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Semi-invariants of Binary Forms and Sylvester's Theorem

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Dedicated to Doron Zeilberger on the Occasion of His Seventieth Birthday

Abstract

We obtain a combinatorial formula related to the shear transformation for semi-invariants of binary forms, which implies the classical characterization of semi-invariants in terms of a differential operator. Then, we present a combinatorial proof of an identity of Hilbert, which leads to a relation of Cayley on semi-invariants. This identity plays a crucial role in the original proof of Sylvester's theorem on semi-invariants in connection with the Gaussian coefficients. Moreover, we show that the additivity lemma of Pak and Panova which yields the strict unimodality of the Gaussian coefficients for $n, k \geq 8$ can be deduced from the ring property of semi-invariants.

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1 Introduction

This work is a continuation of the exploration of the Gaussian coefficients or the q-binomial coefficients by means of semi-invariants of binary forms recently carried out in [3]. We will be mainly concerned with the combinatorial perspectives related to Sylvester's proof of the unimodality conjecture of Cayley. A key identity used by Sylvester is a relation due to Cayley, which turns out to be a consequence of an identity of Hilbert. We shall give a combinatorial interpretation of the identity of Hilbert. Moreover, we show that the additivity lemma of Pak and Panova leading to the strict unimodality of the Gaussian coefficients for $n, k \geq 8$ can be deduced from the ring property of semi-invariants.

Let $p(k,n,m)$ denote the number of partitions of m contained in a $k \times n$ rectangle, then the Gaussian coefficients can be expressed as

$$\begin{bmatrix} n+k \\ k \end{bmatrix} = \sum_{m=0}^{nk} p(k,n,m) q^m, \tag{1.1}$$

see [1, 13].

The Gaussian coefficients are symmetric in q. Cayley [2] conjectured in 1856 that the Gaussian coefficients are unimodal, and it was proved by Sylvester [14] in 1878 resorting to semi-invariants of binary forms. There has been an extensive literature on this subject, see, for example, [7, 10–12, 15, 16]. It is worth mentioning that O'Hara [7] found a constructive proof. Zeilberger [16] discovered an identity, known as the KOH theorem, which justifies the unimodality.

As the first step to bring Sylvester's theorem to a combinatorial ground, let us take a look at the classical characterization of semi-invariants in terms of the differential operator D. A semi-invariant of a binary n-form is a polynomial $I(a_0, a_1, \ldots, a_n)$ with rational coefficients such that

$$I(a_0, a_1, \ldots, a_n) = I(a_0', a_1', \ldots, a_n'), \tag{1.2}$$

where the a_i' are determined by the shear transformation with respect to a variable z, that is, for $0 \leq i \leq n$,

$$a_i' = a_i + \binom{i}{1} a_{i-1} z + \binom{i}{2} a_{i-2} z^2 + \cdots + a_0 z^i. \tag{1.3}$$

Let

$$D = a_0 \frac{\partial}{\partial a_1} + 2a_1 \frac{\partial}{\partial a_2} + 3a_2 \frac{\partial}{\partial a_3} + \cdots + n a_{n-1} \frac{\partial}{\partial a_n}. \tag{1.4}$$

Semi-invariants can be characterized in terms of the differential operator D, see Cayley [2], Sylvester [14], or Hilbert [6]. More precisely, a polynomial $I(a_0, a_1, \ldots, a_n)$ is a semi-invariant of a binary n-form if and only if $D(I) = 0$.

Note that if I and J are two semi-invariants of a binary n-form, then so are $I + J$ and IJ. Based on a combinatorial interpretation of the operator D, we show that $I(a_0', a_1', \ldots, a_n')$ can be expressed in terms of the polynomial $I(a_0, a_1, \ldots, a_n)$ and the operator D, in the spirit of the Taylor expansion. This formula immediately leads to the characterization of semi-invariants in terms of the operator D.

The second objective of this paper is to present a combinatorial proof of an identity of Hilbert involving the operators D and Δ. The operator Δ is defined by

$$\Delta = n a_1 \frac{\partial}{\partial a_0} + (n-1) a_2 \frac{\partial}{\partial a_1} + \cdots + a_n \frac{\partial}{\partial a_{n-1}}. \tag{1.5}$$

The identity of Hilbert [6] reads as follows: For $n, k \geq 0$, and $0 \leq m \leq nk$, let λ be a partition of m contained in a $k \times n$ rectangle, and let $c = nk - 2m$. Then for $i \geq 1$,

$$D\Delta^i(a_\lambda) - \Delta^i D(a_\lambda) = i(c - i + 1)\Delta^{i-1}(a_\lambda). \tag{1.6}$$

As an application of Sylvester's theorem, we show that the additivity lemma of Pak and Panova [8,9] can be deduced from the ring property of semi-invariants. The additivity lemma for the Gaussian coefficients was established via a connection with the Kronecker coefficients in the representation theory of the symmetric group as well as the semigroup property of the Kronecker coefficients due to Christandl, Harrow and Mitchison [4].

Lemma 1.1. *Assume that $k_1, k_2, n \geq 2$, at least one of k_1, k_2 and n is greater than two and at least one of k_1, k_2 and n is even. If the strict unimodality holds for $\left[{n+k_1 \atop n}\right]$ and $\left[{n+k_2 \atop n}\right]$, then it holds for $\left[{n+k_1+k_2 \atop n}\right]$.*

To conclude the introduction, we recall that the strict unimodality proved by Pak and Panova says that for $n, k \geq 8$ and $2 \leq m \leq nk/2$,

$$p(k, n, m) > p(k, n, m-1). \tag{1.7}$$

2 The Operators D and Δ

A binary form of degree n, or a binary n-form, is a homogeneous polynomial in x and y,

$$f(x, y) = a_0 x^n + \binom{n}{1} a_1 x^{n-1} y + \binom{n}{2} a_2 x^{n-2} y^2 + \cdots + a_n y^n, \tag{2.1}$$

where the coefficients $a_0, a_1, \ldots, a_n$ are regarded as variables. Consider the shear transformation: $x = x' + zy'$ and $y = y'$, where z is treated as a variable. Suppose that under this transformation, the binary form $f(x, y)$ becomes

$$f'(x', y') = a'_0 x'^n + \binom{n}{1} a'_1 x'^{n-1} y' + \cdots + a'_n y'^n. \tag{2.2}$$

It is easily checked that for $0 \leq i \leq n$,

$$a'_i = a_i + \binom{i}{1} a_{i-1} z + \binom{i}{2} a_{i-2} z^2 + \cdots + a_0 z^i. \tag{2.3}$$

We say that a polynomial $I(a_0, a_1, \ldots, a_n)$ with rational coefficients is a semi-invariant of the binary form $f(x, y)$ if

$$I(a_0, a_1, \ldots, a_n) = I(a'_0, a'_1, \ldots, a'_n), \tag{2.4}$$

see, for example, [5].

To determine whether a polynomial $I(a_0, a_1, \ldots, a_n)$ satisfies the above condition (2.4), we are led to an expansion of $I(a'_0, a'_1, \ldots, a'_n)$ as a polynomial of z. As expected, the operator D comes to the scene. Recall that D is given by

$$D = a_0 \frac{\partial}{\partial a_1} + 2a_1 \frac{\partial}{\partial a_2} + 3a_2 \frac{\partial}{\partial a_3} + \cdots + n a_{n-1} \frac{\partial}{\partial a_n}. \tag{2.5}$$

Theorem 2.1. *For $n \geq 0$, and for any polynomial $I(a_0, a_1, \ldots, a_n)$ over the rational numbers, we have*

$$I(a_0', a_1', \ldots, a_n') = \sum_{i\geq 0} D^i I(a_0, a_1, \ldots, a_n) \frac{z^i}{i!}. \tag{2.6}$$

While we do not intend to claim that the above formula (2.6) is new due to the lack of accessible literature, at least it is worth noting that such a formulation makes the characterization of semi-invariants transparent in the sense that if $D(I)$ vanishes, then so does $D^i(I)$ for any $i \geq 2$. As will be seen, the idea behind (2.6) serves as an embarkation point to a combinatorial understanding of the identity (1.6) of Hilbert.

Here is an illustration of Theorem 2.1. For a monomial $a^\nu = a_0^{\nu_0} a_1^{\nu_1} \cdots a_n^{\nu_n}$, we define its degree by

$$\nu_0 + \nu_1 + \cdots + \nu_n$$

and its weight by

$$\nu_1 + 2\nu_2 + \cdots + n\nu_n.$$

It is clear that when the operator D is applied to a monomial a^ν, it preserves the degree and decreases the weight by one. For example, let

$$f(x, y) = a_0 x^3 + 3a_1 x^2 y + 3a_2 xy^2 + a_3 y^3,$$

and let

$$I(a_0, a_1, a_2, a_3) = c_1 a_0^2 a_3 + c_2 a_0 a_1 a_2 + c_3 a_1^3. \tag{2.7}$$

Upon the substitution (2.3), we get

$$\begin{aligned} I(a_0', a_1', a_2', a_3') = {} & c_1 a_0^2 a_3 + c_2 a_0 a_1 a_2 + c_3 a_1^3 \\ & + ((3c_1 + c_2) a_0^2 a_2 + (2c_2 + 3c_3) a_0 a_1^2) z \\ & + 3(c_1 + c_2 + c_3) a_0^2 a_1 z^2 \\ & + (c_1 + c_2 + c_3) a_0^3 z^3. \end{aligned}$$

On the other hand,

$$\begin{aligned} D(I) &= (3c_1 + c_2) a_0^2 a_2 + (2c_2 + 3c_3) a_0 a_1^2, \\ D^2(I) &= 6(c_1 + c_2 + c_3) a_0^2 a_1, \\ D^3(I) &= 6(c_1 + c_2 + c_3) a_0^3. \end{aligned}$$

We see that (2.6) holds in this case.

To lay out a combinatorial setting for Theorem 2.1, we adopt the common notation of a partition λ of m contained in a $k \times n$ rectangle, that is, $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, where $n \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 0$ and $\lambda_1 + \lambda_2 + \cdots + \lambda_k = m$. Accordingly, we use a_λ to denote the monomial $a_{\lambda_1} a_{\lambda_2} \cdots a_{\lambda_k}$.

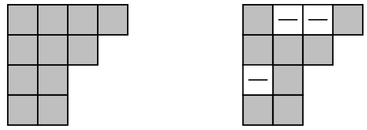


Figure 1: The Young diagram and a semi-diagram of shape $(4, 3, 2, 2)$.

Recall that the Young diagram or the Ferrers diagram of a partition λ is a collection of shaded boxes or cells arranged in left-justified rows with λ_i boxes in the i-th row. The partition λ is referred to as the shape of the diagram. For instance, the Young diagram of shape $(4, 3, 2, 2)$ is illustrated on the left in Figure 1. In this paper, we introduce the notion of a semi-diagram, which is a Young diagram with some shaded cells filled with a minus sign and subsequently turned into hollow cells. The diagram on the right in Figure 1 is a semi-diagram of shape $(4, 3, 2, 2)$ with three minus signs.

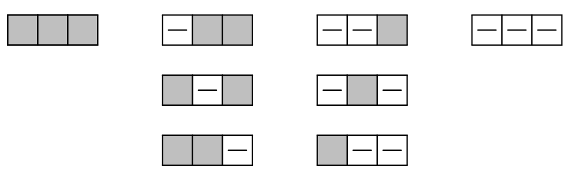
To a semi-diagram T of shape λ, we associate it with a weight as follows. For $1 \leq i \leq k$, if the i-th row contains r_i shaded cells and s_i minus signs, we define its weight as $a_{r_i} z^{s_i}$. The weight of a semi-diagram T, denoted by $w(T)$, is then defined to be the product of weights of all rows. Of course, the weight of an empty row is set to be a_0. For example, the weight of the semi-diagram in Figure 1 equals $a_3 a_2^2 a_1 z^3$.

In terms of semi-diagrams, the substitution (2.3) can be interpreted as filling some of the shaded cells of a Young diagram with a minus sign and turning them into hollow cells. Consider only one row with i shaded cells, which has weight a_i. For $0 \leq j \leq i$, there are $\binom{i}{j}$ ways to turn this row into a semi-diagram by placing a minus sign to j shaded cells and turning them into hollow cells. Any of the resulting semi-diagrams has weight $a_{i-j} z^j$. This operation is in accordance with the substitution (2.3).

For example, under the shear transformation, a_3 becomes

$$a_3' = a_3 + 3a_2 z + 3a_1 z^2 + a_0 z^3,$$

which is the sum of the weights of the semi-diagrams



We are now ready to prove Theorem 2.1.

Proof of Theorem 2.1. It suffices to show that for any monomial a_λ,

$$a_\lambda' = a_{\lambda_1}' a_{\lambda_2}' \cdots a_{\lambda_k}' = \sum_{i=0}^{m} D^i(a_\lambda) \frac{z^i}{i!}. \tag{2.8}$$

Let $K_i(\lambda)$ denote the set of semi-diagrams of λ containing i minus signs, where $0 \leq i \leq m$. Set

$$\sum_{T \in K_i(\lambda)} w(T) = W_i(\lambda). \tag{2.9}$$

Then we have

$$a'_\lambda = \sum_{i=0}^{m} W_i(\lambda). \tag{2.10}$$

In particular, the semi-diagrams containing only one minus sign give rise to the coefficient of z in $I(a'_0, a'_1, \ldots, a'_n)$. This operation of filling only one shaded cell of a Young diagram with a minus sign and turning it into a hollow cell can be described by the action of the operator D on a_λ, and this explains where the operator D comes from combinatorially.

Moreover, one realizes that in general the coefficient of z^i in $I(a'_0, a'_1, \ldots, a'_n)$ can also be expressed in terms of the operator D. Indeed, the action of D^i on a_λ can be interpreted as placing i distinguishable minus signs in a Young diagram and turning them into hollow cells. But for the coefficient of z^i in $I(a'_0, a'_1, \ldots, a'_n)$, the minus signs in a semi-diagram are regarded indistinguishable. This yields the relation

$$D^i(a_\lambda)\frac{z^i}{i!} = W_i(\lambda). \tag{2.11}$$

Combining (2.10) and (2.11) gives (2.8). This completes the proof. ∎

Now we turn to the vertical shear transformation:

$$x = x'', \quad y = zx'' + y'',$$

where z is considered as a variable. Under this transformation, the binary form $f(x, y)$ becomes

$$f''(x'', y'') = a''_0 x''^n + \binom{n}{1} a''_1 x''^{n-1} y'' + \cdots + a''_n y''^n,$$

where, for $0 \le i \le n$,

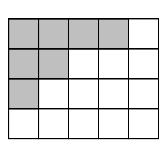
$$a''_i = a_i + \binom{n-i}{1} a_{i+1} z + \binom{n-i}{2} a_{i+2} z^2 + \cdots + a_n z^{n-i}. \tag{2.12}$$

A polynomial $I(a_0,\, a_1, \ldots,\, a_n)$ with rational coefficients is called a semi-invariant with respect to the vertical shear transformation provided that

$$I(a_0,\, a_1, \ldots,\, a_n) = I(a''_0,\, a''_1, \ldots,\, a''_n). \tag{2.13}$$

A polynomial $I(a_0,\, a_1, \ldots,\, a_n)$ is a semi-invariant with respect to the vertical shear transformation if and only if $\Delta(I) = 0$, see Hilbert [6].

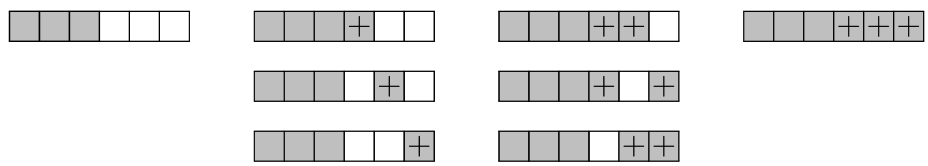
To give a combinatorial interpretation of the operator Δ, we need to have a full picture of the Young diagram of a partition λ contained in a $k \times n$ rectangle. More precisely, we shall use shaded cells for the cells in the shape of λ and use hollow cells for the cells outside the shape of λ. For example, below is the depiction of the partition $\lambda = (4, 2, 1, 0)$ contained in a 4×5 rectangle:



Consider a single row diagram with i shaded cells and $n-i$ hollow cells, whose weight is a_i. For $0 \le j \le n-i$, there are $\binom{n-i}{j}$ ways to turn this row into a semi-diagram by placing a plus sign to j hollow cells and turning them into shaded cells. Any of the resulting semi-diagrams has weight $a_{i+j}z^j$, where we define the weight of a plus sign to be z. For example, for $n=6$, under the vertical shear transformation, a_3 becomes

$$a_3'' = a_3 + 3a_4 z + 3a_5 z^2 + a_6 z^3,$$

which is the sum of the weights of the semi-diagrams



In particular, the action of the operator Δ can be interpreted in terms of the operation of filling only one hollow cell in a Young diagram of λ with a plus sign and turning it into a shaded cell. Analogous to Theorem 2.1, we have the following expansion.

Theorem 2.2. *For $n \ge 0$ and for any polynomial $I(a_0, a_1, \ldots, a_n)$ over the rational numbers, we have*

$$I(a_0'', a_1'', \ldots, a_n'') = \sum_{i\ge 0} \Delta^i I(a_0, a_1, \ldots, a_n)\frac{z^i}{i!}. \tag{2.14}$$

3 Sylvester's Theorem

The following theorem of Sylvester [14] establishes a connection between the Gaussian coefficients and semi-invariants. For $0 \le m \le nk/2$, let

$$\delta(k,n,m) = p(k,n,m) - p(k,n,m-1), \tag{3.1}$$

with the convention that $p(k,n,-1) = 0$.

Theorem 3.1. *For $n,k \ge 0$ and $0 \le m \le nk/2$, the number of semi-invariants of a binary n-form of degree k and weight m equals $\delta(k,n,m)$.*

Theorem 3.1 also takes the following form, as Sylvester [14] chose to work with.

Theorem 3.2. *For $n,k \ge 0$ and $0 \le m \le nk/2$, the number of semi-invariants of a binary n-form of degree k and weight not exceeding m equals $p(k,n,m)$.*

The following identity (3.2) of Cayley [2] crops up in Sylvester's proof of the unimodality of the Gaussian coefficients, see [14]. Cayley used this relation to construct covariants.

It might be worth mentioning that from a combinatorial point of view, the identity of Hilbert is easier to justify than the identity of Cayley for it does not involve the conditions on semi-invariants. It should also be noted that to pass from Hilbert's identity to Cayley's identity, the condition $m \le nk/2$ is required because it is a constraint for semi-invariants, see Cayley [2].

Theorem 3.3. *For $n, k \ge 0$, and $0 \le m \le nk/2$, let I be a semi-invariant of a binary n-form of degree k and weight m, and let $c = nk - 2m$. Then, for $i \ge 1$,*

$$D\Delta^i(I) = i(c - i + 1)\Delta^{i-1}(I). \tag{3.2}$$

Since $D(I) = 0$ for a semi-invariant I, we see that the above relation is a consequence of the identity (1.6) of Hilbert [6] with a_λ being replaced by a semi-invariant I.

To make the paper self-contained, we give an exposition of Sylvester's proof. The consideration of the dimension identity (3.6) makes the argument easier to understand in terms of an equality, instead of arguing with two inequalities in opposite directions as described by Sylvester.

We define $Q_n(k, m)$ as the vector space of polynomials in $a_0, a_1, \ldots, a_n$ over the rational numbers that are homogeneous of degree k and weight m. We shall use $S_n(k, m)$ to denote the vector space of semi-invariants of degree k and weight m, that is,

$$S_n(k, m) = \{I \in Q_n(k, m) \mid D(I) = 0\}. \tag{3.3}$$

The number of semi-invariants of degree k and weight m of a binary n-form is referred to as the dimension of the vector space $S_n(k, m)$. For example, $\dim S_4(4, 6) = 2$. Below are two semi-invariants of degree 4 and weight 6 of a binary 4-form:

$$I_1 = 3a_1^2a_2^2 - 4a_1^3a_3 - 2a_0a_1a_2a_3 + 3a_0^2a_3^2 + 4a_0a_1^2a_4 - 4a_0^2a_2a_4, \tag{3.4}$$

$$I_2 = a_0a_2^3 - 2a_0a_1a_2a_3 + a_0^2a_3^2 + a_0a_1^2a_4 - a_0^2a_2a_4. \tag{3.5}$$

Notice that for $m = 0$, $\dim S_n(k, 0) = p(k, n, 0) = 1$.

Proof of Theorem 3.2. For $0 \le i \le m + 1$, let

$$V_i = D^i(Q_n(k, m)),$$

and let

$$T_i \colon V_{i-1} \to V_i,$$

where $1 \le i \le m + 1$, that is, $T_i(I) = D(I)$ for any $I \in V_{i-1}$. Then, the kernel of T_i, denoted by $\ker T_i$, namely,

$$\ker T_i = \{I \in V_{i-1} \mid D(I) = 0\},$$

is a subspace of V_{i-1}. Hence

$$\dim V_{i-1} = \dim \ker T_i + \dim V_i. \tag{3.6}$$

Notice that while acting on a monomial in $a_0, a_1, \ldots, a_n$, the operator D preserves the degree and lowers the weight by one. On the other hand, $Q_n(k, 0)$ is generated by a_0^k, which is a semi-invariant, and so $D(Q_n(k, 0)) = 0$. It follows that

$$V_{m+1} = D^{m+1}(Q_n(k, m)) = 0.$$

Iterating (3.6) gives

$$\dim V_0 = \dim\ker T_1 + \dim\ker T_2 + \cdots + \dim\ker T_{m+1}. \tag{3.7}$$

It is apparent that

$$\ker T_i \subseteq S_n(k, m - i + 1). \tag{3.8}$$

The real challenge is to show that

$$\ker T_i = S_n(k, m - i + 1), \tag{3.9}$$

that is, the successive applications of the operator D resulting in V_{i-1} do not leave out any semi-invariants in $S_n(k, m - i + 1)$.

For $i = 1$, nothing needs to be said since by definition,

$$\ker T_1 = S_n(k, m). \tag{3.10}$$

But for $i = 2$, what does (3.9) mean? Note that a semi-invariant I in $S_n(k, m - 1)$ should come from a polynomial in $Q_n(k, m - 1)$. However, (3.9) indicates that we can restrict our attention only to $D(Q_n(k, m))$, which is a subspace of $Q_n(k, m - 1)$, and we can still get all the semi-invariants in $S_n(k, m - 1)$.

Sylvester realized that in some sense the operator D is the inverse of the operator Δ, as guaranteed by the identity (3.2) of Cayley. In other words, the action of the operator Δ ensures that every semi-invariant in $S_n(k, m-i+1)$ can be shielded from the annihilation of the operator D.

For example, for $n = 3, k = 3$ and $m = 0$, it is clear that $Q_3(3, 0)$ is generated by a_0^3, which is a semi-invariant of degree three and weight zero. Now, $V_3 = D^3(Q_3(3, 3))$. One may wonder whether a_0^3 is still there in V_3. Employing the operator Δ, we find that

$$\Delta^3(a_0^3) = 18a_0^2a_3 + 324a_0a_1a_2 + 162a_1^3,$$

which is a polynomial in $Q_3(3, 3)$. Then it is easily verified that

$$D^3\Delta^3(a_0^3) = 3024a_0^3.$$

So a_0^3 remains in V_3, and this is in accordance with the fact that $S_3(3, 0)$ is generated by a_0^3.

The above reasoning is valid for the general case. For any semi-invariant I in $S_n(k, m-i + 1)$, $\Delta^{i-1}(I)$ falls into $Q_n(k, m)$. Thanks to the identity (3.2), we deduce that by successively applying the operator D to $\Delta^{i-1}(I)$, one recovers the semi-invariant I if we do not mind the nonzero constant. To be more specific, we deduce that I truly belongs to V_{i-1}, and hence the proof is complete. ∎

Examining the proof of Sylvester, one sees that what Sylvester tried to demonstrate is the following property of D.

Theorem 3.4. *For $n, k \geq 0$ and $1 \leq m \leq nk/2$, we have*

$$Q_n(k, m-1) = D(Q_n(k, m)), \tag{3.11}$$

or equivalently, the transformation D is a surjection from $Q_n(k, m)$ to $Q_n(k, m-1)$.

Once the above surjectivity is in hand, it immediately follows that the number of semi-invariants, namely, the dimension of the kernel of D, is given by $\delta(k, n, m)$. As far as the unimodality of the Gaussian coefficients is concerned, we see that Sylvester's proof also contains a justification of the injectivity of the transformation Δ.

Theorem 3.5. *For $n, k \geq 0$ and $1 \leq m \leq nk/2$, the transformation Δ is an injection from $Q_n(k, m-1)$ to $Q_n(k, m)$.*

Proctor [11] came up with a proof of the unimodality of the Gaussian coefficients by introducing two operators different from D and Δ, while taking a notice of the surjectivity and injectivity of D and Δ. It would be appealing to reach a better understanding of these properties from a combinatorial angle.

4 A Combinatorial Proof of Hilbert's Identity

Based on the combinatorial interpretations of the operators D and Δ, we give a combinatorial proof of the identity of Hilbert [6], as stated below.

Theorem 4.1. *For $n, k \geq 0$, and $0 \leq m \leq nk$, let λ be a partition of m contained in a $k \times n$ rectangle, and let $c = nk - 2m$. Then, for $i \geq 1$,*

$$D\Delta^i(a_\lambda) - \Delta^i D(a_\lambda) = i(c - i + 1)\Delta^{i-1}(a_\lambda). \tag{4.1}$$

The above identity of Hilbert plays a fundamental role in the characterization of invariants as well as the construction of covariants.

To present a combinatorial interpretation of the above relation, we recall that the diagram of a partition λ contained in a $k \times n$ rectangle contains shaded cells inside the shape of λ and hollow cells outside the shape of λ. A semi-diagram will also be represented in the same manner. While we shall encounter some signs filled in semi-diagrams, the weight of a row with r shaded cells will be defined by a_r. So the signs will not affect the weight of a semi-diagram.

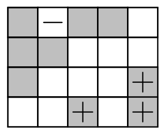
Proof of Theorem 4.1. The action of $D\Delta^i$ on a_λ can be understood as placing i distinguishable plus signs in hollow cells of the Young diagram of λ and turning them into shaded cells, and then placing a minus sign in a shaded cell and turning it into a hollow cell.

Similarly, the application of $\Delta^i D$ to a_λ means to fill a shaded cell of the Young diagram of shape λ with a minus sign and to turn it into a hollow cell, then fill i hollow cells with distinguishable plus signs and turn them into shaded cells.

Now, we use the symbol $\pm$ to denote a cell that is filled with a plus sign first and subsequently filled with a minus sign. Similarly, $\mp$ denotes a cell that is filled with a minus sign first and subsequently filled with a plus sign.

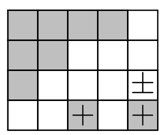
With respect to the computation of $D\Delta^i(a_\lambda) - \Delta^i D(a_\lambda)$, the semi-diagrams that do not contain any $\pm$ or $\mp$ signs would cancel out.

In fact, it is readily seen that there is a one-to-one correspondence between the semi-diagrams in $D\Delta^i(a_\lambda)$ that do not contain any $\pm$ signs and the semi-diagrams in $\Delta^i D(a_\lambda)$ that do not contain any $\mp$ signs. For example, when $k = 4, n = 5$ and $\lambda = (4, 2, 1, 0)$, the following semi-diagram occurs in $D\Delta^3(a_\lambda)$, and it is also in $\Delta^3 D(a_\lambda)$:



Therefore, it suffices to consider the cases when the sign $\pm$ or $\mp$ is involved.

Let us consider the semi-diagrams generated by $D\Delta^i(a_\lambda)$ containing the sign $\pm$. For example, when $k = 4, n = 5$ and $\lambda = (4, 2, 1, 0)$, the following semi-diagram is an illustration of such a configuration generated by $D\Delta^3(a_\lambda)$:

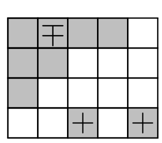


These configurations under consideration can be produced from the Young diagram of λ in a $k \times n$ rectangle by placing i distinguishable plus signs in hollow cells and subsequently adding a minus sign to a shaded cell with a plus sign and turning it into a hollow cell. On the other hand, semi-diagrams with a $\pm$ cell and $i - 1$ distinguishable plus signs (outside the shape of λ, to be precise) can be constructed in an alternative way.

We may choose $i - 1$ plus signs from the i distinguishable plus signs, place them in the hollow cells of the Young diagram of λ contained in a $k \times n$ rectangle, turn them into shaded cells, and finish with filling a hollow cell with the $\pm$ sign and keeping it hollow. Notice that the location of a hollow cell with the $\pm$ sign does not affect the weight of a semi-diagram. That is to say, the cell with the symbol $\pm$ might as well be viewed just as a hollow cell. Therefore, if the $\pm$ sign is not taken into consideration, the semi-diagrams containing $i - 1$ distinguishable plus signs are generated by applying the operator Δ^{i-1} to a_λ. Note that there are $nk - m - (i - 1)$ hollow cells left for the moment, any of which can be chosen as a residence of the $\pm$ sign. It follows that the total contribution of weights in this case amounts to

$$i(nk - m - (i - 1))\Delta^{i-1}(a_\lambda). \tag{4.2}$$

We now consider the semi-diagrams generated by $\Delta^i D(a_\lambda)$ containing the sign $\mp$. For example, when $k = 4, n = 5$ and $\lambda = (4, 2, 1, 0)$, the following semi-diagram arises in $\Delta^3 D(a_\lambda)$:



In general, the involved configurations in this case can be generated from a Young diagram of shape λ contained in a $k \times n$ rectangle by placing a minus sign in a shaded cell, and then distributing i distinguishable plus signs to $i-1$ purely hollow cells and the remaining plus sign to the cell with a minus sign. Note that the cells carrying the $\mp$ sign as well as the plus sign are all shaded in the end.

There is also another way to construct such configurations. We may choose $i-1$ plus signs from the i distinguishable plus signs, place them in the hollow cells and turn them into shaded cells, and then fill a shaded cell with the sign $\mp$ and keep it shaded. Since there are m choices for a shaded cell to be filled with the sign $\mp$, the weights of all the feasible configurations generated by $\Delta^i D(a_\lambda)$ add up to

$$im\Delta^{i-1}(a_\lambda). \tag{4.3}$$

Casting up (4.2) and (4.3), we arrive at

$$i(nk-2m-i+1)\Delta^{i-1}(a_\lambda), \tag{4.4}$$

in agreement with the right hand side of (4.1). This completes the proof. ∎

It is worth mentioning that Hilbert obtained another identity on D and Δ. As before, let $c = nk-2m$. Then, for $i \geq 1$,

$$D^i\Delta(a_\lambda) - \Delta D^i(a_\lambda) = i(c+i-1)D^{i-1}(a_\lambda). \tag{4.5}$$

Using a similar approach to the identity (4.1), it is not hard to provide a combinatorial proof of (4.5). The detailed description is left out. Utilizing (4.5), Hilbert demonstrated that the number of invariants of a binary n-form of degree k and weight $m = nk/2$ equals $\delta(k,n,nk/2)$, where at least one of k and n is even. In fact, Hilbert deduced that the operator Δ is an injection from $Q_n(k,nk/2-1)$ to $Q_n(k,nk/2)$. If not, there would exist a nonzero polynomial I that can be written as a linear combination of the basis elements of $Q_n(k,nk/2-1)$ such that in some way the action of Δ on I yields an invariant in $Q_n(k,nk/2)$, that is, $D\Delta(I) = 0$. On the other hand, there must exist i such that $D^i(I) = 0$ but $D^{i-1}(I) \neq 0$. This incurs a contradiction with (4.5).

We also remark that the argument of Hilbert is valid for semi-invariants with $0 \leq m \leq nk/2$ where the parity constraint on n and k may be lifted. So it can be regarded as an alternative proof of Sylvester's theorem. Note that in the case when at least one of k and n is even and $m = nk/2$, a semi-invariant of degree k and weight m turns out to be an invariant.

5 The Additivity Lemma of Pak and Panova

In this section, we present a derivation of the additivity lemma of Pak and Panova in the context of semi-invariants. As will be seen, it is a consequence of the ring property of semi-invariants. In view of Theorem 3.1, to prove the strict unimodality of $\begin{bmatrix} n+k_1+k_2 \\ n \end{bmatrix}$, it suffices to show that for any $2 \le m \le \lfloor n(k_1+k_2)/2 \rfloor$, there exists a semi-invariant of degree k_1+k_2 and weight m.

Proof of Lemma 1.1. It is not difficult to see that under the conditions on n, k_1, k_2, for any $4 \le m \le \lfloor n(k_1+k_2)/2 \rfloor$, we can always express m as $m_1 + m_2$ such that $2 \le m_1 \le \lfloor nk_1/2 \rfloor$ and $2 \le m_2 \le \lfloor nk_2/2 \rfloor$.

Since at least one of k_1, k_2 and n is even, we find that

$$\lfloor nk_1/2 \rfloor + \lfloor nk_2/2 \rfloor = \lfloor n(k_1+k_2)/2 \rfloor. \tag{5.1}$$

This takes care of the case $m = \lfloor n(k_1+k_2)/2 \rfloor$.

For any $\lfloor nk_1/2 \rfloor + 2 \le m < \lfloor n(k_1+k_2)/2 \rfloor$, taking $m_1 = \lfloor nk_1/2 \rfloor$, a simple computation shows that the corresponding m_2 falls into the right range, that is, $2 \le m_2 < \lfloor nk_2/2 \rfloor$.

For any $4 \le m < \lfloor nk_1/2 \rfloor + 2$, we may choose m_2 to be 2 and set $m_1 = m - 2$. It can be checked that $2 \le m_1 < \lfloor nk_1/2 \rfloor$.

We now assume that $4 \le m \le \lfloor nk_1/2 \rfloor + \lfloor nk_2/2 \rfloor$ and $m = m_1 + m_2$, where $2 \le m_1 \le \lfloor nk_1/2 \rfloor$ and $2 \le m_2 \le \lfloor nk_2/2 \rfloor$. By the assumptions for $\left[{n+k_1 \atop n}\right]$ and $\left[{n+k_2 \atop n}\right]$, we see that there exists a semi-invariant I of a binary n-form of degree k_1 and weight m_1 and a semi-invariant J of a binary n-form of degree k_2 and weight m_2. Consequently, IJ is a semi-invariant of a binary n-form of degree k_1+k_2 and weight m.

We now turn to the remaining cases $m = 2$ and $m = 3$. Since $k_1, k_2, n \ge 2$ and at least one of k_1, k_2 and n is greater than two, it is evident that at least one of $nk_1/2$ and $nk_2/2$ is greater than or equal to three. Let us assume that $nk_1/2 \ge 3$. By the assumption for $\left[{n+k_1 \atop n}\right]$, there is a semi-invariant of degree k_1 and weight m_1 for any $2 \le m_1 \le nk_1/2$. For $m_1 = 2$, suppose that I is a semi-invariant of degree k_1 and weight two. For $m_1 = 3$, suppose that J is a semi-invariant of degree k_1 and weight three. Clearly, $a_0^{k_2}$ is a semi-invariant of degree k_2 and weight zero. It follows that $a_0^{k_2} I$ is a semi-invariant of degree k_1+k_2 and weight two and $a_0^{k_2} J$ is a semi-invariant of degree k_1+k_2 and weight three. This completes the proof. ∎

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