise, it is called a z-leaf. Such leaves are crucial for the study of the γ-positivity. As will be seen, they are the natural ingredients of a Foata-Strehl action.

Furthermore, Janson-Kuba-Panholzer [15] introduced the notion of a j-plateau of a k-Stirling permutation. In this event, an index i is said to be a j-plateau of σ if $\sigma_i = \sigma_{i+1} = r$ and σ_i is the j-th occurrence of r in σ. If a leaf v of T is the j-th child for some $2 \leq j \leq k$, then v is called a z_j-leaf. Janson-Kuba-Panholzer [15] showed that a z_j-leaf in a $(k+1)$-ary increasing tree corresponds to a j-plateau of σ.

By a Gessel tree on M we mean a plane tree with internal vertices $1, 2, \ldots, n$ along with unlabeled external vertices such that the internal vertex i has exactly $k_i + 1$ children and the internal vertices

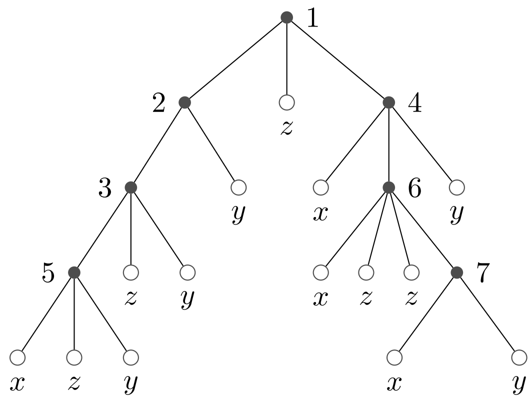


Figure 2: A Gessel tree on $M = \{1^2, 2, 3^2, 4^2, 5^2, 6^3, 7\}$.

Denote by $\mathrm{xleaf}(T)$, $\mathrm{yleaf}(T)$ and $\mathrm{zleaf}(T)$ the numbers of x-leaves, y-leaves and z-leaves, respectively. The following property can be easily deduced from the recursive construction of the Gessel correspondence.

Proposition 2.1 *Let n and M be given as before. The Gessel map ϕ establishes a one-to-one correspondence between Stirling permutations of M and Gessel trees on M. Moreover, let $\sigma \in Q_M$ and $T = \phi(\sigma)$. Then*

$$(\mathrm{asc}(\sigma), \mathrm{des}(\sigma), \mathrm{plat}(\sigma)) = (\mathrm{xleaf}(T), \mathrm{yleaf}(T), \mathrm{zleaf}(T)). \tag{2.1}$$

Instead of reproducing a proof of the above property, we discuss a refined description of the Gessel correspondence that will be needed later in this paper. For this purpose, we shall introduce the notion of the Gessel decomposition of a Stirling permutation on M.

Let σ be a Stirling permutation on M. For any $1 \leq i \leq n$, the i-segment of σ, denoted by $S_i(\sigma)$, is defined to be the unique sequence $\sigma_r\sigma_{r+1}\cdots\sigma_s$ containing the element i, where $1 \leq r \leq s \leq K$, such that $\sigma_{r-1} < \sigma_r$ and $\sigma_s > \sigma_{s+1}$ with the convention $\sigma_0 = \sigma_{K+1} = 0$. It is clear from the definition of a Stirling permutation of M that the i-segment of σ is well-defined. In fact, one sees that $S_i(\sigma)$ contains all the occurrences of i in σ.

For example, for the Stirling permutation $\sigma = 5533211466674$ of $M = \{1^2, 2, 3^2, 4^2, 5^2, 6^3, 7\}$ corresponding to the Gessel tree in Figure 2, we have

$$\begin{array}{rcl} S_1(\sigma) & = & 5533211466674, \\ S_2(\sigma) & = & 55332, \\ S_3(\sigma) & = & 5533, \\ S_4(\sigma) & = & 466674, \\ S_5(\sigma) & = & 55, \\ S_6(\sigma) & = & 6667, \end{array}$$

The idea of the Gessel correspondence can be perceived as a decomposition of an i-segment of a Stirling permutation. Let σ be a Stirling permutation on a multiset M. The Gessel decomposition of the i-segment $S_i(\sigma)$ is defined to be a decomposition

$$S_i(\sigma) = w_0\, i\, w_1\, i\, w_2\, i\, \cdots\, i\, w_{k_i}, \tag{2.2}$$

where for each $0 \leq t \leq k_i$, w_t is either empty or a j-segment for some j.

We now come to the following refined property of the Gessel correspondence.

Proposition 2.2 *Let n and M be given as before. Let σ be a Stirling permutation on M and let T be the corresponding Gessel tree. Assume that σ_p is the first occurrence of i in σ and σ_q is the last occurrence of i in σ. Then the vertex i has an x-leaf in T if and only if p is an ascent and i has a y-leaf if and only if q is a descent of σ.*

Proof. First, assume that i has an x-leaf in T. By the Gessel correspondence, w_0 in the Gessel decomposition of σ as given in (2.2) is empty. Since an i-segment of σ is surrounded by two elements smaller than i, σ_p is immediately preceded by a smaller element. This means that p is an ascent of σ. Conversely, if p is an ascent of σ, then w_0 must be empty, and so i has an x-leaf in T. The same reasoning applies to a descent of σ involving the last occurrence of i and a y-leaf of i in T, and hence the proof is complete. ∎

3 A Foata-Strehl action on Gessel trees

To give a combinatorial interpretation of the γ-coefficients, we shall define an action on a Gessel tree, which plays the same role as the original Foata-Strehl group action for the γ-coefficients. Such an action is often called a modified Foata-Strehl action.

Let T be a Gessel tree on M. We say that an x-leaf in T is balanced if its parent has a y-leaf; otherwise, we say that the x-leaf is unbalanced. Similarly, a y-leaf is said to be balanced if its parent has an x-leaf; otherwise, it is said to be unbalanced.

For $1 \leq i \leq n$, the Foata-Strehl action ψ_i is defined as follows. It does nothing to T unless the internal vertex i possesses an unbalanced y-leaf. In case the vertex i has unbalanced y-leaf, then $\psi_i(T)$ is defined to be the Gessel tree obtained from T by interchanging the first child (along with its the subtree) and the last child of vertex i, and keeping the order of the other children unchanged. As a result, the unbalanced y-leaf becomes an unbalanced x-leaf. Figure 3 depicts the action of ψ_2 on the Gessel tree in Figure 2.

Let G_M be the set of Gessel trees on M. By Proposition 2.1, the Eulerian polynomials $C_M(x, y, z)$ as defined in (1.4) can be expressed in terms of the Gessel trees, namely,

$$C_M(x, y, z) = \sum_{T \in G_M} x^{\mathrm{xleaf}(T)} y^{\mathrm{yleaf}(T)} z^{\mathrm{zleaf}(T)}. \tag{3.1}$$

The above connection is our starting point to arrive at a combinatorial interpretation of the partial

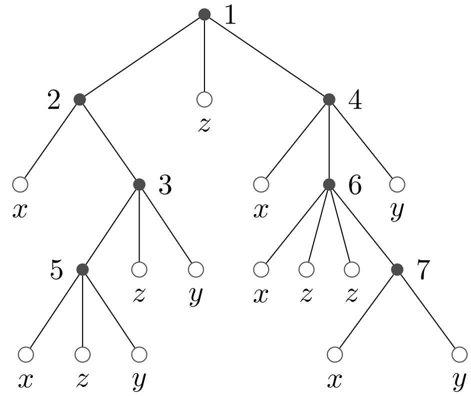


Figure 3: Action of ψ_2 on the Gessel tree in Figure 2.

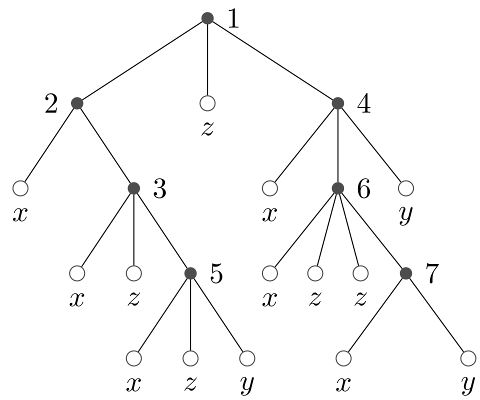


Figure 4: A canonical Gessel tree.

which we call canonical Gessel trees. To be more specific, we say that a Gessel tree is canonical if it does not contain any internal vertex having an unbalanced y-leaf.

Theorem 3.1 *Let n, M and K be given as before. Then*

$$C_M(x, y, z) = \sum_{i=0}^{K-n} z^i \sum_{j=1}^{\lfloor (K+1-i)/2 \rfloor} \gamma_{M,i,j} (xy)^j (x+y)^{K+1-i-2j}, \tag{3.2}$$

where $\gamma_{M,i,j}$ is the number of canonical Gessel trees on M with i z-leaves, j y-leaves.

Proof. Let $G_{M,i}$ denote the set of Gessel trees on M with i z-leaves. Then Theorem 3.1 is equivalent to

$$\sum_{T \in G_{M,i}} x^{\mathrm{xleaf}(T)} y^{\mathrm{yleaf}(T)} z^{\mathrm{zleaf}(T)} = \sum_{T \in H_{M,i}} (xy)^{\mathrm{yleaf}(T)} (x+y)^{K+1-i-2\mathrm{yleaf}(T)} z^i, \tag{3.3}$$

where $H_{M,i}$ denotes the set of canonical trees $T \in G_{M,i}$ without any unbalanced y-leaves.

For each $T \in H_{M,i}$, we define $\mathrm{Orbit}(T)$ to be the set of Gessel trees S that can be transformed to T via the Foata-Strehl action. Clearly, the number of z-leaves is invariant under any Foata-Strehl

$\psi_j(S) = T$. Let $\mathrm{uxleaf}(T)$ and $\mathrm{bxleaf}(T)$ denote the numbers of unbalanced x-leaves and balanced x-leaves of T, respectively. Since there are no unbalanced y-leaves in T,

$$\mathrm{bxleaf}(T) = \mathrm{yleaf}(T).$$

Given that the total number of leaves in T equals $K + 1$, we find that

$$2\,\mathrm{yleaf}(T) + \mathrm{uxleaf}(T) = \mathrm{xleaf}(T) + \mathrm{yleaf}(T) = K + 1 - i,$$

that is,

$$\mathrm{uxleaf}(T) = K + 1 - i - 2\mathrm{yleaf}(T). \tag{3.4}$$

Therefore,

$$\sum_{S \in \mathrm{Orbit}(T)} x^{\mathrm{xleaf(T)}} y^{\mathrm{yleaf(T)}} z^{\mathrm{zleaf}(T)} = (xy)^{\mathrm{yleaf}(T)} (x+y)^{K+1-i-2\mathrm{yleaf}(T)} z^i.$$

Summing over all canonical Gessel trees in $H_{M,i}$ yields (3.3), and hence the proof is complete. ∎

4 A context-free grammar approach

In this section, we present a context-free grammar approach to the partial γ-positivity of $C_M(x, y, z)$. Observe that the construction of Gessel trees implies a recursive formula for the computation of the Eulerian polynomials $C_M(x, y, z)$. For the case of Stirling permutations, the trivariate second order Eulerian polynomials $C_n(x, y, z)$ are defined by (1.4). Dumont [7] deduced the recursion

$$C_n(x, y, z) = xyz \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right) C_{n-1}(x, y, z), \tag{4.1}$$

where $n \geq 1$ and $C_0(x, y, z) = x$, see also Haglund-Visontai [13].

Theorem 4.1 *Let n and M be given as before. Set $M' = \{1^{k_1}, 2^{k_2}, \ldots, (n-1)^{k_{n-1}}\}$. Then*

$$C_M(x, y, z) = xyz^{k_n - 1} \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right) C_{M'}(x, y, z) \tag{4.2}$$

with $C_\emptyset(x, y, z) = x$.

In the language of context-free grammars, for $k \geq 1$, define the grammar

$$G_k = \{x \to xyz^{k-1}, \quad y \to xyz^{k-1}, \quad z \to xyz^{k-1}\}. \tag{4.3}$$

Let D_k denote the formal derivative with respect to the grammar G_k. Then the above relation (4.2) can be rewritten as

$$C_M(x, y, z) = D_{k_n} D_{k_{n-1}} \cdots D_{k_1}(x), \tag{4.4}$$

where $D_{k_n} D_{k_{n-1}} \cdots D_{k_1}$ is meant to apply D_{k_1} first, followed by the applications of D_{k_2} and so on. The grammatical expression is informative to establish a connection to the partial γ-positivity of $C_M(x, y, z)$. Thanks to the idea of change of variables due to Ma-Ma-Yeh [17], we set to get

$$D_k(u) = D_k(xy) = D_k(x)y + xD_k(y) = xy(x+y)z^{k-1} = uvz^{k-1}, \tag{4.5}$$

$$D_k(v) = D_k(x+y) = 2xyz^{k-1} = 2uz^{k-1}, \tag{4.6}$$

and

$$D_k(z) = xyz^{k-1} = uz^{k-1}. \tag{4.7}$$

Now, for the variables u, v, z, the grammar G_k can be recast as

$$G_k = \{u \to uvz^{k-1}, \quad v \to 2uz^{k-1}, \quad z \to uz^{k-1}\}. \tag{4.8}$$

The following theorem can be viewed as an equivalent form of Theorem 3.1 restated on a variation of Gessel trees, called pruned Gessel trees, along with a grammatical labeling by using the variables u, v and z. It is worth mentioning that the grammatical labeling can be thought as a guideline to generate the γ-coefficients. Indeed, it drops a hint in search for a Foata-Strehl action.

Intuitively, a pruned Gessel tree is obtained from a Gessel tree by chopping off the x-leaves and y-leaves. We have to say that this viewpoint alone is of no substantial help. Here comes the idea of characterizing pruned Gessel trees. Assume that T is a Gessel tree on M. Then internal vertices of the pruned Gessel tree obtained from T are of the following four types.

Type 1: i has $k_i + 1$ children, and neither the first nor the last child is a leaf. This means that i has neither an x-leaf nor a y-leaf in T.

Type 2: i has k_i children, and the first child of i is not a leaf. This means that i has a y-leaf, but no x-leaf, in T. In this case, the vertex i is associated with a label y.

Type 3: i has k_i children, and the last child of i is not a leaf. This means that i has an x-leaf, but no y-leaf, in T. In this case, the vertex i is associated with a label x.

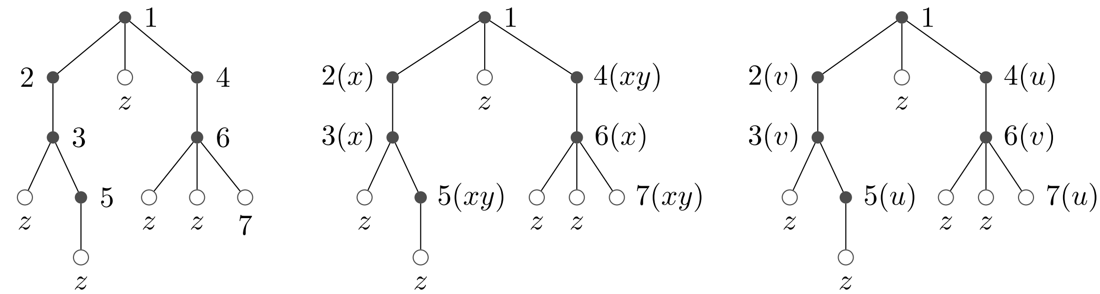
Type 4: i has $k_i - 1$ children. This means that i has both an x-leaf and a y-leaf in T. In this case, the vertex i is associated a label xy.

Conversely, the four types of vertices are sufficient for the generation of pruned Gessel trees. Figure 5 displays the pruned tree of the canonical Gessel tree in Figure 4 along with the (x, y)-labeling and the (u, v)-labeling.

When restricted to pruned canonical Gessel trees, Type 2 vertices are not allowed to show up. This property enables us to substitute the label xy with u, and the label x with v.

The weight of a pruned canonical Gessel tree T, denoted by $w(T)$, is defined to be the product of the (u, v)-labels. As usual, the empty product is meant to be 1. For example, the weight of the pruned Gessel tree in Figure 5 equals $u^3v^3z^5$. The γ-polynomial $\gamma_M(u, v, z)$, called the γ-polynomial of M, is defined to be the generating function of the partial γ-coefficients in (3.2). To be more specific, let

$$\gamma_M(u, v, z) = \sum_{i=0}^{K-n} z^i \sum_{j=1}^{\lfloor (K+1-i)/2 \rfloor} \gamma_{M,i,j} u^j v^{K+1-i-2j}. \tag{4.9}$$



The original labeling The (x,y)-labeling The (u,v)-labeling

Figure 5: A pruned canonical Gessel tree.

Theorem 4.2 *Let n and M be given as before. Then*

$$\gamma_M(u,v,z) = \sum_{T \in P_M} w(T), \tag{4.10}$$

where P_M denotes the set of pruned canonical Gessel trees on M.

For the case of Stirling permutations, the above grammar G_k reduces to the grammar given by Ma-Ma-Yeh [17], namely,

$$G = \{u \to uvz, \quad v \to 2uz, \quad z \to uz\}, \tag{4.11}$$

where we have used z in place of w in [17]. The pruned canonical Gessel trees for Stirling permutations serve as an underlying combinatorial structure for the grammar in (4.11).

The following theorem asserts that pruned canonical Gessel trees on a multiset M with the (u,v)-labeling can be generated in the same way as successively applying the formal derivatives D_{k_1}, D_{k_2}, …, D_{k_n} to x.

Theorem 4.3 *Let n and M be given as before. Then*

$$\gamma_M(u,v,z) = D_{k_n} D_{k_{n-1}} \cdots D_{k_1}(x). \tag{4.12}$$

The proof is in the same line as the argument in [5] for the grammatical labeling of 0-1-2 plane trees. Instead of presenting a proof in full detail, it suffices to focus on the action corresponding to the rule $v \to 2uz^{k-1}$ of the grammar G_k as in (4.8).

Assume that T is a pruned canonical Gessel tree on

$$M' = \{1^{k_1}, 2^{k_2}, \ldots, (n-1)^{k_{n-1}}\}.$$

As before, we have $M = \{1^{k_1}, 2^{k_2}, \ldots, n^{k_n}\}$. Put $k = k_n$. Suppose that T has a vertex i with the label v. This means that i has k children with the last child not being a leaf. There are two ways to generate a pruned canonical Gessel tree S from T by appending the vertex n to T as a child of i.

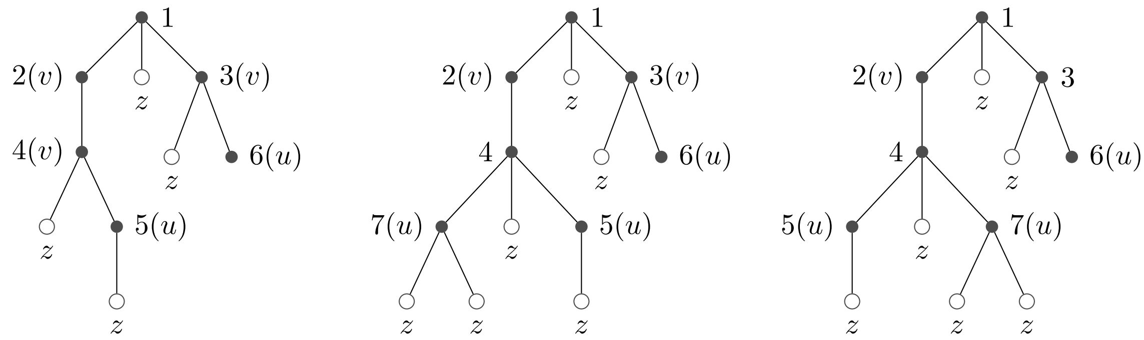


Figure 6: The two possibilities for the rule $v \to 2uz^{k-1}$.

Case 1. Make n the first child of i. Then i will no longer have a label, and n will be assigned the label u. Meanwhile, n will have $k-1$ z-leaves. We see that this operation corresponds to the rule $v \to uz^{k-1}$.

Case 2. Swap the first child and the last child along with their subtrees in the pruned canonical Gessel tree S obtained in Case 1. Then we get a pruned canonical Gessel tree on M. Summing up, these two cases correspond to the rule $v \to 2uz^{k-1}$.

For example, for the pruned canonical Gessel tree T on $M' = \{1^2, 2, 3^2, 4^2, 5^2, 6\}$ in Figure 6, there are two ways to append the vertex 7 to T as a child of 4 to produce a pruned canonical Gessel tree on $M = M' \cup \{7^3\}$.

It must be understood, however, that the notion of pruned canonical Gessel trees is somewhat cosmetic, in the sense that whereas it does not change anything in nature, it may have an effect on the impression. In fact, we might as well keep the original (x, y)-leaves, and transport the labels of the (x, y)-leaves to their parents. Nevertheless, the pruned version bears the advantage of taking a simpler form especially for permutations where the structure of 0-1-2 increasing plane trees comes into play.

To conclude this section, we claim that the grammar G_k captures all the possibilities of constructing pruned canonical Gessel trees on M from the ones on M'. It is only a matter of exercise to verify that this is indeed the case.

5 The partial γ-coefficients

In this section, we demonstrate that the combinatorial interpretation of the partial γ-coefficients of the Eulerian polynomials of Stirling permutations on a multiset falls into the scheme of canonical Gessel trees.

First, we need to translate the defining property of an unbalanced y-leaf into the language of Stirling permutations. This goal can be achieved with the aid of Proposition 2.2 on the implications

Let σ be a Stirling permutation of M. Observe that a descent i arises only when σ_i is the last occurrence. Similarly, an ascent i arises only when σ_i is the first occurrence. Assume that an index i is a descent of σ, and assume that σ_p is the first occurrence of σ_i, where $p \leq i$. As an extension of the notion of a double descent of a permutation of $[n]$, we say that i is a double fall of σ if i is a descent and $p-1$ is a descent as well. Keep in mind the convention that $\sigma_0 = \sigma_{K+1} = 0$. Denote by $\mathrm{dfall}(\sigma)$ the number of double falls of σ. For example, for the Stirling permutation $\sigma = 2533114664$, the descents $2, 9, 10$ are not double falls, whereas the descent 4 is a double fall.

Proposition 5.1 *Let n and M be given as before. Assume that σ is a Stirling permutation on M and T is the corresponding Gessel tree of σ. Then an index i is a double fall of σ if and only if the vertex σ_i in T has an unbalanced y-leaf.*

Proof. Let $j = \sigma_i$, and let $S_j(\sigma)$ be the j-segment with the first occurrence of j at position p and the last occurrence of j at position q. Moreover, let

$$S_j(\sigma) = w_0\, j\, w_1\, j\, \cdots\, w_{k_j-1}\, j\, w_{k_j} \tag{5.1}$$

be the Gessel decomposition of $S_j(\sigma)$. Assume that j has an unbalanced y-leaf. We proceed to show that i is a double fall of σ. The y-leaf indicates that the last factor w_{k_j} is empty, from which it follows that i is a descent, since $S_j(\sigma)$ is surrounded by smaller elements at both ends. Suppose the first occurrence of j appears at position p. It remains to confirm that $p-1$ is also a descent. Since the y-leaf is unbalanced, the segment w_0 in (5.1) is nonempty. By the definition of the Gessel decomposition, any element in w_0 is greater than j, so that $p-1$ must be a descent. This completes the proof. ∎

Given the above characterization of unbalanced y-leaves in terms of Stirling permutations, we obtain the following interpretation of the partial γ-coefficients.

Theorem 5.2 *Let n, M and K be given as before. For $0 \leq i \leq K-n$ and $1 \leq j \leq \lfloor (K+1-i)/2 \rfloor$, we have*

$$\gamma_{M,i,j} = |\{\sigma \in Q_M \mid \mathrm{plat}(\sigma) = i,\ \mathrm{des}(\sigma) = j,\ \mathrm{dfall}(\sigma) = 0\}|. \tag{5.2}$$

Notice that a double fall of a Stirling permutation of a multiset boils down to a double descent of a permutation of $[n]$. Therefore, when specialized to $M = [n]$, Theorem 5.2 reduces to the γ-expansion (1.2) of the bivariate Eulerian polynomials $A_n(x, y)$ due to Foata and Schüzenberger [9].

6 A theorem of Ma-Ma-Yeh

In this section, we show that our combinatorial interpretation of the partial γ-coefficients given the preceding section reduces to a theorem of Ma-Ma-Yeh [17] subject to a restatement prompted by a symmetry consideration.

Recall that the partial γ-coefficients $\gamma_{n,i,j}$ for $C_n(x,y,z)$ are defined by

$$C_n(x,y,z) = \sum_{i=1}^{n} z^i \sum_{j=0}^{\lfloor (2n+1-i)/2 \rfloor} \gamma_{n,i,j}(xy)^j (x+y)^{2n+1-i-2j}. \tag{6.1}$$

Let $\sigma \in Q_n$. An index $1 \le i \le 2n$ is called an ascent-plateau if $\sigma_{i-1} < \sigma_i = \sigma_{i+1}$ and a descent-plateau if $\sigma_{i-1} > \sigma_i = \sigma_{i+1}$. Let $\operatorname{aplat}(\sigma)$ and $\operatorname{dplat}(\sigma)$ denote the numbers of ascent-plateaux and descent-plateaux, respectively. The following result is due to Ma-Ma-Yeh [17].

Theorem 6.1 *For $n \ge 1$, $0 \le i \le n$ and $1 \le j \le \lfloor (2n+1-i)/2 \rfloor$, we have*

$$\gamma_{n,i,j} = |\{\sigma \in Q_n \,|\, \operatorname{des}(\sigma) = i,\ \operatorname{aplat}(\sigma) = j,\ \operatorname{dplat}(\sigma) = 0\}|. \tag{6.2}$$

For the case of Stirling permutations, that is, $M = [n]_2$, we write G_n for G_M. By Theorem 3.1, $\gamma_{n,i,j}$ equals the number of trees $T \in G_n$ with i z-leaves and j y-leaves but with no unbalanced y-leaves. In the meantime, Theorem 5.2 provides a formulation resorting to the notion of a double fall of a Stirling permutation. In this situation, a double fall can be described as follows. Let $n \ge 1$ and $\sigma \in Q_n$. Assume that i is a descent of σ. Let $\sigma_i = j$. Then σ_i must be the second occurrence of j in σ. Assume that σ_p is the first occurrence of j in σ. Now, the descent i is called a double fall if $p-1$ is a descent of σ as well.

While there is no doubt that Theorem 5.2 offers a legitimate combinatorial statement, one could not help wondering about the connection to the Ma-Ma-Yeh story. It turns out that the answer lies in the symmetry of the polynomials $C_n(x,y,z)$. As far as Stirling permutations are concerned, Theorem 5.2 can be reassembled with regards to a slight twist of canonical Gessel trees. In fact, one only needs to interchange the roles of the x-leaves and the y-leaves. Equivalently, we define a canonical Gessel tree on $[n]_2$, or a canonical ternary increasing tree, by imposing the constraint that there are no internal vertices having a z-leaf, but no x-leaf.

Theorem 6.2 *For $n \ge 1$, the partial γ-coefficient $\gamma_{n,i,j}$ for Stirling permutations as defined by (6.1) equals the number of canonical ternary increasing trees on $[n]_2$ with i y-leaves and j x-leaves.*

Notice that the above explanation of $\gamma_{n,i,j}$ can be directly justified in the same manner as the proof of Theorem 3.1. The following proposition gives a characterization of Stirling permutations corresponding to canonical ternary increaing trees.

Proposition 6.3 *Let $n \ge 1$ and $\sigma \in Q_n$. Let T be the corresponding ternary increasing tree of σ. Then σ has a descent-plateau if and only if T contains an internal vertex having a z-leaf, but no x-leaf. That is to say, σ has no descent-plateaux if and only if T is canonical.*

For example, the vertex 2 in the ternary increasing tree in Figure 1 has a z-leaf, but no x-leaf. The corresponding Stirling permutation $\sigma = 33552217714664$ has a descent-plateau 522. On the other hand, Figure 7 furnishes a canonical ternary increasing tree. The corresponding Stirling permutation

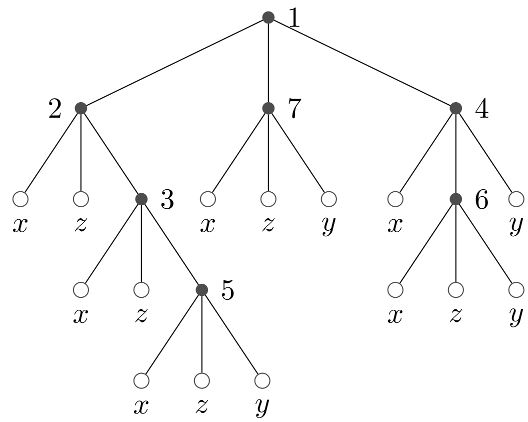


Figure 7: A canonical ternary increasing tree.

The proof of the above proposition is analogous to that of Proposition 5.1. One more thing, we must add that the statistic $\mathrm{aplat}(\sigma)$ reflects the number of internal vertices in the corresponding ternary increasing tree that have an x-leaf and a z-leaf simultaneously. Thus we have reconfirmed the assertion of Ma-Ma-Yeh in the context of canonical ternary increasing trees in connection with Stirling permutations.

Acknowledgments. We wish to thank the referees for helpful suggestions. This work was supported by the National Science Foundation of China.

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