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Average size of a self-conjugate (s,t)-core partition

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Abstract. Armstrong, Hanusa and Jones conjectured that if s,t are coprime integers, then the average size of an (s,t)-core partition and the average size of a self-conjugate (s,t)-core partition are both equal to $\frac{(s+t+1)(s-1)(t-1)}{24}$. Stanley and Zanello showed that the average size of an $(s,s+1)$-core partition equals $\binom{s+1}{3}/2$. Based on a bijection of Ford, Mai and Sze between self-conjugate (s,t)-core partitions and lattice paths in $\lfloor \frac{s}{2} \rfloor \times \lfloor \frac{t}{2} \rfloor$ rectangle, we obtain the average size of a self-conjugate (s,t)-core partition as conjectured by Armstrong, Hanusa and Jones.

Keywords: (s,t)-core partition, self-conjugate partition, lattice path

AMS Classification: 05A17, 05A15

1 Introduction

In this paper, employing a bijection of Ford, Mai and Sze between self-conjugate (s,t)-core partitions and lattice paths, we prove a conjecture of Armstrong, Hanusa and Jones on the average size of a self-conjugate (s,t)-core partition.

A partition is called a t-core partition, or simply a t-core, if its Ferrers diagram contains no cells with hook length t. A partition is called an (s,t)-core partition, or simply an (s,t)-core, if it is simultaneously an s-core and a t-core. Let $r = gcd(s,t)$. If $r > 1$, then each r-core is an (s,t)-core, and so there are infinitely many (s,t)-cores. When s and t are coprime, Anderson [1] showed that the number of (s,t)-core partitions equals

$$\frac{1}{s+t}\binom{s+t}{s}.$$

For coprime integers s and t, Ford, Mai and Sze [4] characterized the set of hook lengths of diagonal cells in self-conjugate (s,t)-core partitions, and they showed that the number of self-conjugate (s,t)-core partitions is

$$\binom{\lfloor \frac{s}{2} \rfloor + \lfloor \frac{t}{2} \rfloor}{\lfloor \frac{s}{2} \rfloor}. \tag{1.1}$$

A partition is of size n if it is a partition of n. Aukerman, Kane and Sze [3] conjectured that the largest size of an (s,t)-core partition for coprime numbers s and t is $\frac{(s^2-1)(t^2-1)}{24}$. Olsson and Stanton [5] proved this conjecture and obtained the following uniqueness property.

Theorem 1.1 *If s and t are coprime, then there is a unique largest (s,t)-core partition of size*

$$\frac{(s^2-1)(t^2-1)}{24}, \tag{1.2}$$

which turns out to be self-conjugate.

A short proof for the conjecture of Aukerman, Kane and Sze was given by Tripathi [7]. Vandehey [8] obtained the following characterization of the largest (s,t)-core partition.

Theorem 1.2 *There exists a largest (s,t)-core partition λ with respect to the partial order of containment, that is, for each (s,t)-core μ, $\lambda_i \geq \mu_i$ for $1 \leq i \leq l(\mu)$.*

It is clear that the largest (s,t)-core in the above theorem is unique. It is the (s,t)-core of the largest size, and it is also an (s,t)-core of the longest length.

Armstrong, Hanusa and Jones [2] proposed the following conjecture concerning the average size of an (s,t)-core and the average size of a self-conjugate (s,t)-core.

Conjecture 1.3 *Assume that s and t are coprime. Then the average size of an (s,t)-core and the average size of a self-conjugate (s,t)-core are both equal to*

$$\frac{(s+t+1)(s-1)(t-1)}{24}.$$

Stanley and Zanello [6] showed that the conjecture for the average size of an (s,t)-core holds for $(s,s+1)$-cores. More precisely, they showed that the average size of an $(s,s+1)$-core equals $\binom{s+1}{3}/2$. In this paper, we prove the case of Conjecture 1.3 concerning the average size of a self-conjugate (s,t)-core.

2 The average size of a self-conjugate (s,t)-core

In this section, we give a proof of the case of Conjecture (1.3) for self-conjugate (s,t)-cores, which is stated as follows.

Theorem 2.1 *Assume that s and t are coprime. Then the average size of a self-conjugate (s,t)-core equals*

$$\frac{(s+t+1)(s-1)(t-1)}{24}.$$

69	53	37	21	5
47	31	15	−1	−17
25	9	−7	−23	−39
3	−13	−29	−45	−61

Figure 2.1: A lattice path P in the array $A(8, 11)$

Before we present the proof, let us recall a characterization of self-conjugate (s, t)-cores obtained by Ford, Mai and Sze [4]. They introduced an array $A(s, t) = (A_{i,j})_{1\leq i\leq \lfloor s/2\rfloor, 1\leq j\leq \lfloor t/2\rfloor}$, where

$$A_{i,j} = st - (2j - 1)s - (2i - 1)t. \tag{2.1}$$

Let $\mathcal{P}(A(s, t))$ be the set of lattice paths in $A(s, t)$ from the lower-left corner to the upper-right corner. For example, Figure 2.1 gives an array $A(s, t)$ for $s = 8$ and $t = 11$, where the solid lines represent a lattice path in $\mathcal{P}(A(s, t))$. For a lattice path P in $\mathcal{P}(A(s, t))$, let $M_{A(s,t)}(P)$ denote the set of positive entries $A_{i,j}$ below P along with the absolute values of negative entries above P. The following theorem of Ford, Mai and Sze [4] establishes a connection between self-conjugate (s, t)-cores and lattice pathes in $A(s, t)$.

Theorem 2.2 *Assume that s and t are coprime. Let $A(s, t)$ be the array as given in (2.1). Then there is a bijection Φ between the set $\mathcal{P}(A(s, t))$ and the set of self-conjugate (s, t)-core partitions such that for any $P \in \mathcal{P}(A(s, t))$, the set of main diagonal hook lengths of $\Phi(P)$ is given by $M_{A(s,t)}(P)$.*

For example, in Figure 2.1, 5 is the only positive entry below P, while -7 and -13 are the negative entries above P. Thus $M_{A(8,11)}(P) = \{5, 7, 13\}$. So we have $\Phi(P) = (7, 5, 5, 3, 3, 1, 1)$, which is an $(8, 11)$-core partition.

The following lemma gives a formula for the size of a self-conjugate (s, t)-core partition λ corresponding to a lattice P in $\mathcal{P}(A(s, t))$.

Lemma 2.3 *For any lattice path P in $\mathcal{P}(A(s, t))$, we have*

$$|\Phi(P)| = \frac{(s^2 - 1)(t^2 - 1)}{24} - \sum_{(i,j) \text{ is above } P} A_{i,j}.$$

Proof. For a self-conjugate partition λ, define

$$MD(\lambda) = \{h | h \text{ is the hook length of a cell on the main diagonal of } \lambda\}.$$

Clearly, the main diagonal cells have distinct hook lengths and the size of a self-conjugate partition equals the sum of elements in $MD(\lambda)$. Let P be a lattice path in $\mathcal{P}(A(s, t))$. By

Theorem 2.2, we find that

$$
\begin{aligned}
|\Phi(P)| &= \sum_{h\in MD(\Phi(P))} h \\
&= \sum_{(i,j) \text{ is below } P,\, A_{i,j}>0} A_{i,j} - \sum_{(i,j) \text{ is above } P,\, A_{i,j}<0} A_{i,j} \\
&= \sum_{A_{i,j}>0} A_{i,j} - \sum_{(i,j) \text{ is above } P} A_{i,j}.
\end{aligned}
$$

To show that

$$\sum_{A_{i,j}>0} A_{i,j} = \frac{(s^2-1)(t^2-1)}{24}, \tag{2.2}$$

let Q be the lattice path along the left and upper borders of $A(s,t)$. Note that $M_{A(s,t)}(Q)$ consists of positive entries of $A(s,t)$. Let $\lambda = \Phi(Q)$. By Theorem 2.2, the set of main diagonal hook lengths of λ equals $M_{A(s,t)}(Q)$. It follows that

$$|\lambda| = \sum_{A_{i,j}>0} A_{i,j}. \tag{2.3}$$

We now proceed to show that

$$|\lambda| = \frac{(s^2-1)(t^2-1)}{24}. \tag{2.4}$$

By Theorem 1.1, there is a unique (s,t)-core μ with the largest size $\frac{(s^2-1)(t^2-1)}{24}$. To prove (2.4), it suffices to show that $\mu = \lambda$. Let $l(\lambda)$ and $l(\mu)$ denote the lengths of λ and μ respectively. By Theorem 2.2, there is a lattice path $R \in \mathcal{P}(A(s,t))$ such that $\mu = \Phi(R)$. Using Theorem 1.2, we find that

$$l(\mu) \geq l(\lambda) \tag{2.5}$$

and

$$\mu_i \geq \lambda_i \tag{2.6}$$

for all i. Combining (2.5) and (2.6), we obtain that

$$\mu_1 + l(\mu) - 1 \geq \lambda_1 + l(\lambda) - 1. \tag{2.7}$$

Next we show that

$$\lambda_1 + l(\lambda) - 1 \geq \mu_1 + l(\mu) - 1. \tag{2.8}$$

Notice that the largest main diagonal hook length of λ is $\lambda_1 + l(\lambda) - 1$, that is,

$$\max MD(\lambda) = \lambda_1 + l(\lambda) - 1. \tag{2.9}$$

Since $\lambda = \Phi(Q)$, by Theorem 2.2, we deduce that

$$MD(\lambda) = M_{A(s,t)}(Q) = \{A_{i,j} | A_{i,j} > 0, 1 \leq i \leq \lfloor s/2 \rfloor, 1 \leq j \leq \lfloor t/2 \rfloor\}. \tag{2.10}$$

Clearly, $A_{1,1}$ is largest among all positive entries in $A(s,t)$. It follows from (2.9) and (2.10) that

$$A_{1,1} = \lambda_1 + l(\lambda) - 1. \tag{2.11}$$

On the other hand, since $\mu_1 + l(\mu) - 1$ is the hook length of the cell in the upper-left corner of μ, Theorem 2.2 ensures the existence of an entry $A_{i,j}$ of $M_{A(s,t)}(R)$ such that

$$|A_{i,j}| = \mu_1 + l(\mu) - 1. \tag{2.12}$$

We claim that

$$A_{1,1} \geq |A_{i,j}|, \tag{2.13}$$

for any entry $A_{i,j}$. Observe that

$$A_{1,1} > |A_{\lfloor s/2\rfloor,\lfloor t/2\rfloor}|, \tag{2.14}$$

since

$$A_{1,1} + A_{\lfloor s/2\rfloor,\lfloor t/2\rfloor} = (st - s - t) + (st + s + t - 2t\lfloor s/2\rfloor - 2s\lfloor t/2\rfloor) > 0.$$

Notice that $A_{1,1}$ is the largest entry in $A(s,t)$ and $A_{\lfloor s/2\rfloor,\lfloor t/2\rfloor}$ is the smallest entry in $A(s,t)$. Thus (2.14) implies (2.13). This proves (2.8).

Combining (2.7) and (2.8), we deduce that

$$\lambda_1 + l(\lambda) - 1 = \mu_1 + l(\mu) - 1. \tag{2.15}$$

In view of (2.11) and (2.15), we see that

$$A_{1,1} = \mu_1 + l(\mu) - 1.$$

Thus $A_{1,1}$ lies in $MD(\mu)$. By Theorem 2.2, $A_{1,1}$ belongs to $M_{A(s,t)}(R)$. Since $A_{1,1} > 0$, R is the lattice path along the left and upper borders, namely, $Q = R$ and $\lambda = \mu$. So we conclude that λ is the largest (s,t)-core. This completes the proof. ∎

As to the case of Conjecture 1.3 for self-conjugate cores, we need some identities on the number of lattice paths in a rectangular region. Let m and n be positive integers, and B_{mn} be an $m \times n$ diagram. The positions of the cells of the first row are $(1,1),(1,2),\ldots,(1,n)$, and so on. The set of lattice paths from the lower-left corner to the upper-right corner of B_{mn} is denoted by $\mathcal{P}(B_{mn})$. Let $f(i,j)$ be the number of lattice paths in $\mathcal{P}(B_{mn})$ that lie below the cell (i,j), possibly containing the right or lower border of the cell (i,j).

Lemma 2.4 *For positive integers m and n, we have*

$$\sum_{1\leq i\leq m,\,1\leq j\leq n} f(i,j) = \frac{mn}{2}\binom{m+n}{m}. \tag{2.16}$$

Proof. It is clear that the number of lattice paths in $\mathcal{P}(B_{mn})$ below the cell (i,j) equals the number of lattice paths above the cell $(m-i+1, n-j+1)$. Hence we have

$$f(i,j) + f(m-i+1, n-j+1) = |\mathcal{P}(B_{mn})|.$$

Note that the total number of lattice paths in $\mathcal{P}(B_{mn})$ equals $\binom{m+n}{m}$. So we get

$$f(i,j) + f(m-i+1, n-j+1) = \binom{m+n}{m}. \tag{2.17}$$

Summing (2.17) over i and j, we obtain (2.16). ∎

Lemma 2.5 *For positive integers m and n, we have*

$$\sum_{1\le i\le m,\, 1\le j\le n} if(i,j) = \binom{m+2}{3}\binom{m+n}{m+1} \tag{2.18}$$

and

$$\sum_{1\le i\le m,\, 1\le j\le n} jf(i,j) = \binom{n+2}{3}\binom{m+n}{n+1}. \tag{2.19}$$

Proof. Let

$$G(m,n) = \sum_{1\le i\le m,\, 1\le j\le n} if(i,j).$$

To prove (2.18), we claim that for $m, n \geq 2$,

$$G(m,n) = G(m-1,n) + G(m,n-1) + \binom{m+1}{2}\binom{m+n-1}{m}. \tag{2.20}$$

To prove (2.20), let T be the set of triples (P, C_1, C_2), where P is a path in $\mathcal{P}(B_{mn})$, C_1 and C_2 are two cells above P satisfying that they are in the same column and C_1 is at least as high as C_2. Notice that C_1 and C_2 are allowed to be the same cell.

We proceed to compute $|T|$ in two ways. It is easily seen that $if(i,j)$ is the number of triples (P, C_1, C_2) in T with $C_1 = (i,j)$. For $m, n \geq 1$, we have $|T| = G(m,n)$.

For a given lattice path P in $\mathcal{P}(B_{mn})$, the cells above P form the Ferrers diagram of a partition, denoted by μ. Let μ' be the conjugate of μ. In the j-th column of the Ferrers diagram of μ, there are $\binom{\mu'_j+1}{2}$ ways to choose C_1 and C_2 such that C_1 is not lower than C_2. It follows that for given lattice path P in $\mathcal{P}(B_{mn})$, there are $\sum_{1\le j\le \mu_1}\binom{\mu'_j+1}{2}$ choices for C_1 and C_2. Thus, for $m, n \geq 1$, we have

$$|T| = \sum_{\mu:\, 1\le\mu_1\le n,\, 1\le\mu'_1\le m}\ \sum_{1\le j\le\mu_1}\binom{\mu'_j+1}{2}. \tag{2.21}$$

So we deduce that for $m, n \geq 1$,

$$G(m,n) = \sum_{\mu:\, 1\le\mu_1\le n,\, 1\le\mu'_1\le m}\ \sum_{1\le j\le\mu_1}\binom{\mu'_j+1}{2}. \tag{2.22}$$

Next, we use the above expression (2.22) for $G(m,n)$ to derive the recurrence relation (2.20). For $m, n \geq 2$, the right hand side of (2.22) can be written as

$$\sum_{\mu:\, 1\le\mu_1\le n,\, \mu'_1=m}\ \sum_{1\le j\le\mu_1}\binom{\mu'_j+1}{2} + \sum_{\mu:\, 1\le\mu_1\le n,\, 1\le\mu'_1\le m-1}\ \sum_{1\le j\le\mu_1}\binom{\mu'_j+1}{2}. \tag{2.23}$$

It is evident from (2.22) that the second double sum in (2.23) equals $G(m-1,n)$. The first double sum in (2.23) can be rewritten as

$$\sum_{\mu:\,1\le\mu_1\le n,\,\mu_1'=m}\ \sum_{2\le j\le\mu_1}\binom{\mu_j'+1}{2}+\sum_{\mu:\,1\le\mu_1\le n,\,\mu_1'=m}\binom{m+1}{2}. \tag{2.24}$$

Clearly, the number of partitions μ with $1\le\mu_1\le n$ and $\mu_1'=m$ equals the number of lattice paths from the lower-left corner to the upper-right corner in $B_{m,n-1}$, which is $\binom{m+n-1}{m}$. Hence the second sum in (2.24) simplifies to

$$\binom{m+1}{2}\binom{m+n-1}{m}. \tag{2.25}$$

To compute the double sum in (2.24), let $\tilde{\mu}$ denote the partition obtained from μ by deleting the first column of the Ferrers diagram of μ. In this notation, we have

$$\sum_{\mu:\,1\le\mu_1\le n,\,\mu_1'=m}\ \sum_{2\le j\le\mu_1}\binom{\mu_j'+1}{2} = \sum_{\tilde{\mu}:\,0\le\tilde{\mu}_1\le n-1,\,\tilde{\mu}_1'\le m}\ \sum_{1\le j\le\tilde{\mu}_1}\binom{\tilde{\mu}_j'+1}{2}. \tag{2.26}$$

Notice that the right hand side of (2.26) equals $G(m,n-1)$. Combining (2.25) and (2.26), we see that the first double sum in (2.23) equals

$$G(m,n-1)+\binom{m+1}{2}\binom{m+n-1}{m}.$$

This proves the recurrence relation (2.20).

For $m,n\ge 1$, let

$$F(m,n)=\binom{m+2}{3}\binom{m+n}{m+1}.$$

Clearly, $F(1,n)=G(1,n)$ and $F(m,1)=G(m,1)$ for $m,n\ge 1$. Moreover, it is easily verified that $F(m,n)$ also satisfies the recurrence relation (2.20). So we obtain (2.18), which can be rewritten in the form of (2.19). This completes the proof. ∎

Now we are ready to prove Theorem 2.1.

Proof of Theorem 2.1. Let $SC(s,t)$ denote the set of self-conjugate (s,t)-cores. We aim to show that

$$\sum_{\lambda\in SC(s,t)}|\lambda| = \frac{(s+t+1)(s-1)(t-1)}{24}\binom{\lfloor\frac{s}{2}\rfloor+\lfloor\frac{t}{2}\rfloor}{\lfloor\frac{s}{2}\rfloor}. \tag{2.27}$$

By Theorem 2.2, we find that

$$\sum_{\lambda\in SC(s,t)}|\lambda| = \sum_{P\in\mathcal{P}(A(s,t))}|\Phi(P)|. \tag{2.28}$$

Using Lemma 2.3, we get

$$\sum_{P\in\mathcal{P}(A(s,t))}|\Phi(P)| = \frac{(s^2-1)(t^2-1)}{24}\binom{\lfloor\frac{s}{2}\rfloor+\lfloor\frac{t}{2}\rfloor}{\lfloor\frac{t}{2}\rfloor} - \sum_{P\in\mathcal{P}(A(s,t))}\ \sum_{(i,j)\text{ is above }P}A_{i,j}. \tag{2.29}$$

Combining (2.28) and (2.29), we obtain that

$$\sum_{\lambda\in SC(s,t)} |\lambda| = \frac{(s^2-1)(t^2-1)}{24}\binom{\lfloor \frac{s}{2}\rfloor + \lfloor \frac{t}{2}\rfloor}{\lfloor \frac{t}{2}\rfloor} - \sum_{P\in\mathcal{P}(A(s,t))}\ \sum_{(i,j) \text{ is above } P} A_{i,j}. \tag{2.30}$$

By the definition (2.1) of the array $A(s,t)$, we deduce that

$$\begin{aligned}\sum_{P\in\mathcal{P}(A(s,t))}\ \sum_{(i,j) \text{ is above } P} A_{i,j} &= \sum_{P\in\mathcal{P}(A(s,t))}\ \sum_{(i,j) \text{ is above } P} (st+s+t-2sj-2ti)\\ &= (st+s+t)\sum_{1\le i\le\lfloor \frac{s}{2}\rfloor,1\le j\le\lfloor \frac{t}{2}\rfloor} f(i,j) - 2s\sum_{1\le i\le\lfloor \frac{s}{2}\rfloor,1\le j\le\lfloor \frac{t}{2}\rfloor} jf(i,j)\\ &\quad -2t\sum_{1\le i\le\lfloor \frac{s}{2}\rfloor,1\le j\le\lfloor \frac{t}{2}\rfloor} if(i,j). \end{aligned}\tag{2.31}$$

Using Lemma 2.4 and Lemma 2.5 with $m=\lfloor \frac{s}{2}\rfloor$ and $n=\lfloor \frac{t}{2}\rfloor$, (2.31) becomes

$$\begin{aligned}\sum_{P\in\mathcal{P}(A(s,t))}\ \sum_{(i,j) \text{ is above } P} A_{i,j} &= (st+s+t)\binom{\lfloor \frac{s}{2}\rfloor + \lfloor \frac{t}{2}\rfloor}{\lfloor \frac{s}{2}\rfloor}\frac{\lfloor \frac{s}{2}\rfloor\lfloor \frac{t}{2}\rfloor}{2} - 2s\binom{\lfloor \frac{t}{2}\rfloor+2}{3}\binom{\lfloor \frac{s}{2}\rfloor + \lfloor \frac{t}{2}\rfloor}{\lfloor \frac{s}{2}\rfloor-1}\\ &\quad -2t\binom{\lfloor \frac{s}{2}\rfloor+2}{3}\binom{\lfloor \frac{s}{2}\rfloor + \lfloor \frac{t}{2}\rfloor}{\lfloor \frac{t}{2}\rfloor-1}. \end{aligned}\tag{2.32}$$

We claim that

$$\begin{aligned}\frac{(s^2-1)(t^2-1)}{24}\binom{\lfloor \frac{s}{2}\rfloor + \lfloor \frac{t}{2}\rfloor}{\lfloor \frac{t}{2}\rfloor} &= \frac{(s+t+1)(s-1)(t-1)}{24}\binom{\lfloor \frac{s}{2}\rfloor + \lfloor \frac{t}{2}\rfloor}{\lfloor \frac{t}{2}\rfloor}\\ &\quad +(st+s+t)\frac{\lfloor \frac{s}{2}\rfloor\lfloor \frac{t}{2}\rfloor}{2}\binom{\lfloor \frac{s}{2}\rfloor + \lfloor \frac{t}{2}\rfloor}{\lfloor \frac{t}{2}\rfloor}\\ &\quad -2t\binom{\lfloor \frac{s}{2}\rfloor+2}{3}\binom{\lfloor \frac{s}{2}\rfloor + \lfloor \frac{t}{2}\rfloor}{\lfloor \frac{t}{2}\rfloor-1}\\ &\quad -2s\binom{\lfloor \frac{t}{2}\rfloor+2}{3}\binom{\lfloor \frac{s}{2}\rfloor + \lfloor \frac{t}{2}\rfloor}{\lfloor \frac{s}{2}\rfloor-1}, \end{aligned}\tag{2.33}$$

which simplifies to

$$\frac{st(s-1)(t-1)}{24} = (st+s+t)\frac{\lfloor \frac{s}{2}\rfloor\lfloor \frac{t}{2}\rfloor}{2} - \frac{t}{3}\left(\left\lfloor\frac{s}{2}\right\rfloor+2\right)\left\lfloor\frac{s}{2}\right\rfloor\left\lfloor\frac{t}{2}\right\rfloor - \frac{s}{3}\left(\left\lfloor\frac{t}{2}\right\rfloor+2\right)\left\lfloor\frac{s}{2}\right\rfloor\left\lfloor\frac{t}{2}\right\rfloor.$$

When s and t are coprime, at least one of s and t is odd. Thus, we may assume, without loss of generality, that s is odd. In this case, the above relation can be easily verified. So the claim holds. Combining (2.30), (2.32) and (2.33), we arrive at (2.27), and hence the proof is complete. ∎

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