Math. Z.
277 (2014), no. 3-4, 1017-1025

Combinatorial Proof of the Inversion Formula on the Kazhdan-Lusztig R-Polynomials

William Y.C. Chen[1], Neil J.Y. Fan[2], Alan J.X. Guo[3], Peter L. Guo[4]
Harry H.Y. Huang[5], Michael X.X. Zhong[6]

[1,3,4,5,6]Center for Combinatorics, LPMC-TJKLC
Nankai University, Tianjin 300071, P.R. China

[2]Department of Mathematics
Sichuan University, Chengdu, Sichuan 610064, P.R. China

emails: [1]chen@nankai.edu.cn, [2]fan@scu.edu.cn,
[3]aalen@mail.nankai.edu.cn
[4]lguo@nankai.edu.cn, [5]hhuang@cfc.nankai.edu.cn
[6]michaelzhong@mail.nankai.edu.cn

Abstract. In this paper, we present a combinatorial proof of the inversion formula on the Kazhdan-Lusztig R-polynomials. This problem was raised by Brenti. As a consequence, we obtain a combinatorial interpretation of the equi-distribution property due to Verma stating that any nontrivial interval of a Coxeter group in the Bruhat order has as many elements of even length as elements of odd length. The same argument leads to a combinatorial proof of an extension of Verma's equi-distribution to the parabolic quotients of a Coxeter group obtained by Deodhar. As another application, we derive a refinement of the inversion formula for the symmetric group by restricting the summation to permutations ending with a given element.

Keywords: Kazhdan-Lusztig R-polynomial, inversion formula, Bruhat order

AMS Classification: 05A19, 05E15, 20F55

1 Introduction

Let (W,S) be a Coxeter system. For $u,v \in W$, let $R_{u,v}(q)$ be the Kazhdan-Lusztig R-polynomial indexed by u and v. The following inversion formula was obtained by

Kazhdan and Lusztig [8]:

$$\sum_{u\leq w\leq v}(-1)^{\ell(w)-\ell(u)}R_{u,w}(q)R_{w,v}(q)=\delta_{u,v}, \tag{1.1}$$

where $\leq$ is the Bruhat order and ℓ is the length function, see also Humphreys [6]. The aim of this paper is to present a combinatorial interpretation of this formula. This problem was raised by Brenti [3].

To give a combinatorial proof of (1.1), we start with Dyer's combinatorial description of the R-polynomials in terms of increasing Bruhat paths [5]. Then we reformulate the inversion formula in terms of V-paths. For $u \leq w \leq v$, by a V-path from u to v with bottom w we mean a pair (Δ_1, Δ_2) of Bruhat paths such that Δ_1 is a decreasing path from u to w and Δ_2 is an increasing path from w to v. We construct an involution on V-paths. This leads to a combinatorial proof of (1.1).

We give two applications of the involution. First, we restrict the involution to V-paths from u to v with maximal length. This induces an involution on the interval $[u, v]$ with $u < v$, which leads to a combinatorial proof of the equi-distribtution property that any nontrivial interval $[u, v]$ has as many elements of even length as elements of odd length. This property was proved inductively by Verma [12], which was used to deduce the Möbius function of the Bruhat order. Other proofs of the Möbius function formula for the Bruhat order can be found in [2, 4, 9, 11]. Recently, Jones [7] found a combinatorial proof for the equi-distribution property by constructing an involution on the intervals of a Coxeter group W. When W is finite, Jones [7] showed that this involution agrees with the construction of Rietsch and Williams [10] in their study of discrete Morse theory and totally nonnegative flag varieties.

The idea that we have used to prove Verma's equi-distribution can also be applied to Deodhar's [4] extension to parabolic quotients. For $J \subseteq S$, let W_J be the parabolic subgroup of W generated by J, and let W^J be the quotient of W consisting of minimal representatives of the left cosets of W_J in W, that is,

$$W^J = \{w \in W \mid \ell(ws) > \ell(w) \;\text{ for any }\; s \in J\}.$$

The quotient W^J forms a subposet of W in the Bruhat order. For $u \leq v \in W^J$, let

$$[u, v]^J = [u, v] \cap W^J$$

and let

$$K_J(u, v) = \{w \in [u, v]^J \mid [w, v]^J = [w, v]\}.$$

When $u < v$, Deodhar [4] showed that $K_J(u, v)$ contains as many elements of even length as elements of odd length, from which the Möbius function of the Bruhat order on W^J can be easily deduced. When $J = \emptyset$, Deodhar's assertion reduces to Verma's euqi-distribution. The Möbius function on W^J was rederived by Björner and Wachs [2] with the aid of topological techniques, and by Stembridge [11] by an algebraic approach. We construct an involution on $K_J(u,v)$ that leads to a simple combinatorial interpretation of Deodhar's equi-distribution.

As a second application, we find a refinement of the inversion formula when W is the symmetric group S_n. For a permutation $w \in S_n$, we write $w = w(1)w(2)\cdots w(n)$, where $w(i)$ denotes the element in the i-th position. Let u and v be two permutations in S_n such that $u < v$ in the Bruhat order. For $1 \le k \le n$, let $[u,v]_k$ denote the set of permutations in the interval $[u,v]$ that end with k, that is,

$$[u,v]_k = \{w \in [u,v] \mid w(n) = k\}.$$

By using a variation of the involution, we show that the summation

$$\sum_{w \in [u,v]_k} (-1)^{\ell(w)-\ell(u)} R_{u,w}(q) R_{w,v}(q)$$

equals zero or a power of q up to a sign.

2 An involution on V-paths

Our combinatorial proof of the inversion formula is based on an equivalent formulation of (1.1) in terms of the $\widetilde{R}$-polynomials. Let (W,S) be a Coxeter system. For $u,v \in W$ with $u \le v$, the $\widetilde{R}$-polynomials $\widetilde{R}_{u,v}(q)$ were introduced by Dyer [5], which are connected to the R-polynomials via the following relation

$$R_{u,v}(q) = q^{\frac{\ell(v)-\ell(u)}{2}} \widetilde{R}_{u,v}(q^{\frac{1}{2}} - q^{-\frac{1}{2}}),$$

see also Björner and Brenti [1]. Thus the inversion formula (1.1) can be restated as

$$\sum_{u \le w \le v} (-1)^{\ell(w)-\ell(u)} \widetilde{R}_{u,w}(q) \widetilde{R}_{w,v}(q) = \delta_{u,v}. \tag{2.1}$$

To give a bijective proof of (2.1), we need a combinatorial interpretation of the $\widetilde{R}$-polynomials due to Dyer [5] in terms of increasing Bruhat paths of a Coxeter group. For a Coxeter system (W,S), let

$$T = \{wsw^{-1} \mid s \in S,\ w \in W\}$$

be the set of reflections. The Bruhat graph $BG(W)$ of W is a directed graph whose nodes are the elements of W such that there is an arc from u to v if $v = ut$ for some $t \in T$ and $\ell(u) < \ell(v)$. We use $u \xrightarrow{t} v$ to denote the arc from u to v, where t is the reflection such that $v = ut$. An increasing path in the Bruhat graph is defined based on the reflection ordering on the positive roots of W. Let Φ be the root system of W, and Φ^+ be the positive root system. A total ordering $\prec$ on Φ^+ is called a reflection ordering if for any $\alpha \prec \beta \in \Phi^+$ and two nonnegative real numbers λ, μ such that

$\lambda\alpha+\mu\beta \in \Phi^+$, then we have $\alpha \prec \lambda\alpha+\mu\beta \prec \beta$. Since positive roots in Φ^+ are in one-to-one correspondence with reflections, a reflection ordering induces a total ordering on the reflection set T.

Let $\Delta = u_0 \xrightarrow{t_1} u_1 \xrightarrow{t_2} \cdots \xrightarrow{t_r} u_r$ be a path from u to v, where $u_0 = u$ and $u_r = v$. We say that Δ is increasing if $t_1 \prec t_2 \prec \cdots \prec t_r$, and Δ is decreasing if $t_1 \succ t_2 \succ \cdots \succ t_r$. Let $\ell(\Delta)$ denote the length of Δ, that is, the number of arcs in Δ. Dyer [5] showed that for any fixed reflection ordering $\prec$ on T, we have

$$\widetilde{R}_{u,v}(q) = \sum_{\Delta} q^{\ell(\Delta)}, \tag{2.2}$$

where the sum ranges over increasing Bruhat paths from u to v with respect to $\prec$, see also Björner and Brenti [1]. By definition, the reverse of a reflection ordering is also a reflection ordering. So (2.2) can be restated as

$$\widetilde{R}_{u,v}(q) = \sum_{\Delta'} q^{\ell(\Delta')},$$

where the sum ranges over decreasing Bruhat paths from u to v with respect to $\prec$.

By a V-path from u to v with bottom w, we mean a pair (Δ_1, Δ_2) of Bruhat paths such that Δ_1 is a decreasing path from u to w and Δ_2 is an increasing path from w to v. The sign of a V-path (Δ_1, Δ_2) is defined as

$$\mathrm{sgn}(\Delta_1, \Delta_2) = (-1)^{\ell(\Delta_1)}.$$

The length of a Bruhat path from u to w has the same parity as $\ell(w) - \ell(u)$, see, e.g., Björner and Brenti [1]. It follows that

$$\mathrm{sgn}(\Delta_1, \Delta_2) = (-1)^{\ell(w)-\ell(u)},$$

and so (2.1) can be rewritten as

$$\sum_{u \le w \le v} (-1)^{\ell(w)-\ell(u)} \widetilde{R}_{u,w}(q) \widetilde{R}_{w,v}(q) = \sum_{(\Delta_1,\Delta_2)} \mathrm{sgn}(\Delta_1, \Delta_2) q^{\ell(\Delta_1)+\ell(\Delta_2)} = \delta_{u,v}, \tag{2.3}$$

where the second sum ranges over V-paths from u to v.

We now define an involution Φ on V-paths, which preserves the length, but reverses the sign of a V-path. This leads to a combinatorial proof of (2.3).

An Involution Φ on V-Paths: For $u < v$, let (Δ_1, Δ_2) be a V-path from u to v with bottom w. Write

$$\Delta_1 = u_0 \xrightarrow{t_1} u_1 \xrightarrow{t_2} \cdots \xrightarrow{t_i} u_i \quad \text{and} \quad \Delta_2 = v_0 \xrightarrow{t'_1} v_1 \xrightarrow{t'_2} \cdots \xrightarrow{t'_j} v_j,$$

where $u_0 = u$, $u_i = v_0 = w$ and $v_j = v$. The V-path $\Phi(\Delta_1, \Delta_2) = (\Delta'_1, \Delta'_2)$ is constructed according to the following two cases.

Case 1: $u = w$ or $t_i \succ t'_1$. Set

$$\Delta'_1 = u_0 \xrightarrow{t_1} u_1 \xrightarrow{t_2} \cdots \xrightarrow{t_i} u_i \xrightarrow{t'_1} v_1 \quad \text{and} \quad \Delta'_2 = v_1 \xrightarrow{t'_2} \cdots \xrightarrow{t'_j} v_j.$$

Case 2: $v = w$ or $t_i \prec t'_1$. Set

$$\Delta'_1 = u_0 \xrightarrow{t_i} u_1 \xrightarrow{t_2} \cdots \xrightarrow{t_{i-1}} u_{i-1} \quad \text{and} \quad \Delta'_2 = u_{i-1} \xrightarrow{t_i} v_0 \xrightarrow{t'_1} v_1 \xrightarrow{t'_2} \cdots \xrightarrow{t'_j} v_j.$$

It turns out that the involution Φ yields a simple combinatorial interpretation of the following parity property of Verma [12].

Theorem 2.1 (Verma [12]) *Let (W, S) be a Coxeter system and $u < v \in W$. Then the interval $[u, v]$ has the same number of elements of odd length as elements of even length.*

Indeed, for $u < v \in W$, there exists a unique maximal increasing (or, decreasing) Bruhat path from u to v [5]. Thus, for any $w \in [u, v]$ there is a unique maximal V-path from u to v with bottom w. So the maximal V-paths from u to v are in one-to-one correspondence with elements in the interval $[u, v]$. Restricting the involution Φ to the maximal V-paths from u to v induces an involution on the interval $[u, v]$, which reverses the parity of the length of each element in $[u, v]$. This proves Theorem 2.1.

The above argument also serves as a combinatorial interpretation of the following equi-distribution due to Deodhar [4]. Let us recall the common notation as mentioned in Introduction. For $J \subseteq S$, let

$$W^J = \{w \in W \mid \ell(ws) > \ell(w) \ \text{ for any } \ s \in J\}.$$

For $u \leq v \in W^J$, let

$$[u, v]^J = [u, v] \cap W^J$$

and let

$$K_J(u, v) = \{w \in [u, v]^J \mid [w, v]^J = [w, v]\}.$$

Theorem 2.2 (Deodhar [4]) *Let (W, S) be a Coxeter system, and $J \subseteq S$. Then, for $u < v \in W$, the set $K_J(u, v)$ has the same number of elements of odd length as elements of even length.*

To construct an involution on $K_J(u, v)$, we recall a labeling on the edges of the poset $[u, v]^J$ introduced by Björner and Wachs [2], see also Björner and Brenti [1]. Let $v = s_1 s_2 \cdots s_q$ be a given reduced expression of v. We read a maximal chain in $[u, v]^J$ from top to bottom. Let $v = w_0 \to w_1 \to \cdots \to w_r = u$ be a maximal chain in $[u, v]^J$, where $r = \ell(v) - \ell(u)$. Then there is a unique sequence $(i_1, i_2, \ldots, i_r)$ of distinct integers such that for $1 \leq k \leq r$, w_k has a reduced expression obtained from $s_1 s_2 \cdots s_q$ by deleting simple reflections indexed by $i_1, i_2, \ldots, i_k$. Label the edge from w_{k-1} to w_k by i_k. We denote the maximal chain with such a labeling by $v = w_0 \xrightarrow{i_1} w_1 \xrightarrow{i_2} \cdots \xrightarrow{i_r} w_r = u$, and say that the chain $v = w_0 \xrightarrow{i_1} w_1 \xrightarrow{i_2} \cdots \xrightarrow{i_r} w_r = u$ is increasing if $i_1 < i_2 < \cdots < i_r$, and it is decreasing if $i_1 > i_2 > \cdots > i_r$. The following theorem is due to Björner and Wachs [2], see also Björner and Brenti [1].

Theorem 2.3 (Björner and Wachs [2]) *Let $u < v \in W^J$, and let $v = s_1 s_2 \cdots s_q$ be a given reduced expression of v. Then there is a unique increasing maximal chain from v to u in $[u, v]^J$.*

We remark that when $J = \emptyset$, the proof of Theorem 2.3 can be employed to show that for any given reduced expression of v, there is a unique decreasing maximal chain from v to u in $[u, v]^{\emptyset} = [u, v]$.

We are now ready to present an involution Ψ on $K_J(u, v)$, which reverses the parity of the length. This leads to a combinatorial proof of Theorem 2.2.

An Involution Ψ on $K_J(u, v)$: Let $w \in K_J(u, v)$, and let $v = s_1 s_2 \cdots s_q$ be a fixed reduced expression of v. Since $[w, v]^J = [w, v]$, by the above remark, there exists a unique decreasing maximal chain $v = v_0 \xrightarrow{i_1} v_1 \xrightarrow{i_2} \cdots \xrightarrow{i_m} v_m = w$ from v to w in $[u, v]^J$. Let $w = s_{k_1} s_{k_2} \cdots s_{k_p}$ be the reduced expression of w obtained from $s_1 s_2 \cdots s_q$ by deleting the generators indexed by $i_1, i_2, \ldots, i_m$, that is, $1 \leq k_1 < k_2 < \cdots < k_p \leq q$ and $\{k_1, k_2, \ldots, k_p\} = \{1, 2, \ldots, q\} \backslash \{i_1, i_2, \ldots, i_m\}$. Assume that $w = w_0 \xrightarrow{k_{j_1}} w_1 \xrightarrow{k_{j_2}} \cdots \xrightarrow{k_{j_t}} w_t = u$ is the unique increasing maximal chain in $[u, w]^J$ with respect to the reduced expression $w = s_{k_1} s_{k_2} \cdots s_{k_p}$. Note that $1 \leq j_1 < \cdots < j_t \leq p$. Then $\Psi(w)$ is defined according to the following two cases:

Case 1: $u = w$ or $i_m < k_{j_1}$. Set $\Psi(w) = v_{m-1}$;

Case 2: $v = w$ or $i_m > k_{j_1}$. Set $\Psi(w) = w_1$.

The following theorem shows that Ψ is an involution on $K_J(u, v)$. The proof relies on the following properties of the Bruhat order, see, for example, Björner and Brenti [1].

The Subword Property: Let $u, v \in W$. Then $u \leq v$ in the Bruhat order if and only if every reduced expression of v has a subword that is a reduced expression of u.

The Lifting Property: Suppose that $u < v \in W$, and $s \in S$ is a simple reflection. If $\ell(sv) < \ell(v)$ and $\ell(su) > \ell(u)$, then $u \leq sv$ and $su \leq v$. Similarly, if $\ell(vs) < \ell(v)$ and $\ell(us) > \ell(u)$, then $u \leq vs$ and $us \leq v$.

Theorem 2.4 *The map Ψ is an involution on $K_J(u, v)$.*

Proof. By the construction of Ψ, it suffices to show that for $w \in K_J(u, v)$, $\Psi(w)$ also belongs to $K_J(u, v)$. This is trivial when $u = w$ or $i_m < k_{j_1}$. Now we consider the case when $v = w$ or $i_m > k_{j_1}$. Let $w' = \Psi(w)$. Assume that $w = s_1 \cdots \hat{s}_{i_m} \cdots \hat{s}_{i_2} \cdots \hat{s}_{i_1} \cdots s_q$ and $w' = s_1 \cdots \hat{s}_{k_{j_1}} \cdots \hat{s}_{i_m} \cdots \hat{s}_{i_1} \cdots s_q$, where for a simple reflection $s \in S$, $\hat{s}$ means that s is missing. We aim to prove that $w' \in K_J(u, v)$.

Suppose to the contrary that $w' \notin K_J(u, v)$. Then there exists an element $w'' \in [w', v]$ such that $w'' \notin [w', v]^J$. By definition, there exists $s \in J$ such that $\ell(w''s) < \ell(w'')$. Since $\ell(w's) > \ell(w')$, the lifting property implies that $w's \leq w''$. Thus we have $w's \leq v$. Since $\ell(vs) > \ell(v)$, we see that $w's \neq v$. It follows that $w's < v$, that is,

$$s_1 \cdots \hat{s}_{k_{j_1}} \cdots \hat{s}_{i_m} \cdots \hat{s}_{i_1} \cdots s_q s < s_1 s_2 \cdots s_q.$$

It is easily checked that $\hat{s}_{k_{j_1}} \cdots \hat{s}_{i_m} \cdots \hat{s}_{i_1} \cdots s_q s < s_{k_{j_1}} \cdots s_q$. By the lifting property, we deduce that $s_{k_{j_1}} \cdots \hat{s}_{i_m} \cdots \hat{s}_{i_1} \cdots s_q s \leq s_{k_{j_1}} \cdots s_q$. Thus we have

$$ws = s_1 \cdots \hat{s}_{i_m} \cdots \hat{s}_{i_1} \cdots s_q s \leq s_1 \cdots s_q = v,$$

which implies that $ws \in [w, v]$. On the other hand, it is obvious that $ws \notin [w, v]^J$. So we conclude that $w \notin K_J(u, v)$, contradicting the assumption that $w \in K_J(u, v)$. This completes the proof. ∎

From the proof of Theorem 2.4, we see that for $w \in [u, v]^J$, $w \in K_J(u, v)$ if and only if there does not exist any $s \in J$ such that $ws \in [u, v]$. Notice that this characterization has been observed by Deodhar [4, Lemma 3].

3 A refinement of the inversion formula for S_n

In this section, we use a variation of the involution Φ to give a refinement of the inversion formula for the symmetric group S_n. We introduce the notion of an S-interval. Let u, v be two permutations in S_n with $u < v$. Let

$$D(u, v) = \{1 \leq i \leq n \,|\, u(i) \neq v(i)\}.$$

Suppose that $D(u, v) = \{i_1, i_2, \ldots, i_j\}_<$, that is, $D(u, v) = \{i_1, i_2, \ldots, i_j\}$ and $i_1 < i_2 < \cdots < i_j$. Let $b_1 < b_2 < \cdots < b_j$ be the values of $u(i_1), u(i_2), \ldots, u(i_j)$ listed in increasing order. We say that $[u, v]$ is an S-interval if it satisfies the following conditions:

(1) $i_j = n$ and $u(i_j) = b_j$;

(2) The values in $\{b_1, b_2, \ldots, b_j\}$ that are greater than $u(i_1)$ appear in increasing order in u, whereas the values in $\{b_1, b_2, \ldots, b_j\}$ that are less than $u(i_1)$ appear in decreasing order in u;

(3) In the cycle notation, $v = (b_1, b_2, \ldots, b_j)\, u$, that is, v is obtained from u by rotating the elements $b_1, b_2, \ldots, b_j$ in u.

Recall that for $u < v \in S_n$, $[u,v]_k$ denotes the set of permutations in $[u,v]$ that end with k. The following theorem gives a refinement of the inversion formula for S_n.

Theorem 3.1 *Assume that $u < v \in S_n$. Let m be the smallest index such that $u(m) \neq v(m)$. If $[u,v]$ is an S-interval, and $k = u(m)$ or $k = v(m)$, then we have*

$$\sum_{w \in [u,v]_k} (-1)^{\ell(w)-\ell(u)} \widetilde{R}_{u,w}(q) \widetilde{R}_{w,v}(q) = (-1)^r q^{s-1},$$

where $s = |D(u,v)|$ and

$$r = |\{j \in D(u,v) \mid u(j) > k\}|;$$

Otherwise, we have

$$\sum_{w \in [u,v]_k} (-1)^{\ell(w)-\ell(u)} \widetilde{R}_{u,w}(q) \widetilde{R}_{w,v}(q) = 0.$$

For $1 \leq k \leq n$, let $P_k(u,v)$ denote the set of V-paths from u to v with bottoms contained in $[u,v]_k$. To prove Theorem 3.1, we shall construct an involution Ω on $P_k(u,v)$. The reflection set T of S_n consists of transpositions of S_n, that is,

$$T = \{(i,j) \mid 1 \leq i < j \leq n\}.$$

For two permutations u, v in S_n, it is known that there is an arc from u to v in the Bruhat graph of S_n if $v = u(i,j)$ and $u(i) < u(j)$, see Björner and Brenti [1].

From now on, we choose the reflection ordering $\prec$ on T to be the lexicographic ordering:

$$(1,2) \prec (1,3) \prec \cdots \prec (1,n) \prec (2,3) \prec \cdots \prec (n-1,n). \tag{3.1}$$

For a Bruhat path $\Delta = u_0 \xrightarrow{t_1} u_1 \xrightarrow{t_2} \cdots \xrightarrow{t_r} u_r$, let

$$L(\Delta) = (t_1, t_2, \ldots, t_r).$$

An Involution Ω on $P_k(u,v)$: Let (Δ_1, Δ_2) be a V-path in $P_k(u,v)$ with bottom w. Write $\Delta_1 = u_0 \xrightarrow{t_1} u_1 \xrightarrow{t_2} \cdots \xrightarrow{t_i} u_i$ and $\Delta_2 = v_0 \xrightarrow{t'_1} v_1 \xrightarrow{t'_2} \cdots \xrightarrow{t'_j} v_j$, where $u_0 = u$, $u_i = v_0 = w$ and $v_j = v$. Let $t = \min\{t_i, t'_1\}$. Then the V-path $\Omega(\Delta_1, \Delta_2) = (\Delta'_1, \Delta'_2)$ is defined as follows. We consider three cases.

Case 1: t is an internal transposition, that is, $t = (a,b)$ and $1 \leq a < b < n$. In this case, set $(\Delta'_1, \Delta'_2) = \Phi(\Delta_1, \Delta_2)$.

Case 2: t is a boundary transposition, that is, $t = (a,n)$ for some $a < n$, and there is an internal transposition among the transpositions $t_1, \ldots, t_i, t'_1, \ldots, t'_j$. Let $\widetilde{t}$ be the smallest internal transposition among $t_1, \ldots, t_i, t'_1, \ldots, t'_j$. By the choice of the reflection ordering in (3.1), it is easy to check that $\widetilde{t}$ belongs to either $\{t_1, \ldots, t_i\}$ or $\{t'_1, \ldots, t'_j\}$, but not both. So we have the following two subcases.

Subcase 1: $\widetilde{t}$ belongs to $\{t_1, \ldots, t_i\}$. Assume that $t_{i_0} = \widetilde{t}$, where $1 \le i_0 \le i$. Let Δ'_1 be the path such that $L(\Delta'_1)$ is the sequence obtained from $L(\Delta_1)$ by deleting t_{i_0}, and let Δ'_2 be the path such that $L(\Delta'_2)$ is the sequence obtained from $L(\Delta_2)$ by inserting t_{i_0} such that $L(\Delta'_2)$ remains increasing.

Subcase 2: $\widetilde{t}$ belongs to $\{t'_1, \ldots, t'_j\}$. Assume that $t'_{j_0} = \widetilde{t}$, where $1 \le j_0 \le j$. Let Δ'_2 be the path such that $L(\Delta'_2)$ is the sequence obtained from $L(\Delta_2)$ by deleting t'_{j_0}, and let Δ'_1 be the path such that $L(\Delta'_1)$ is the sequence obtained from $L(\Delta_1)$ by inserting t_{j_0} such that $L(\Delta'_1)$ remains decreasing.

Case 3: The transpositions $t_1, \ldots, t_i, t'_1, \ldots, t'_j$ are all boundary transpositions. In this case, set $(\Delta'_1, \Delta'_2) = (\Delta_1, \Delta_2)$.

It is easy to verify that Ω is a length preserving involution on $P_k(u, v)$, and it is clear that Ω reverses the sign of (Δ_1, Δ_2) unless (Δ_1, Δ_2) is a fixed point. To prove Theorem 3.1, we also need the following property.

Proposition 3.2 *Assume that $u < v \in S_n$ and $1 \le k \le n$. Then the involution Ω on $P_k(u, v)$ has at most one fixed point. Moreover, Ω has a fixed point if and only if $[u, v]$ is an S-interval and $k = u(m)$ or $k = v(m)$, where m is the smallest integer such that $u(m) \neq v(m)$.*

Proof. To prove that Ω has at most one fixed point, assume that $(\Delta_1, \Delta_2) \in P_k(u, v)$ is a V-path that is fixed by Ω. We proceed to show that (Δ_1, Δ_2) is uniquely determined. Let $\Delta_1 = u_0 \xrightarrow{t_1} u_1 \xrightarrow{t_2} \cdots \xrightarrow{t_i} u_i$ and $\Delta_2 = v_0 \xrightarrow{t'_1} v_1 \xrightarrow{t'_2} \cdots \xrightarrow{t'_j} v_j$. By the construction of Ω, we see that $t_1, \ldots, t_i$ and $t'_1, \ldots, t'_j$ are all boundary transpositions. Assume that $t_1 = (p_1, n), \ldots, t_i = (p_i, n)$ and $t'_1 = (p'_1, n), \ldots, t'_j = (p'_j, n)$. Since Δ_1 and Δ_2 are Bruhat paths, we see that

$$u(n) > u(p_1) > \cdots > u(p_i) = k = w(n) > w(p'_1) > \cdots > w(p'_j). \tag{3.2}$$

Noting that $t_1 \succ t_2 \succ \cdots \succ t_i$ and $t'_1 \prec t'_2 \prec \cdots \prec t'_j$, we find that $n > p_1 > \cdots > p_i$ and $p'_1 < \cdots < p'_j < n$.

By (3.2) together with the relation $w = u\,(p_1, n) \cdots (p_i, n)$, it is easily seen that

$$\{p_1, \ldots, p_i\} \cap \{p'_1, \ldots, p'_j\} = \emptyset.$$

This yields that $w(p'_1) = u(p'_1), \ldots, w(p'_j) = u(p'_j)$, and so (3.2) becomes

$$u(n) > u(p_1) > \cdots > u(p_i) = k = w(n) > u(p'_1) > \cdots > u(p'_j). \tag{3.3}$$

Observe that

$$\{p_1, \ldots, p_i\} \cup \{p'_1, \ldots, p'_j\} \cup \{n\} = D(u, v).$$

In view of (3.3), we deduce that given u, v and k, the values of i, j as well as the elements $p_1, \ldots, p_i, p'_1, \ldots, p'_j$ are uniquely determined. In other words, the V-path (Δ_1, Δ_2) is uniquely determined.

It remains to prove that Ω has a fixed point if and only if $[u, v]$ is an S-interval and $k = u(m)$ or $k = v(m)$. By the above argument, we see that if Ω has a fixed point, then $[u, v]$ is an S-interval and $k = u(p_i) = v(p'_1)$. Since $m = \min\{p_i, p'_1\}$, we obtain that $k = u(m)$ if $p_i < p'_1$ and $k = v(m)$ if $p_i > p'_1$. Conversely, if $[u, v]$ is an S-interval, it is easy to construct a V-path in $P_k(u, v)$ fixed by Ω, where $k = u(m)$ or $k = v(m)$. This completes the proof. ∎

We are now ready to complete the proof of Theorem 3.1.

Proof of Theorem 3.1. By Proposition 3.2, we only need to consider the case when $[u, v]$ is an S-interval and $k = u(m)$ or $k = v(m)$. In this case, we have

$$\sum_{w \in [u,v]_k} (-1)^{\ell(w)-\ell(u)} \widetilde{R}_{u,w}(q) \widetilde{R}_{w,v}(q) = (-1)^{\ell(\Delta_1)} q^{\ell(\Delta_1)+\ell(\Delta_2)},$$

where (Δ_1, Δ_2) is the unique V-path in $P_k(u, v)$ that is fixed by Ω. Evidently,

$$\ell(\Delta_1) + \ell(\Delta_2) = |D(u, v)| - 1.$$

It is also clear that

$$\ell(\Delta_1) = |\{j \in D(u, v) \mid u(j) > k\}|.$$

Hence the proof is complete. ∎

Acknowledgments. We wish to thank the referee for valuable suggestions. This work was supported by the 973 Project and the National Science Foundation of China.

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