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Cubic Equations Through the Looking Glass of Sylvester

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Abstract

One can hardly believe that there is still something to be said about cubic equations. To dodge this doubt, we will instead try and say something about Sylvester. He doubtless found a way of solving cubic equations. As mentioned by Rota, it was the only method in this vein that he could remember. We realize that in the generic case Sylvester's magnificent approach aimed at reduced cubic equations boils down to an easy identity expressing a cubic polynomial as a sum of two third powers of linear forms. This leads to Cardano's formula for cubic equations involving the third roots of unity.

A special case of a remarkable discovery of Sylvester [7] states that in the generic case, a cubic binary form can be represented as a sum of two third powers of linear forms. This result has been presented in the contexts of invariant theory of binary forms, Waring's problem for binary forms, the apolarity of polynomials and the umbral method, see [2–6]. However, it appears that no account of this ingenious idea has been given in full detail. Let us get to the point.

As is well known, by substituting x with $x - a_1$, a cubic polynomial

$$x^3 + 3a_1x^2 + 3a_2x + a_3 \tag{1}$$

can be written in the reduced form

$$f(x) = x^3 - 3px + q. \tag{2}$$

First, for the case $p = 0$, the equation can be readily solved. From now on, we assume that $p \neq 0$. For the moment, please do not ask why. Let us write $f(x)$ as

$$f(x) = x^3 - 3rsx + rs(r + s). \tag{3}$$

In doing so, the parameters r and s are determined by the relations

$$rs = p, \quad rs(r + s) = q.$$

That is to say, r and s are the roots of the quadratic equation

$$x^2 - \frac{q}{p}x + p = 0. \tag{4}$$

If $q^2 = 4p^3$, the above equation (4) has double roots, that is,

$$r = s = \frac{q}{2p}.$$

In this case, (3) reduces to

$$f(x) = x^3 - 3r^2x + 2r^3,$$

and the following identity

$$x^3 - 3r^2x + 2r^3 = (x-r)^2(x+2r) \tag{5}$$

serves as a key to the roots of $f(x)$.

Now we are left with the generic case when the quadratic equation (4) has two distinct roots r and s. Under this circumstance, here emerges an identity:

$$x^3 - 3rsx + rs(r+s) = \frac{s}{s-r}(x-r)^3 - \frac{r}{s-r}(x-s)^3, \tag{6}$$

which spells out why we choose to write $f(x)$ in the form of (3). This relation also ensures that the cubic equation $f(x) = 0$ can be solved with ease.

The assumption that $p \neq 0$ implies that $s \neq 0$, and hence the cubic equation $f(x) = 0$ can be reformulated as

$$(x-r)^3 = \frac{r}{s}(x-s)^3.$$

Let $u_1,\ u_2,\ u_3$ be the three cubic roots of r/s. Clearly, $u_1, u_2, u_3 \neq 1$ since $r \neq s$. Thus, the solutions of the equation $f(x) = 0$ are given by

$$x_j = \frac{r - su_j}{1 - u_j}, \tag{7}$$

where $\ j = 1, 2, 3$.

Consider the example in [1]. Let

$$f(x) = x^3 - 6x - 6.$$

Then we are led to the relations $rs = 2$ and $r + s = -3$. Solve the quadratic equation

$$x^2 + 3x + 2 = 0$$

to get

$$r = -2, \quad s = -1.$$

It follows that $f(x)$ can be expressed as

$$f(x) = -(x+2)^3 + 2(x+1)^3, \tag{8}$$

so that the cubic equation $f(x) = 0$ takes the form

$$(x+2)^3 = 2(x+1)^3.$$

Let

$$\omega = -\frac{1}{2} + \frac{\sqrt{3}}{2} i$$

be the third root of unity, and let u_1, u_2, u_3 be the three cubic roots of 2, namely, for $j = 0, 1, 2,$

$$u_j = \sqrt[3]{2} e^{j \frac{2\pi i}{3}} = \sqrt[3]{2} \omega^j.$$

Thus the solutions of the equation $f(x) = 0$ are furnished by

$$x_j = \frac{-2 + u_j}{1 - u_j} = \frac{\sqrt[3]{2}(\omega^j - \sqrt[3]{4})}{1 - \omega^j \sqrt[3]{2}} = \omega^j \sqrt[3]{2} + \omega^{-j} \sqrt[3]{4},$$

where $j = 0, 1, 2$. It is no accident that the above solutions are in agreement with Cardano's formula as presented in [1].

Last but not least, we should not forget that we owe our thanks to Sylvester.

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References

[1] Bewersdorff, J. (2006). *Galois Theory for Beginners: A Historical Perspective*. Providence, RI: American Mathematical Society.

[2] Ehrenborg, R. and Rota, G.-C. (1993). Apolarity and canonical forms for homogeneous polynomials. European J. Combin. 14: 157–181.

[3] Kung, J.P.S. and Rota, G.-C. (1984). The theory of binary forms, Bull. Amer. Math. Soc. (N.S.) 10: 27–85.

[4] Kung, J.P.S. (1989). Canonical forms of binary forms: Variations on a theme of Sylvester, In: D. Stanton, Ed., *Invariant Theory and Tableaux*, Springer-Verlag, pp. 46–58.

[5] Kung, J.P.S., Rota, G.-C. and Yan, C.H.F. (2009). *Combinatorics: The Rota Way*. New York: Cambridge University Press.

[6] Rota, G.-C. (1999). Two turning points of invariant theory. Math. Intelligencer 21: 20–27.

[7] Sylvester, J.J. (1851). On a remarkable discovery in the theory of canonical forms and of hyperdeterminants. Math. Mag. 2: 391–410.