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## An Overpartition Analogue of Bressoud's Theorem of Rogers-Ramanujan-Gordon Type

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**Abstract.** For  $k \geq 2$  and  $k \geq i \geq 1$ , let  $B_{k,i}(n)$  denote the number of partitions of  $n$  such that part 1 appears at most  $i - 1$  times, two consecutive integers  $l$  and  $l + 1$  appear at most  $k - 1$  times and if  $l$  and  $l + 1$  appear exactly  $k - 1$  times then the sum of the parts  $l$  and  $l + 1$  is congruent to  $i - 1$  modulo 2. Let  $A_{k,i}(n)$  denote the number of partitions with parts not congruent to  $i$ ,  $2k - i$  and  $2k$  modulo  $2k$ . Bressoud's theorem states that  $A_{k,i}(n) = B_{k,i}(n)$ . Corteel, Lovejoy, and Mallet found an overpartition analogue of Bressoud's theorem for  $i = 1$ , that is, for partitions not containing non-overlined part 1. We obtain an overpartition analogue of Bressoud's theorem in the general case. For  $k \geq 2$  and  $k \geq i \geq 1$ , let  $D_{k,i}(n)$  denote the number of overpartitions of  $n$  such that the non-overlined part 1 appears at most  $i - 1$  times, for any integer  $l$ ,  $l$  and non-overlined  $l + 1$  appear at most  $k - 1$  times and if the parts  $l$  and the non-overlined part  $l + 1$  together appear exactly  $k - 1$  times then the sum of the parts  $l$  and non-overlined parts  $l + 1$  has the same parity as the number of overlined parts that are less than  $l + 1$  plus  $i - 1$ . Let  $C_{k,i}(n)$  denote the number of overpartitions of  $n$  with the non-overlined parts not congruent to  $\pm i$  and  $2k - 1$  modulo  $2k - 1$ . We show that  $C_{k,i}(n) = D_{k,i}(n)$ . Note

that this relation can also be considered as a Rogers-Ramanujan-Gordon type theorem for overpartitions.

**Keywords:** the Rogers-Ramanujan-Gordon theorem, overpartition, Bressoud's theorem

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## 1 Introduction

The Rogers-Ramanujan-Gordon theorem is a combinatorial generalization of the Rogers-Ramanujan identities [15, 16], see Gordon [10]. It establishes the equality between the number of partitions of  $n$  with parts satisfying certain residue conditions and the number of partitions of  $n$  with certain difference conditions. Gordon found an involution for an equivalent form of the generating function identity for this relation. An algebraic proof was given by Andrews [1] by using a recursive approach. It should be noted that the Rogers-Ramanujan-Gordon theorem is concerned only with odd moduli. Bressoud [4] succeeded in finding a theorem of Rogers-Ramanujan-Gordon type for even moduli by using an algebraic approach in the spirit of Andrews [1].

The objective of this paper is to give an overpartition analogue of Bressoud's theorem. We derive the equality between the number of overpartitions of  $n$  such that the non-overlined parts belong to certain residue classes modulo an odd positive integer and the number of overpartitions of  $n$  with parts satisfying certain difference conditions. A special case of this relation has been discovered by Corteel, Lovejoy, and Mallet [8].

An overpartition analogue of the Rogers-Ramanujan-Gordon theorem was obtained by Chen, Sang and Shi [6], which states that the number of overpartitions of  $n$  with non-overlined parts belonging to certain residue classes modulo an even positive integer equals the number of overpartitions of  $n$  with parts satisfying certain difference conditions. However, as will be seen, the proof of the overpartition analogue of the Rogers-Ramanujan-Gordon theorem does not seem to be directly applicable to the case for the overpartition analogue of Bressoud's theorem.

Let us give an overview of some definitions. A partition  $\lambda$  of a positive integer  $n$  is a non-increasing sequence of positive integers  $\lambda_1 \geq \cdots \geq \lambda_s > 0$  such that  $n = \lambda_1 + \cdots + \lambda_s$ . The partition of zero is defined to be the partition with no parts. An overpartition  $\lambda$  of a positive integer  $n$  is also a non-increasing sequence of positive integers  $\lambda_1 \geq \cdots \geq \lambda_s > 0$  such that  $n = \lambda_1 + \cdots + \lambda_s$  and the first occurrence of each integer may be overlined. For example,  $(\bar{7}, 7, 6, \bar{5}, 2, \bar{1})$  is an overpartition of 28. Many  $q$ -series identities have combinatorial interpretations in terms of overpartitions, see, for example, Corteel and Lovejoy [7]. Furthermore, overpartitions possess many analogous properties to ordinary partitions, see Lovejoy [11, 13]. For example, various overpartition theorems of the Rogers-Ramanujan-Gordon type have been obtained by Corteel and Lovejoy [9], Corteel, Lovejoy and Mallet [8] and Lovejoy [11, 12, 14]. For a partition or an overpartition  $\lambda$  and for any integer  $l$ , let  $f_l(\lambda)(\bar{f}_l(\lambda))$  denote the number of occurrences of a non-overlined

part  $l$  (an overlined part  $\bar{l}$ ) in  $\lambda$ . Let  $V_\lambda(l)$  denote the number of overlined parts in  $\lambda$  that are less than or equal to  $l$ .

We shall adopt the common notation as used in Andrews [3]. Let

$$(a)_\infty = (a; q)_\infty = \prod_{i=0}^{\infty} (1 - aq^i),$$

and

$$(a)_n = (a; q)_n = \frac{(a)_\infty}{(aq^n)_\infty}.$$

We also write

$$(a_1, \dots, a_k; q)_\infty = (a_1; q)_\infty \cdots (a_k; q)_\infty.$$

The Rogers-Ramanujan-Gordon theorem reads as follows.

**Theorem 1.1** (*Rogers-Ramanujan-Gordon*) For  $k \geq 2$  and  $k \geq i \geq 1$ , let  $F_{k,i}(n)$  denote the number of partitions of  $n$  of the form  $\lambda_1 + \lambda_2 + \cdots + \lambda_s$ , where  $\lambda_j \geq \lambda_{j+1}$ ,  $\lambda_j - \lambda_{j+k-1} \geq 2$  and part 1 appears at most  $i - 1$  times. Let  $E_{k,i}(n)$  denote the number of partitions of  $n$  into parts  $\not\equiv 0, \pm i \pmod{2k+1}$ . Then for any  $n \geq 0$ , we have

$$E_{k,i}(n) = F_{k,i}(n). \tag{1.1}$$

In the algebraic proof of the above relation, Andrews [1, 3] introduced a hypergeometric function  $J_{k,i}(a; x; q)$  as given by

$$J_{k,i}(a; x; q) = H_{k,i}(a; xq; q) - axqH_{k,i-1}(a; xq; q), \tag{1.2}$$

where

$$H_{k,i}(a; x; q) = \sum_{n=0}^{\infty} \frac{x^{kn} q^{kn^2 + n - in} a^n (1 - x^i q^{2ni}) (axq^{n+1})_\infty (1/a)_n}{(q)_n (xq^n)_\infty}. \tag{1.3}$$

To prove (1.1), Andrews considered a refinement of  $F_{k,i}(n)$ , that is, the number of partitions enumerated by  $F_{k,i}(n)$  with exactly  $m$  parts, denoted by  $F_{k,i}(m, n)$ , and he showed that  $J_{k,i}(0; x; q)$  and the generating function of  $F_{k,i}(m, n)$  satisfy the same recurrence relation with the same initial values. Setting  $x = 1$  and using Jacobi's triple product identity, Andrews deduced that  $J_{k,i}(0; 1; q)$  equals the generating function for  $E_{k,i}(n)$ . This yields  $E_{k,i}(n) = F_{k,i}(n)$ .

The following Rogers-Ramanujan-Gordon type theorem for even moduli is due to Bressoud [4].

**Theorem 1.2** For  $k \geq 2$  and  $k \geq i \geq 1$ , let  $B_{k,i}(n)$  denote the number of partitions of  $n$  of the form  $\lambda = \lambda_1 + \lambda_2 + \cdots + \lambda_s$  such that

$$(i) \ f_1(\lambda) \leq i - 1,$$

(ii)  $f_l(\lambda) + f_{l+1}(\lambda) \leq k - 1$ ,

(iii) if  $f_l(\lambda) + f_{l+1}(\lambda) = k - 1$ , then  $lf_l(\lambda) + (l + 1)f_{l+1}(\lambda) \equiv i - 1 \pmod{2}$ .

Let  $A_{k,i}(n)$  denote the number of partitions of  $n$  with parts not congruent to  $0, \pm i$  modulo  $2k$ . Then we have

$$A_{k,i}(n) = B_{k,i}(n). \quad (1.4)$$

In the proof of Bressoud, he also used the hypergeometric function  $J_{k,i}(a; x; q)$  and used a recurrence relation for  $(-xq)_\infty J_{(k-1)/2, i/2}(a; x^2; q^2)$ .

Lovejoy [11] found the following overpartition analogues of Rogers-Ramanujan-Gordon theorem for the cases  $i = 1$  and  $i = k$ .

**Theorem 1.3** For  $k \geq 2$ , let  $\overline{B}_k(n)$  denote the number of overpartitions of  $n$  of the form  $\lambda_1 + \lambda_2 + \cdots + \lambda_s$  such that  $\lambda_j - \lambda_{j+k-1} \geq 1$  if  $\lambda_j$  is overlined and  $\lambda_j - \lambda_{j+k-1} \geq 2$  otherwise. Let  $\overline{A}_k(n)$  denote the number of overpartitions of  $n$  into parts not divisible by  $k$ . Then we have

$$\overline{A}_k(n) = \overline{B}_k(n). \quad (1.5)$$

**Theorem 1.4** For  $k \geq 2$ , let  $\overline{D}_k(n)$  denote the number of overpartitions of  $n$  of the form  $\lambda_1 + \lambda_2 + \cdots + \lambda_s$  such that 1 cannot occur as a non-overlined part, and  $\lambda_j - \lambda_{j+k-1} \geq 1$  if  $\lambda_j$  is overlined and  $\lambda_j - \lambda_{j+k-1} \geq 2$  otherwise. Let  $\overline{C}_k(n)$  denote the number of overpartitions of  $n$  whose non-overlined parts are not congruent to  $0, \pm 1$  modulo  $2k$ . Then we have

$$\overline{C}_k(n) = \overline{D}_k(n). \quad (1.6)$$

Chen, Sang and Shi [6] obtained an overpartition analogue of the Rogers-Ramanujan-Gordon theorem in the general case.

**Theorem 1.5** For  $k \geq 2$  and  $k \geq i \geq 1$ , let  $P_{k,i}(n)$  denote the number of overpartitions of  $n$  of the form  $\lambda = \lambda_1 + \lambda_2 + \cdots + \lambda_s$  such that part 1 occurs as a non-overlined part at most  $i - 1$  times, and  $\lambda_j - \lambda_{j+k-1} \geq 1$  if  $\lambda_j$  is overlined and  $\lambda_j - \lambda_{j+k-1} \geq 2$  otherwise. For  $k \geq 2$  and  $k > i \geq 1$ , let  $Q_{k,i}(n)$  denote the number of overpartitions of  $n$  whose non-overlined parts are not congruent to  $0, \pm i$  modulo  $2k$  and let  $Q_{k,k}(n)$  denote the number of overpartitions of  $n$  with parts not divisible by  $k$ . Then we have

$$P_{k,i}(n) = Q_{k,i}(n). \quad (1.7)$$

As an overpartition analogue of Bressoud's theorem for the case  $i = 1$ , Corteel, Lovejoy, and Mallet [8] obtained the following overpartition theorem.

**Theorem 1.6** For  $k \geq 2$ , let  $\overline{A}_k^3(n)$  denote the number of overpartitions whose non-overlined parts are not congruent to  $0, \pm 1$  modulo  $2k - 1$ . Let  $\overline{B}_k^3(n)$  denote the number of overpartitions  $\lambda$  of  $n$  such that

- (i)  $f_1(\lambda) = 0$ ,
- (ii)  $f_l(\lambda) + f_{\bar{l}}(\lambda) + f_{l+1}(\lambda) \leq k - 1$ ,
- (iii) if  $f_l(\lambda) + f_{\bar{l}}(\lambda) + f_{l+1}(\lambda) = k - 1$ , then  $lf_l(\lambda) + lf_{\bar{l}}(\lambda) + (l+1)f_{l+1}(\lambda) \equiv V_\lambda(l) \pmod{2}$ .

Then we have

$$\overline{A}_k^3(n) = \overline{B}_k^3(n). \quad (1.8)$$

In this paper, we give an overpartition analogue of the Bressoud's theorem in the general case.

## 2 The Main Result

The main result of this paper can be stated as follows.

**Theorem 2.1** For  $k \geq 2$  and  $k \geq i \geq 1$ , let  $D_{k,i}(n)$  denote the number of overpartitions of  $n$  of the form  $\lambda = \lambda_1 + \lambda_2 + \cdots + \lambda_s$  such that

- (i)  $f_1(\lambda) \leq i - 1$ ;
- (ii)  $f_l(\lambda) + f_{\bar{l}}(\lambda) + f_{l+1}(\lambda) \leq k - 1$ ;
- (iii) if  $f_l(\lambda) + f_{\bar{l}}(\lambda) + f_{l+1}(\lambda) = k - 1$ , then  $lf_l(\lambda) + lf_{\bar{l}}(\lambda) + (l+1)f_{l+1}(\lambda) \equiv V_\lambda(l) + i - 1 \pmod{2}$ .

Let  $C_{k,i}(n)$  denote the number of overpartitions of  $n$  whose non-overlined parts are not congruent to  $0, \pm i$  modulo  $2k - 1$ . Then we have

$$C_{k,i}(n) = D_{k,i}(n). \quad (2.9)$$

Instead of using the function  $\tilde{J}_{k,i}(a; x; q)$  as in the proof of Theorem 1.6 given by Corteel, Lovejoy, and Mallet [8], we find that the function  $\tilde{H}_{k,i}(a; x; q)$ , also introduced by Corteel, Lovejoy, and Mallet [8], is related to the generating functions of the numbers  $C_{k,i}(n)$  and  $D_{k,i}(n)$ . Recall that

$$\tilde{J}_{k,i}(a; x; q) = \tilde{H}_{k,i}(a; xq; q) + axq\tilde{H}_{k,i-1}(a; xq; q), \quad (2.10)$$

where

$$\tilde{H}_{k,i}(a; x; q) = \sum_{n \geq 0} \frac{(-a)^n q^{kn^2 - \binom{n}{2} + n - in} x^{(k-1)n} (1 - x^i q^{2ni}) (-x, -1/a)_n (-axq^{n+1})_\infty}{(q^2; q^2)_n (xq^n)_\infty}. \quad (2.11)$$

It should be noticed that the function  $\tilde{J}_{k,i}(a; x; q)$  can be expressed as  $F_{1,k,i}(-q, \infty; -1/a; q)$  in the notation of Bressoud [5], and the function  $(-q)_\infty \tilde{H}_{k,i}(a; x; q)$  can be written as  $H_{k,i}(-1/a, -x; x; q)_2$  in the notation of Andrews [2].

Let  $\tilde{B}_k^3(m, n)$  denote the number of overpartitions enumerated by  $\tilde{B}_k^3(n)$  with exactly  $m$  parts. Corteel, Lovejoy and Mallet [8] have shown that the coefficients of  $x^m q^n$  in  $\tilde{J}_{k,1}(1/q; x; q)$  and  $\tilde{B}_k^3(m, n)$  satisfy the same recurrence relation with the same initial values. Moreover, they proved that the generating function of  $\tilde{B}_k^3(m, n)$  also equals  $\tilde{J}_{k,1}(1/q; x; q)$ , that is,

$$\sum_{m,n \geq 0} \overline{B}_k^3(m, n) x^m q^n = \tilde{J}_{k,1}(-1/q; x; q). \quad (2.12)$$

Setting  $a = 1/q$ ,  $x = 1$  and using Jacobi's triple product identity, the function  $\tilde{J}_{k,i}(a; x; q)$  can be expressed as an infinite product, namely,

$$\tilde{J}_{k,1}(1/q; 1; q) = \frac{(q, q^{2k-2}, q^{2k-1}; q^{2k-1})_\infty (-q)_\infty}{(q)_\infty}.$$

Clearly, this is the generating function for  $\overline{A}_k^3(n)$ . It follows that  $\overline{A}_k^3(n) = \overline{B}_k^3(n)$ .

However, the proof of Corteel, Lovejoy and Mallet does not seem to apply to the general case, since  $\tilde{J}_{k,i}(1/q; x; q)$  does not seem to have an infinite product expression for  $i \geq 2$ . Our strategy goes as follows. For  $C_{k,i}(n)$ , we show that the generating function for  $C_{k,i}(n)$  can be expressed in terms of  $\tilde{H}_{k,i}(a; x; q)$  with  $a = 1/q$  and  $x = q$ . For  $D_{k,i}(n)$ , let  $D_{k,i}(m, n)$  denote the number of overpartitions enumerated by  $D_{k,i}(n)$  with exactly  $m$  parts. We find a combinatorial interpretation of  $D_{k,i}(m, n) - D_{k,i-1}(m, n)$  from which we can derive a recurrence relation for  $D_{k,i}(m, n)$ . Furthermore, we see that the recurrence relation and initial values of  $D_{k,i}(m, n)$  coincide with the recurrence relation and the initial values of the coefficients of  $x^m q^n$  in  $\tilde{H}_{k,i}(1/q; xq; q)$ . Thus we reach the conclusion that the generating function of  $D_{k,i}(m, n)$  equals  $\tilde{H}_{k,i}(-1/q; xq; q)$ . Setting  $x = 1$ , we deduce that the generating function of  $D_{k,i}(n)$  equals the generating function of  $C_{k,i}(n)$ .

For convenience, we write  $W_{k,i}(x; q)$  for  $\tilde{H}_{k,i}(1/q; xq; q)$ , that is,

$$W_{k,i}(x; q) = \sum_{n \geq 0} \frac{(-1)^n q^{(2k-1)\binom{n+1}{2} - in} x^{(k-1)n} (1 - x^i q^{(2n+1)i}) (-xq)_\infty}{(q)_n (xq^{n+1})_\infty}. \quad (2.13)$$

Recall that Andrews found the following recurrence relation for  $H_{k,i}(a; x; q)$ :

$$H_{k,i}(a; x; q) - H_{k,i-1}(a; x; q) = x^{i-1} H_{k,k-i+1}(a; xq; q) - ax^i q H_{k,k-i}(a; xq; q). \quad (2.14)$$

A recurrence relation for  $W_{k,i}(x; q)$  is given below.

**Theorem 2.2** *For  $k \geq 2$  and  $k \geq i \geq 1$ , we have*

$$W_{k,i}(x; q) - W_{k,i-1}(x; q) = (1 + xq)(xq)^{i-1} W_{k,k-i}(xq; q). \quad (2.15)$$

*Proof.* Since

$$q^{-in} - x^i q^{(n+1)i} - q^{(-i+1)n} + x^{i-1} q^{(n+1)(i-1)} = q^{-in}(1 - q^n) + x^{i-1} q^{(n+1)(i-1)}(1 - xq^{n+1}),$$

it can be checked that  $W_{k,i}(x; q) - W_{k,i-1}(x; q)$  can be written as

$$\sum_{n=1}^{\infty} q^{-in} \frac{(-1)^n x^{(k-1)n} q^{(2k-1)\binom{n+1}{2}} (-xq)_{\infty}}{(q)_{n-1} (xq^{n+1})_{\infty}} + \sum_{n=0}^{\infty} (xq^{n+1})^{i-1} \frac{(-1)^n x^{(k-1)n} q^{(2k-1)\binom{n+1}{2}} (-xq)_{\infty}}{(q)_n (xq^{n+2})_{\infty}}. \quad (2.16)$$

Now, replacing  $n$  with  $n + 1$ , the first sum in (2.16) can be expressed as

$$\sum_{n=0}^{\infty} q^{-i(n+1)} \frac{(-1)^{(n+1)} x^{(k-1)(n+1)} q^{(2k-1)\binom{n+2}{2}} (-xq)_{\infty}}{(q)_n (xq^{n+2})_{\infty}}. \quad (2.17)$$

Hence  $W_{k,i}(x; q) - W_{k,i-1}(x; q)$  equals

$$\begin{aligned} & - (xq)^{i-1} \sum_{n=0}^{\infty} \frac{(-1)^n (xq)^{(k-1)n} q^{(2k-1)\binom{n+1}{2}} x^{k-i} q^{(2k-1)(n+1)-in-2i+1-(k-1)n} (-xq)_{\infty}}{(q)_n (xq^{n+2})_{\infty}} \\ & + (xq)^{i-1} \sum_{n=0}^{\infty} \frac{(-1)^n (xq)^{(k-1)n} q^{(2k-1)\binom{n+1}{2}+(i-1)n-(k-1)n} (-xq)_{\infty}}{(q)_n (xq^{n+2})_{\infty}} \\ & = (1 + xq)(xq)^{i-1} \sum_{n \geq 0} \frac{(-1)^n (xq)^{(k-1)n} q^{(2k-1)\binom{n+1}{2}-(k-i)n} (1 - x^{k-i} q^{(2n+2)(k-i)}) (-xq^2)_{\infty}}{(q)_n (xq^{n+2})_{\infty}} \\ & = (1 + xq)(xq)^{i-1} W_{k,k-i}(xq), \end{aligned}$$

as desired. ■

The following relation can be considered as a combinatorial interpretation of  $D_{k,i}(m, n) - D_{k,i-1}(m, n)$ .

**Theorem 2.3** *For  $k \geq 2$ ,  $k \geq i \geq 1$  and for  $m, n \geq 0$ , let  $S_{k,i}(m, n)$  denote the set of overpartitions enumerated by  $D_{k,i}(m, n)$  with exactly one overlined part 1 and exactly  $i - 1$  non-overlined part 1. Let  $T_{k,i}(m, n)$  denote the set of overpartitions enumerated by  $D_{k,i}(m, n)$  with exactly one overlined part 1 and exactly  $i - 2$  non-overlined part 1. Then we have*

$$D_{k,i}(m, n) - D_{k,i-1}(m, n) = |S_{k,i}(m, n)| + |T_{k,i}(m, n)|. \quad (2.18)$$

*Proof.* Let  $U_{k,i}(m, n)$  denote the set of overpartitions enumerated by  $D_{k,i}(n)$  with exactly  $m$  parts. By the definitions of  $D_{k,i}(m, n)$  and  $D_{k,i-1}(m, n)$ , it can be easily seen that  $U_{k,i-1}(m, n)$  is not contained in  $U_{k,i}(m, n)$ . To compute  $D_{k,i}(m, n) - D_{k,i-1}(m, n)$ , we wish to construct an injection  $\varphi$  from overpartitions in  $U_{k,i-1}(m, n)$  to overpartitions in

$U_{k,i}(m, n)$ . We proceed to give a characterization of the images of this map, which leads to relation (2.18).

Let  $\lambda$  be an overpartition in  $U_{k,i-1}(m, n)$ . If there exists an overlined part of  $\lambda$  with the smallest underlying part, then we switch this overlined part to a non-overlined part, otherwise we choose a smallest non-overlined part and switch it to an overlined part. Let  $\lambda'$  denote the resulting overpartition. It can be checked that this map is an injection. It is not difficult to verify that  $\lambda' \in U_{k,i}(m, n)$ . Hence the number  $D_{k,i}(m, n) - D_{k,i-1}(m, n)$  can be interpreted as the number of overpartitions in  $U_{k,i}(m, n)$  which cannot be obtained by using the above map.

By the construction of the map  $\varphi$ , we may generate all the overpartitions in  $U_{k,i}(m, n)$  with no overlined part equal to 1 and all the overpartitions in  $U_{k,i}(m, n)$  with an overlined 1 and with at most  $i - 3$  non-overlined part 1. Therefore,  $D_{k,i}(m, n) - D_{k,i-1}(m, n)$  is exactly the number of overpartitions in  $U_{k,i}(m, n)$  with exactly one overlined part 1 such that the non-overlined part 1 appears either  $i - 1$  times or  $i - 2$  times. This completes the proof.  $\blacksquare$

**Theorem 2.4** For  $k \geq 2$ ,  $k \geq i \geq 1$ , and  $m, n \geq 0$ , we have

$$|S_{k,i}(m, n)| = D_{k,k-i}(m - i, n - m). \quad (2.19)$$

*Proof.* We define a bijection  $\phi$  from  $S_{k,i}(m, n)$  to  $U_{k,k-i}(m - i, n - m)$  which implies (2.19). Let  $\lambda$  be an overpartition in  $S_{k,i}(m, n)$ , the map  $\phi$  is defined as follows.

Step 1. Remove all the  $i$  parts with underlying part 1.

Step 2. Subtract 1 from each part.

Clearly, the resulting overpartition  $\lambda'$  is an overpartition of  $n - m$  with  $m - i$  parts. Moreover, we claim that  $\lambda' \in U_{k,k-i}(m - i, n - m)$ .

We first show that  $f_1(\lambda') \leq k - i - 1$ . By the construction of  $\phi$ , it is easy to see that  $f_1(\lambda') = f_2(\lambda)$  and  $f_1(\lambda) = i - 1$ . From the condition (ii) in Theorem 2.1, that is,  $f_1(\lambda) + f_{\overline{1}}(\lambda) + f_2(\lambda) \leq k - 1$ , we find that  $f_2(\lambda) \leq k - 1 - i$ .

We still need to verify that if there is an integer  $l$  such that

$$f_l(\lambda') + f_{\overline{l}}(\lambda') + f_{l+1}(\lambda') = k - 1, \quad (2.20)$$

then we have

$$lf_l(\lambda') + lf_{\overline{l}}(\lambda') + (l + 1)f_{l+1}(\lambda') \equiv V_{\lambda'}(l) + k - i - 1 \pmod{2}. \quad (2.21)$$

By the construction of  $\phi$ , it is easily checked that (2.20) implies

$$f_{l+1}(\lambda) + f_{\overline{l+1}}(\lambda) + f_{l+2}(\lambda) = k - 1.$$

Since  $\lambda \in S_{k,i}(m, n)$ , we have

$$(l + 1)f_{l+1}(\lambda) + (l + 1)f_{\overline{l+1}}(\lambda) + (l + 2)f_{l+2}(\lambda) \equiv V_{\lambda}(l + 1) + i - 1 \pmod{2},$$

Clearly,  $f_t(\lambda') = f_{t+1}(\lambda)$  and  $f_{\bar{t}}(\lambda') = f_{\overline{t+1}}(\lambda)$  for any  $t \geq 1$ . Thus we deduce that

$$lf_i(\lambda') + lf_{\bar{i}}(\lambda') + (l+1)f_{l-1}(\lambda') \equiv V_\lambda(l+1) + i - 1 - (k-1) \pmod{2}.$$

Again, by the construction of  $\phi$ , we find  $V_\lambda(l+1) = V_{\lambda'}(l) + 1$ . So we arrive at relation (2.21), which implies  $\lambda' \in U_{k,k-i}(m-i, n-m)$ .

It is not difficult to verify that the above construction is reversible, that is, from any overpartition in  $U_{k,k-i}(m-i, n-m)$ , we can recover an overpartition in  $S_{k,i}(m, n)$ . This completes the proof.  $\blacksquare$

For example, let  $k = 5$  and  $i = 3$ , and let  $\lambda = (\overline{9}, 9, 9, 9, 8, 8, 7, 7, 7, 6, 6, \overline{5}, 5, 5, 4, 4, 4, \overline{3}, 3, 2, 2, \overline{1}, 1, 1)$  be an overpartition in  $S_{5,3}(24, 125)$ . Then we get  $\phi(\lambda) = (\overline{8}, 8, 8, 8, 7, 7, 6, 6, 6, 5, 5, \overline{4}, 4, 4, 3, 3, 3, \overline{2}, 2, 1, 1)$ , which is an overpartition in  $U_{5,2}(21, 101)$ .

**Theorem 2.5** For  $k \geq 2$ ,  $k \geq i \geq 1$ , and  $m, n \geq 0$ , we have

$$|T_{k,i}(m, n)| = D_{k,k-i}(m-i+1, n-m). \quad (2.22)$$

*Proof.* We proceed to construct a bijection  $\chi$  from  $T_{k,i}(m, n)$  to  $U_{k,k-i}(m-i+1, n-m)$ . Let  $\lambda$  be an overpartition in  $T_{k,i}(m, n)$ , the map  $\chi$  is defined as follows.

Step 1. Remove all  $i-1$  parts equal to 1.

Step 2. Subtract 1 from each part.

Clearly, the resulting overpartition  $\lambda'$  is an overpartition of  $n-m$  with  $m-i+1$  parts. We shall show that  $\lambda' \in U_{k,k-i}(m-i+1, n-m)$ .

We first verify that  $f_1(\lambda') \leq k-i-1$ . It is obvious that  $f_1(\lambda') = f_2(\lambda)$ . So it suffices to prove that  $f_2(\lambda) \leq k-i-1$ . Since  $\lambda \in T_{k,i}(m, n)$ , we have  $f_1(\lambda) = i-2$ ,  $f_{\overline{1}}(\lambda) = 1$  and

$$f_1(\lambda) + f_{\overline{1}}(\lambda) + f_2(\lambda) \leq k-1. \quad (2.23)$$

It follows that  $f_2(\lambda) \leq k-i$ .

It remains to show that the non-overlined part 2 cannot occur  $k-i$  times. Assume to the contrary that  $f_2(\lambda) = k-i$ . Then the equality in (2.23) holds, that is,

$$f_1(\lambda) + f_{\overline{1}}(\lambda) + f_2(\lambda) = k-1.$$

We proceed to derive a contradiction to the condition (iii) in Theorem 2.1. By the facts  $f_1(\lambda) = i-2$  and  $f_{\overline{1}}(\lambda) = 1$ , we find

$$1f_1(\lambda) + 1f_{\overline{1}}(\lambda) + 2f_2(\lambda) = 2k-i-1. \quad (2.24)$$

Since  $V_\lambda(1) = 1$ , from (2.24) it follows that

$$1f_1(\lambda) + 1f_{\overline{1}}(\lambda) + 2f_2(\lambda) \not\equiv V_\lambda(1) + i - 1 \pmod{2},$$

which is a contradiction to assumption that the non-overlined 2 occurs  $k - i$  times. Thus we reach the conclusion that the non-overlined part 2 occurs at most  $k - i - 1$  times in  $\lambda$ , or equivalently, the non-overlined part 1 occurs at most  $k - i - 1$  times in  $\lambda'$ .

Next, we check condition (ii) in Theorem 2.1 for  $\lambda'$ . For any  $l \geq 1$ , we see that

$$f_{l+1}(\lambda) = f_l(\lambda') \quad \text{and} \quad f_{\overline{l+1}}(\lambda) = f_{\overline{l}}(\lambda'). \quad (2.25)$$

From condition (ii) for  $\lambda$ , we get

$$f_l(\lambda') + f_{\overline{l}}(\lambda') + f_{l+1}(\lambda') \leq k - 1.$$

Finally, we verify the condition that if there is an integer  $l$  such that

$$f_l(\lambda') + f_{\overline{l}}(\lambda') + f_{l+1}(\lambda') = k - 1, \quad (2.26)$$

then we have

$$lf_l(\lambda') + lf_{\overline{l}}(\lambda') + (l+1)f_{l+1}(\lambda') \equiv V_{\lambda'}(l) + k - i - 1 \pmod{2}. \quad (2.27)$$

Notice that (2.26) implies

$$f_{l+1}(\lambda) + f_{\overline{l+1}}(\lambda) + f_{l+2}(\lambda) = k - 1. \quad (2.28)$$

Since  $\lambda \in T_{k,i}(m, n)$ , by condition (iii) for  $\lambda$ , we have

$$(l+1)f_{l+1}(\lambda) + (l+1)f_{\overline{l+1}}(\lambda) + (l+2)f_{l+2}(\lambda) \equiv V_{\lambda}(l+1) + i - 1 \pmod{2}. \quad (2.29)$$

Substituting (2.25) into (2.29), we obtain

$$lf_l(\lambda') + lf_{\overline{l}}(\lambda') + (l+1)f_{l+1}(\lambda') \equiv V_{\lambda}(l+1) + i - 1 - (k - 1) \pmod{2}. \quad (2.30)$$

Observing that  $V_{\lambda}(l+1) = V_{\lambda'}(l) + 1$ , (2.30) can be rewritten as (2.27). This leads to the conclusion that  $\lambda' \in U_{k,k-i}(m - i + 1, n - m)$ .

It is routine to verify that the above procedure is reversible, that is, from any overpartition in  $U_{k,k-i}(m - i + 1, n - m)$ , one can recover an overpartition in  $T_{k,i}(m, n)$ . This completes the proof.  $\blacksquare$

By relations (2.18), (2.19) and (2.22), we obtain a recurrence relation of  $D_{k,i}(m, n)$ .

**Theorem 2.6** *For  $k \geq 2$ ,  $k \geq i \geq 1$  and for  $m, n \geq 0$ , we have*

$$D_{k,i}(m, n) - D_{k,i-1}(m, n) = D_{k,k-i}(m - i, n - m) + D_{k,k-i}(m - i + 1, n - m). \quad (2.31)$$

By Theorem 2.2 and Theorem 2.6, we obtain a combinatorial interpretation of  $W_{k,i}(x; q)$  in terms of overpartitions.

**Theorem 2.7** For  $k \geq 2$ ,  $k \geq i \geq 1$ , we have

$$W_{k,i}(x; q) = \sum_{m,n \geq 0} D_{k,i}(m, n) x^m q^n. \quad (2.32)$$

*Proof.* For  $m, n \geq 0$ , and for  $k \geq 2$  and  $k \geq i \geq 1$ , let  $w_{k,i}(m, n)$  denote the coefficient of  $x^m q^n$  in  $W_{k,i}(x; q)$ , that is,

$$W_{k,i}(x; q) = \sum_{m,n \geq 0} w_{k,i}(m, n) x^m q^n. \quad (2.33)$$

We proceed to show that  $D_{k,i}(m, n)$  and  $w_{k,i}(m, n)$  satisfy the same recurrence relation with the same initial values.

Clearly, we have  $w_{k,i}(0, 0) = 1$  for  $k \geq 2$  and  $k \geq i \geq 1$ , and  $w_{k,0}(m, n) = 0$  for  $k \geq 2, m, n \geq 0$ . Moreover, we have  $w_{k,i}(m, n) = 0$  if  $m$  or  $n$  is zero but not both. By Theorem 2.2, we find that

$$w_{k,i}(m, n) - w_{k,i-1}(m, n) = w_{k,k-i}(m-i, n-m) + w_{k,k-i}(m-i+1, n-m), \quad (2.34)$$

which is the same recurrence relation as  $D_{k,i}(m, n)$  as given in Theorem 2.6.

It is clear that  $D_{k,i}(0, 0) = 1$  for  $k \geq 2$  and  $k \geq i \geq 1$ , and  $D_{k,0}(m, n) = 0$  for  $k \geq 2$  and  $m, n \geq 0$ . Moreover,  $D_{k,i}(m, n) = 0$  if  $m$  or  $n$  is zero but not both. Now, we see that  $D_{k,i}(m, n)$  and  $w_{k,i}(m, n)$  have the same recurrence relation and the same initial values. This completes the proof.  $\blacksquare$

We are now ready to finish the proof of Theorem 2.1.

*Proof of Theorem 2.1.* Setting  $x = 1$  in (2.32), we find that the generating function for  $D_{k,i}(n)$  equals  $W_{k,i}(1; q)$ . In other words,

$$\sum_{n \geq 0} D_{k,i}(n) q^n = \sum_{n=0}^{\infty} \frac{(-1)^n q^{(2k-1)\binom{n+1}{2} - in} (1 - q^{(2n+1)i}) (-q)_{\infty}}{(q)_n (q^{n+1})_{\infty}}. \quad (2.35)$$

The right hand side of (2.35) can be expressed as

$$\frac{(-q)_{\infty}}{(q)_{\infty}} \sum_{n=0}^{\infty} (-1)^n q^{(2k-1)\binom{n+1}{2} - in} + \frac{(-q)_{\infty}}{(q)_{\infty}} \sum_{n=0}^{\infty} (-1)^n q^{(2k-1)\binom{n+1}{2} + i(n+1)}. \quad (2.36)$$

By substituting  $n$  with  $-(n+1)$  in the second sum of (2.36), we get

$$\frac{(-q)_{\infty}}{(q)_{\infty}} \sum_{n=-\infty}^{\infty} (-1)^n q^{(2k-1)\binom{n+1}{2} - in}.$$

In view of Jacobi's triple product identity, we obtain

$$\sum_{n \geq 0} D_{k,i}(n) q^n = \frac{(q^i, q^{2k-1-i}, q^{2k-1}; q^{2k-1})_{\infty} (-q)_{\infty}}{(q)_{\infty}}. \quad (2.37)$$

By the definition of  $C_{k,i}(n)$ , it is easily seen that

$$\sum_{n=0}^{\infty} C_{k,i}(n)q^n = \frac{(q^i, q^{2k-1-i}, q^{2k-1}; q^{2k-1})_{\infty}(-q)_{\infty}}{(q)_{\infty}}. \quad (2.38)$$

Comparing (2.37) and (2.38) we deduce that  $C_{k,i}(n) = D_{k,i}(n)$  for  $k \geq 2$ ,  $k \geq i \geq 1$  and  $n \geq 0$ . This completes the proof.  $\blacksquare$

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## References

- [1] G.E. Andrews, An analytic proof of the Rogers-Ramanujan-Gordon identities, *Amer. J. Math.* 88 (1966) 844–846.
- [2] G.E. Andrews, On q-difference equations for certain well-poised basic hypergeometric series, *Quart. J. Math. Oxford Ser. (2)* 19 (1968) 433–447.
- [3] G.E. Andrews, *The Theory of Partitions*, Addison-Wesley Publishing Co., 1976.
- [4] D.M. Bressoud, A generalization of the Rogers-Ramanujan identities for all moduli, *J. Combin. Theory, Ser. A* 27 (1979) 64–68.
- [5] D.M. Bressoud, Analytic and combinatorial generalizations of the Rogers-Ramanujan identities, *Memoir Amer. Math. Soc.*, No. 227 (1980).
- [6] W.Y.C. Chen, D.D.M. Sang and D.Y.H. Shi, The Rogers-Ramanujan-Gordon theorem for overpartitions, *Proc. London Math. Soc.*, to appear.
- [7] S. Corteel and J. Lovejoy, Overpartitions, *Trans. Amer. Math. Soc.* 356 (4) (2004) 1623–1635.
- [8] S. Corteel, J. Lovejoy and O. Mallet, An extension to overpartitions of the Rogers-Ramanujan identities for even moduli, *J. Number Theory* 128 (2008) 1602–1621 .
- [9] S. Corteel and O. Mallet, Overpartitions, lattice paths and Rogers-Ramanujan identities, *J. Combin. Theory Ser. A.* 114 (8) (2007) 1407–1437.
- [10] B. Gordon, A combinatorial generalization of the Rogers–Ramanujan identities, *Amer. J. Math.* 83 (1961) 393–399.
- [11] J. Lovejoy, Gordon’s theorem for overpartitions, *J. Combin. Theory Ser. A.* 103 (2003) 393–401.

- [12] J. Lovejoy, Overpartition theorems of the Rogers-Ramanujan type, *J. London Math. Soc.* 69 (2004) 562–574 .
- [13] J. Lovejoy, Overpartition pairs, *Ann. Inst. Fourier* 56 (2006) 781–794.
- [14] J. Lovejoy, Partitions and overpartitions with attached parts, *Arch. Math. (Basel)* 88 (2007) 316–322.
- [15] S. Ramanujan and L.J. Rogers, Proof of certain identities in combinatory analysis, *Proc. Cambridge Phil. Soc.*, 19 (1919) 211–216.
- [16] L.J. Rogers, Second memoir on the expansion of certain infinite products, *Proc. London Math. Soc.* 25 (1894) 318–343.