Proc. Edinb. Math. Soc.
56 (2013), no. 3, 701-722

2-Log-concavity of the Boros-Moll Polynomials

William Y. C. Chen[1] and Ernest X. W. Xia[2]

[1]Center for Combinatorics, LPMC-TJKLC
Nankai University, Tianjin 300071, P. R. China

[2]Department of Mathematics
Jiangsu University, Jiangsu, Zhenjiang 212013, P. R. China

emails: [1]chen@nankai.edu.cn, [2]ernestxwxia@163.com

Abstract. The Boros-Moll polynomials $P_m(a)$ arise in the evaluation of a quartic integral. It has been conjectured by Boros and Moll that these polynomials are infinitely log-concave. In this paper, we show that $P_m(a)$ is 2-log-concave for any $m \geq 2$. Let $d_i(m)$ be the coefficient of a^i in $P_m(a)$. We also show that the sequence $\{i(i+1)(d_i^2(m)-d_{i-1}(m)d_{i+1}(m))\}_{1\leq i\leq m}$ is log-concave.

AMS Subject Classification: 05A10, 05A20; 33F10

Keywords: 2-log-concavity, Boros-Moll polynomial, quartic integral

1 Introduction

The objective of this paper is to prove the 2-log-concavity of the Boros-Moll polynomials. Recall that a sequence $\{a_i\}_{0\leq i\leq n}$ of real numbers is said to be unimodal if there exists an index $0 \leq j \leq n$ such that

$$a_0 \leq a_1 \leq \cdots \leq a_{j-1} \leq a_j \geq a_{j+1} \geq \cdots \geq a_n.$$

Set $a_{-1} = 0$ and $a_{n+1} = 0$. We say that $\{a_i\}_{0\leq i\leq n}$ is log-concave if

$$a_i^2 - a_{i+1}a_{i-1} \geq 0, \qquad 1 \leq i \leq n.$$

A polynomial is said to be unimodal (resp., log-concave) if the sequence of its coefficients is unimodal (resp., log-concave). It is easy to see that for a positive sequence, the log-concavity is stronger than the unimodality. For a sequence $A = \{a_i\}_{0 \le i \le n}$, define the operator $\mathcal{L}$ by $\mathcal{L}(A) = \{b_i\}_{0 \le i \le n}$, where

$$b_i = a_i^2 - a_{i-1}a_{i+1}, \quad 0 \le i \le n. \tag{1.1}$$

Boros and Moll [8] introduced the notion of infinite log-concavity. We say that the sequence $\{a_i\}_{0 \le i \le n}$ is k-log-concave if the sequence $\mathcal{L}^j\left(\{a_i\}_{0 \le i \le n}\right)$ is log-concave for every $0 \le j \le k-1$, and we say that $\{a_i\}_{0 \le i \le n}$ is ∞-log-concave if $\mathcal{L}^k\left(\{a_i\}_{0 \le i \le n}\right)$ is log-concave for any $k \ge 0$.

Boros and Moll [8] conjectured that the binomial coefficients $\binom{n}{k}$ are infinitely log-concave for any n. An generalization of this conjecture was given independently by Fisk [20], McNamara and Sagan [23], and Stanley, see [9], which states that if a polynomial $a_0+a_1x+\cdots+a_nx^n$ has only real zeros, then the polynomial $b_0+b_1x+\cdots+b_nx^n$ also has only real zeros, where $b_i = a_i^2 - a_{i-1}a_{i+1}$. This conjecture has been proved by Brändén [9]. While Brändén's theorem does not directly apply to the Boros-Moll polynomials, the 2-log-concavity and 3-log-concavity can be recasted in terms of the real rootedness of certain polynomials derived from the Boros-Moll polynomials, as conjectured by Brändén. It is worth mentioning that McNamara and Sagan [23] conjectured that for fixed k, the q-Gaussian coefficients $\begin{bmatrix} n \\ k \end{bmatrix}$ are infinitely q-log-concave. Chen, Wang and Yang [15] proved the strong q-log-concavity of the q-Narayana numbers $N_q(n,k)$ for fixed k, which turns out to be equivalent to the 2-fold q-log-concavity of the Gaussian coefficients.

Recall that Boros and Moll [4–8, 24] have studied the following quartic integral and have shown that for any $a > -1$ and any nonnegative integer m,

$$\int_0^\infty \frac{1}{(x^4+2ax^2+1)^{m+1}}dx = \frac{\pi}{2^{m+3/2}(a+1)^{m+1/2}}P_m(a),$$

where

$$P_m(a) = \sum_{j,k} \binom{2m+1}{2j}\binom{m-j}{k}\binom{2k+2j}{k+j}\frac{(a+1)^j(a-1)^k}{2^{3(k+j)}}. \tag{1.2}$$

Using Ramanujan's Master Theorem, Boros and Moll [7, 24] obtained the following formula for $P_m(a)$:

$$P_m(a) = 2^{-2m}\sum_k 2^k \binom{2m-2k}{m-k}\binom{m+k}{k}(a+1)^k, \tag{1.3}$$

which implies that $P_m(a)$ is a polynomial in a with positive coefficients. Chen, Pang and Qu [12] gave a combinatorial argument to show that the double sum (1.2) can be reduced to the single sum (1.3). Let $d_i(m)$ be the coefficient of a^i of $P_m(a)$, that is,

$$P_m(a) = \sum_{i=0}^{m} d_i(m)a^i. \tag{1.4}$$

For any m, $P_m(a)$ is called a Boros-Moll polynomial, and the sequence $\{d_i(m)\}_{0\leq i\leq m}$ is called a Boros-Moll sequence. From (1.3), we know that $d_i(m)$ can be expressed as

$$d_i(m) = 2^{-2m} \sum_{k=i}^{m} 2^k \binom{2m-2k}{m-k}\binom{m+k}{k}\binom{k}{i}. \tag{1.5}$$

Several proofs of the above formula can be found in the survey of Amdeberhan and Moll [2,3].

Boros and Moll [5] proved that the sequence $\{d_i(m)\}_{0\leq i\leq m}$ is unimodal and the maximum element appears in the middle. In other words,

$$d_0(m) < d_1(m) < \cdots < d_{[\frac{m}{2}]-1}(m) < d_{[\frac{m}{2}]}(m) > d_{[\frac{m}{2}]+1}(m) > \cdots > d_m(m).$$

They also established the unimodality by a different approach [6]. Moll [24] conjectured that the sequence $\{d_i(m)\}_{0\leq i\leq m}$ is log-concave. Kauers and Paule [21] proved this conjecture based on recurrence relations which were derived by using a computer algebra approach. Chen, Pang and Qu [13] found a combinatorial proof of the log-concavity of $P_m(a)$ by introducing the structure of partially 2-colored permutations. Chen and Gu [11] proved the reverse ultra log-concavity of the sequence $\{d_i(m)\}_{0\leq i\leq m}$. Amdeberhan, Manna and Moll [1] studied the 2-adic valuation of an integer sequence and obtained a combinatorial interpretation of the valuations of the integer sequence which is related to the Boros-Moll sequences. Chen and Xia [16] showed that the sequence $\{d_i(m)\}_{0\leq i\leq m}$ satisfies the strongly ratio monotone property which implies the log-concavity and the spiral property. Furthermore, Chen, Yang and Zhou [18] proved that if $f(x)$ is a polynomial with nondecreasing and nonnegative coefficients, then $f(1+x)$ is ratio monotone. From (1.3), it is easily seen that the coefficients of $P_n(x-1)$ are nondecreasing and nonnegative. Hence the polynomials $P_n(x)$ are log-concave and ratio monotone. Recently, Chen, Wang and Xia [14] introduced the notion of interlacing log-concavity and proved that the Boros-Moll polynomials possess this property.

Boros and Moll [8] made the following conjecture.

Conjecture 1.1. *The Boros-Moll sequence $\{d_i(m)\}_{0\leq i\leq m}$ is ∞-log-concave.*

As shown by Boros and Moll [5], in general, $P_m(a)$ are not polynomials with only real zeros. Thus the theorem of Brändén [9] does not apply to $P_m(a)$. Nevertheless, Brändén

[9] made the following conjectures on the real rootedness of polynomials derived from $P_m(a)$. These conjectures imply the 2-log-concavity and the 3-log-concavity of the Boros-Moll polynomials.

Conjecture 1.2 (Brändén)**.** *For each positive integer m, the polynomial*

$$Q_m(x) = \sum_{i=0}^{m} \frac{d_i(m)}{i!} x^i$$

has only real zeros.

Conjecture 1.3 (Brändén)**.** *For each positive integer m, the polynomial*

$$R_m(x) = \sum_{i=0}^{m} \frac{d_i(m)}{(i+2)!} x^i$$

has only real zeros.

Note that $Q_m(x) = \frac{d}{dx^2}(x^2 R_m(x))$. Hence $Q_m(x)$ has only real zeros if $R_m(x)$ does. This yields that Conjecture 1.3 is stronger than Conjecture 1.2. Based on a result of of Craven and Csordas [19], it can be seen that Conjecture 1.2 implies that $P_m(a)$ is 2-log-concave and Conjecture 1.3 implies that $P_m(a)$ is 3-log-concave. After this work was done, Chen, Dou and Yang [10] proved Conjectures 1.2 and 1.3 by showing that both $Q_n(x)$ and $R_n(x)$ form Sturm sequences.

In another direction, Kauers and Paule [21] considered using the approach of recurrence relations to prove the 2-log-concavity of $P_m(a)$, and they indicated that there is little hope to make it work since the recurrence relations are too complicated.

Roughly speaking, the main idea of this paper is to find an intermediate function $f(m,i)$ so that we can reduce quartic inequalities for the 2-log-concavity to quadratic inequalities. To be precise, the 2-log-concavity is stated as follows.

Theorem 1.4. *The Boros-Moll sequences are 2-log-concave, that is, for $1 \le i \le m-1$,*

$$\frac{d_{i-1}^2(m) - d_{i-2}(m)d_i(m)}{d_i^2(m) - d_{i-1}(m)d_{i+1}(m)} < \frac{d_i^2(m) - d_{i-1}(m)d_{i+1}(m)}{d_{i+1}^2(m) - d_i(m)d_{i+2}(m)}. \tag{1.6}$$

The intermediate function $f(m,i)$ is given by

$$f(m,i) = \frac{(i+1)(i+2)(m+i+3)^2}{(m+1-i)(m+2-i)(m+i+2)^2}. \tag{1.7}$$

Using this intermediate function, we can divide the 2-log-concavity into two quadratic inequalities, which are stated below.

Theorem 1.5. *For $1 \leq i \leq m-1$, we have*

$$\frac{(i+1)(i+2)(m+i+3)^2}{(m+1-i)(m+2-i)(m+i+2)^2} < \frac{d_i^2(m) - d_{i-1}(m)d_{i+1}(m)}{d_{i+1}^2(m) - d_i(m)d_{i+2}(m)}. \tag{1.8}$$

Theorem 1.6. *For $1 \leq i \leq m-1$, we have*

$$\frac{d_{i-1}^2(m) - d_{i-2}(m)d_i(m)}{d_i^2(m) - d_{i-1}(m)d_{i+1}(m)} < \frac{(i+1)(i+2)(m+i+3)^2}{(m+1-i)(m+2-i)(m+i+2)^2}. \tag{1.9}$$

As will be seen, the 2-log-concavity of $P_m(a)$ implies the log-concavity of a sequence considered by Moll [22, 25].

Theorem 1.7. *For $m \geq 2$, the sequence $\{i(i+1)(d_i^2(m) - d_{i-1}(m)d_{i+1}(m)\}_{1\leq i\leq m}$ is log-concave.*

Since log-concavity implies unimodality, the above property leads to another proof of Moll's minimum conjecture [25] for $\{i(i+1)(d_i^2(m) - d_{i-1}(m)d_{i+1}(m)\}_{1\leq i\leq m}$. By comparing the first entry with the last entry, we deduce that this sequence attains its minimum at $i = m$ which equals $2^{-2m}m(m+1)\binom{2m}{m}^2$. This conjecture was confirmed by Chen and Xia [17] by using a result of Chen and Gu [11] and the spiral property of the Boros-Moll sequences [16].

2 How to guess the intermediate function $f(m, i)$

In this section, we explain how we found the intermediate function $f(m, i)$. We begin with a brief review of Kauers and Paule's approach to proving the log-concavity of the Boros-Moll polynomials [21], because we need the recurrence relations and an inequality established by Kauers and Paule. Here are the four recurrence relations

$$d_i(m+1) = \frac{m+i}{m+1}d_{i-1}(m) + \frac{(4m+2i+3)}{2(m+1)}d_i(m), \qquad 0 \leq i \leq m+1, \tag{2.1}$$

$$\begin{aligned} d_i(m+1) = &\frac{(4m-2i+3)(m+i+1)}{2(m+1)(m+1-i)}d_i(m) \\ &- \frac{i(i+1)}{(m+1)(m+1-i)}d_{i+1}(m), \qquad 0 \leq i \leq m, \end{aligned} \tag{2.2}$$

$$d_i(m+2) = \frac{-4i^2 + 8m^2 + 24m + 19}{2(m+2-i)(m+2)}d_i(m+1)$$

$$-\frac{(m+i+1)(4m+3)(4m+5)}{4(m+2-i)(m+1)(m+2)}d_i(m), \qquad 0 \le i \le m+1, \quad (2.3)$$

and for $0 \le i \le m+1$,

$$(m+2-i)(m+i-1)d_{i-2}(m) - (i-1)(2m+1)d_{i-1}(m) + i(i-1)d_i(m) = 0. \quad (2.4)$$

These recurrences are derived by Kauers and Paule [21]. In fact, the relations (2.3) and (2.4) are derived independently by Moll [25] via the WZ-method [26], and the other two relations (2.1) and (2.2) can be easily deduced from (2.3) and (2.4). Based on the four recurrence relations, Kauers and Paule [21] proved the following inequality from which the log-concavity of the Boros-Moll sequences can be deduced.

Theorem 2.1. (Kauers and Paule [21]) *Let m, i be integers with $m \ge 2$. For $0 < i < m$, we have*

$$\frac{d_i(m+1)}{d_i(m)} \ge \frac{4m^2+7m+i+3}{2(m+1-i)(m+1)}. \quad (2.5)$$

Chen and Gu [11] showed that $\{i!d_i(m)\}_{0\le i\le m}$ is log-concave and the sequence $\{d_i(m)\}_{0\le i\le m}$ is reverse ultra log-concave. They established the following upper bound for $d_i(m+1)/d_i(m)$.

Theorem 2.2. (Chen and Gu [11]) *Let m, i be integers and $m \ge 2$. We have for $0 \le i \le m$,*

$$\frac{d_i(m+1)}{d_i(m)} \le \frac{4m^2+7m+3+i\sqrt{4m+4i^2+1}-2i^2}{2(m+1)(m+1-i)}. \quad (2.6)$$

Theorems 2.1 and 2.2 are needed in the proofs of Theorems 1.5 and 1.6, and they are also needed to have a good guess of the intermediate function $f(m,i)$. We start with an approximation of

$$\frac{d_{i-1}^2(m) - d_{i-2}(m)d_i(m)}{d_i^2(m) - d_{i-1}(m)d_{i+1}(m)}.$$

Recall that the following relation was proved by Chen and Gu [11],

$$\lim_{m\to+\infty} \frac{d_i^2(m)}{\left(1+\frac{1}{i}\right)\left(1+\frac{1}{m-i}\right)d_{i-1}(m)d_{i+1}(m)} = 1.$$

This implies that

$$\frac{d_{i-1}^2(m) - d_{i-2}(m)d_i(m)}{d_i^2(m) - d_{i-1}(m)d_{i+1}(m)} \approx \frac{(i+1)(m+1-i)d_{i-1}^2(m)}{i(m+2-i)d_i^2(m)}. \quad (2.7)$$

Using the recurrence relation (2.1), we find

$$\frac{d_{i-1}^2(m)}{d_i^2(m)} = \frac{(m+1)^2 d_i^2(m+1)}{(m+i)^2 d_i^2(m)} - \frac{(4m+2i+3)(m+1)d_i(m+1)}{(m+i)^2 d_i(m)} + \frac{(4m+2i+3)^2}{4(m+i)^2}. \tag{2.8}$$

On the other hand, by Theorems 2.1 and 2.2, we get

$$\lim_{m\to+\infty} \frac{2(m+1)(m+1-i)d_i(m+1)}{(4m^2+7m+i+3)d_i(m)} = 1.$$

It follows that

$$\frac{d_i(m+1)}{d_i(m)} \approx \frac{4m^2+7m+i+3}{2(m+1)(m+1-i)}. \tag{2.9}$$

Substituting (2.9) into (2.8) yields

$$\frac{d_{i-1}^2(m)}{d_i^2(m)} \approx \frac{i^2(i+1+m)^2}{(m+1-i)^2(m+i)^2}. \tag{2.10}$$

Combining (2.7) and (2.10), we deduce that

$$\frac{d_{i-1}^2(m) - d_{i-2}(m)d_i(m)}{d_i^2(m) - d_{i-1}(m)d_{i+1}(m)} \approx \frac{i(i+1)(m+1+i)^2}{(m+1-i)(m+2-i)(m+i)^2}. \tag{2.11}$$

It turns out that the above expression is not an intermediate function that we are looking form. Naturally, we should try to make it a little bigger. The above expression gives a guideline for a suitable adjustment. Let us consider the shifts of the factors in the expression (2.11). After a few trials, we find that the function below serves the purpose as a desired intermediate function

$$\frac{(i+1)(i+2)(m+i+3)^2}{(m+1-i)(m+2-i)(m+i+2)^2}, \tag{2.12}$$

which is the function $f(m,i)$ as given by (1.7).

3 Proof of Theorem 1.5

In this section, we give a proof of Theorem 1.5. The idea goes as follows. We wish to prove an equivalent form of Theorem 1.5, that is, the difference

$$\begin{aligned}(m+1-i)(m+2-i)(m+i+2)^2 &\left(d_i^2(m) - d_{i-1}(m)d_{i+1}(m)\right) \\ &- (i+1)(i+2)(m+i+3)^2 \left(d_{i+1}^2(m) - d_i(m)d_{i+2}(m)\right)\end{aligned} \tag{3.1}$$

is positive. As will be seen, in view of the recurrence relations of $d_i(m)$, (3.1) can be written as

$$A(m,i)d_i^2(m+1) + B(m,i)d_i(m+1)d_i(m) + C(m,i)d_i^2(m), \tag{3.2}$$

where $A(m,i)$, $B(m,i)$ and $C(m,i)$ are given by (3.4), (3.5) and (3.6). To prove that the quadratic form (3.2) is positive, we consider the quadratic polynomial in $d_i(m+1)/d_i(m)$

$$A(m,i)\frac{d_i^2(m+1)}{d_i^2(m)} + B(m,i)\frac{d_i(m+1)}{d_i(m)} + C(m,i). \tag{3.3}$$

It will be shown that $A(m,i) < 0$ for $1 \le i \le m$. Moreover, we shall show that the above polynomial has distinct real roots x_1 and x_2. Assume that $x_1 < x_2$. If the relation

$$x_1 < \frac{d_i(m+1)}{d_i(m)} < x_2$$

holds, then the quadratic polynomial (3.3) is positive.

Let

$$A(m,i) = -\frac{(m+1)^2(m+1-i)^2D(m,i)}{(m+i)i^2(i+1)}, \tag{3.4}$$

$$B(m,i) = \frac{(i-m-1)(m+1)E(m,i)}{(i+m)i^2(i+1)}, \tag{3.5}$$

$$C(m,i) = \frac{F(m,i)}{4(i+m)i^2(i+1)}, \tag{3.6}$$

$$\begin{aligned}\Delta_1(m,i) &= B^2(m,i) - 4A(m,i)C(m,i)\\ &= \frac{(m+1-i)^2(m+1)^2\left(4(m+i)^2G(m,i) + H(m,i)\right)}{i^2(i+m)^2(i+1)^2},\end{aligned} \tag{3.7}$$

where $D(m,i)$, $E(m,i)$, $F(m,i)$, $G(m,i)$ and $H(m,i)$ are given in the Appendix.

Theorem 3.1. *For $1 \le i \le m-1$ and $m \ge 126$, we have*

$$\frac{-B(m,i) + \sqrt{\Delta_1(m,i)}}{2A(m,i)} < \frac{d_i(m+1)}{d_i(m)} < \frac{-B(m,i) - \sqrt{\Delta_1(m,i)}}{2A(m,i)}. \tag{3.8}$$

In order to prove Theorem 3.1, it is necessary to show that $\Delta_1(m,i) > 0$.

Lemma 3.2. *For $1 \le i \le m-1$ and $m \ge 126$, we have $\Delta_1(m,i) > 0$.*

Proof. In view of the definition (3.7) of $\Delta_1(m,i)$ and the fact that $H(m,i)$ is positive, it suffices to show that $G(m,i) > 0$ for $1 \le i \le m-1$. We consider three cases with respect to the range of i.

Case 1: $i^3 \ge \frac{3}{7}m^2$. In this case, we have

$$m^2(2i^3 - m^2)^2 \ge 0, \quad 56i^6m - 24i^3m^3 \ge 0, \quad 20i^5m^2 - 2i^2m^4 > 0,$$

and so $G(m,i) > 0$.

Case 2: $\frac{m^2}{10} < i^3 < \frac{3}{7}m^2$. In this case, we have

$$m^2(2i^3 - m^2)^2 \ge \frac{m^6}{49}, \quad 56i^6m - 24i^3m^3 \ge -\frac{18}{7}m^5, \quad 20i^5m^2 - 2i^2m^4 > 0.$$

Thus, for $m \ge 126$,

$$G(m,i) \ge \frac{m^6}{49} - \frac{18}{7}m^5 > 0.$$

Case 3: $1 \le i^3 \le \frac{m^2}{10}$. In this case, we have

$$m^2(2i^3 - m^2)^2 \ge \frac{16m^6}{25}, \quad 56i^6m - 24i^3m^3 \ge -\frac{46}{25}m^5, \quad 20i^5m^2 - 2i^2m^4 > -2m^{16/3}.$$

It follows that

$$G(m,i) \ge \frac{16m^6}{25} - \frac{46}{25}m^5 - 2m^{16/3}. \tag{3.9}$$

It is easily checked that the right-hand side of (3.9) is positive for $m \ge 10$. This completes the proof. ∎

We are now ready to prove Theorem 3.1.

Proof of Theorem 3.1. We first consider the lower bound of $d_i(m+1)/d_i(m)$, namely,

$$\frac{d_i(m+1)}{d_i(m)} > \frac{-B(m,i) + \sqrt{\Delta_1(m,i)}}{2A(m,i)}. \tag{3.10}$$

From the inequality (2.5) of Kauers and Paule [21], we see that (3.10) is a consequence of the relation

$$\frac{4m^2 + 7m + i + 3}{2(m+1)(m+1-i)} > \frac{-B(m,i) + \sqrt{\Delta_1(m,i)}}{2A(m,i)}. \tag{3.11}$$

Since $A(m,i) < 0$ for $1 \le i \le m$, the inequality (3.11) can be rewritten as

$$A(m,i)\frac{4m^2 + 7m + i + 3}{(m+1)(m+1-i)} + B(m,i) < \sqrt{\Delta_1(m,i)}. \tag{3.12}$$

To verify (3.12), we calculate the difference of the squares of both sides. It is easily checked that

$$\Delta_1(m,i) - \left(A(m,i)\frac{4m^2+7m+i+3}{(m+1)(m+1-i)} + B(m,i)\right)^2 = \frac{(m+1-i)^2(m+1)^2K(m,i)}{i^2(i+m)^2(i+1)^2},$$

where $K(m,i)$ is given in the Appendix. It is easy to check that $K(m,i)$ is positive for $1 \le i \le m-1$. Hence, by Lemma 3.2, we obtain (3.12). This yields (3.10).

It remains to the consider the upper bound of $d_i(m+1)/d_i(m)$, namely,

$$\frac{d_i(m+1)}{d_i(m)} < \frac{-B(m,i) - \sqrt{\Delta_1(m,i)}}{2A(m,i)}. \tag{3.13}$$

By Theorem 2.2 of Chen and Gu [11], we see that (3.13) is a consequence of the following relation

$$\frac{4m^2+7m+i\sqrt{4i^2+4m+1}-2i^2+3}{2(m+1)(m+1-i)} < \frac{-B(m,i) - \sqrt{\Delta_1(m,i)}}{2A(m,i)}. \tag{3.14}$$

Since $A(m,i) < 0$ for $1 \le i \le m-1$, (3.14) can be rewritten as

$$A(m,i)\frac{4m^2+7m+i\sqrt{4i^2+4m+1}-2i^2+3}{(m+1)(m+1-i)} + B(m,i) > -\sqrt{\Delta_1(m,i)}. \tag{3.15}$$

As before, we can check (3.15) by computing the difference of the squares of both sides. It is readily seen that

$$\begin{aligned}\Delta_1(m,i) - &\left(A(m,i)\frac{4m^2+7m+i\sqrt{4i^2+4m+1}-2i^2+3}{(m+1)(m+1-i)} + B(m,i)\right)^2\\ &= \frac{(m+1-i)^2(m+1)^2L(m,i)}{i^2(i+m)(i+1)^2},\end{aligned}$$

where $L(m,i)$ is given in the Appendix. It is easy to verify that

$$\begin{aligned}&(2i^2m+4im^2+2m^3+i^2+24m+14im+13m^2+6i+9)^2(4i^2+4m+1)-(-4i^3m\\ &\quad -8i^2m^2-4im^3-20i^2m-24im^2-4m^3+7i^2-28im-19m^2+20i-20m+7)^2\\ =&16i^6m+96i^5m^2+176i^4m^3+128i^3m^4+48i^2m^5+32im^6+16m^7\\ &+4i^6+264i^5m+972i^4m^2+1088i^3m^3+492i^2m^4+312im^5+196m^6\\ &+48i^5+1456i^4m+3248i^3m^2+2064i^2m^3+1184im^4+960m^5+168i^4\end{aligned}$$

$$+ 3508i^3m + 4368i^2m^2 + 2372im^3 + 2384m^4 + 164i^3 + 3876i^2m$$
$$+ 3036im^2 + 3196m^3 - 120i^2 + 2164im + 2404m^2 - 172i + 1036m + 32,$$

which is positive for $1 \le i \le m-1$. So we reach the conclusion that $L(m,i) > 0$. Therefore, we obtain (3.15) which implies (3.13). In view of (3.10) and (3.13), we arrive at (3.8). This completes the proof. ∎

To conclude this section, we present a proof of Theorem 1.5.

Proof of Theorem 1.5. First we show that the difference (3.1) can be represented in terms of $d_i(m)$ and $d_i(m+1)$. In view of (2.1), (2.2) and (2.4), we find that for $1 \le i \le m-1$,

$$d_{i+1}(m) = \frac{(4m-2i+3)(m+i+1)}{2i(i+1)}d_i(m) - \frac{(m+1-i)(m+1)}{i(i+1)}d_i(m+1), \quad (3.16)$$

$$d_{i+2}(m) = \frac{2m+1}{i+2}d_{i+1}(m) - \frac{(m-i)(m+i+1)}{(i+1)(i+2)}d_i(m), \quad (3.17)$$

$$d_{i-1}(m) = \frac{m+1}{m+i}d_i(m+1) - \frac{4m+2i+3}{2(m+i)}d_i(m). \quad (3.18)$$

Applying the above recurrence relations, we get

$$\begin{aligned}(m+1-i)(m+2-i)(m+i+2)^2 &\left(d_i^2(m) - d_{i-1}(m)d_{i+1}(m)\right)\\ - (i+1)(i+2)(m+i+3)^2 &\left(d_{i+1}^2(m) - d_i(m)d_{i+2}(m)\right)\\ &= A(m,i)d_i^2(m+1) + B(m,i)d_i(m+1)d_i(m) + C(m,i)d_i^2(m). \quad (3.19)\end{aligned}$$

It is easily verified that Theorem 1.5 holds for $2 \le m \le 125$. By Theorem 3.1, we conclude that the difference (3.2) is positive for $m \ge 126$ and $1 \le i \le m-1$. This completes the proof. ∎

4 Proof of Theorem 1.6

In this section, we give a proof of Theorem 1.6. The main steps can be described as follows. To prove the theorem, we wish to show that the difference

$$\begin{aligned}(i+1)(i+2)(m+i+3)^2 &\left(d_i^2(m) - d_{i-1}(m)d_{i+1}(m)\right)\\ - (m+1-i)(m+2-i)(m+i+2)^2 &\left(d_{i-1}^2(m) - d_{i-2}(m)d_i(m)\right) \quad (4.1)\end{aligned}$$

is positive for $1 \le i \le m-1$. By the recurrence relations of $d_i(m)$, the difference (4.1) can be restated as

$$U(m,i)d_i^2(m+1) + V(m,i)d_i(m+1)d_i(m) + W(m,i)d_i^2(m), \tag{4.2}$$

where $U(m,i)$, $V(m,i)$ and $W(m,i)$ are given by (4.3), (4.4) and (4.5), respectively. We need to consider five cases for the range of i. The conclusion in each case implies that (4.2) is positive. Notice that the definition of $\Delta_2(m,i)$ is given in (4.6), which can be either positive or negative depending on the range of i.

Case 1: $1 \le i < \left(\frac{m^2}{2}\right)^{1/3} - m^{1/3}$. In this case, $\Delta_2(m,i)$ can be either nonnegative or negative. We need to consider the case when $\Delta_2(m,i)$ is nonnegative. Theorem 4.1 is established for this purpose.

Case 2: $\left(\frac{m^2}{2}\right)^{1/3} - m^{1/3} \le i \le \left(\frac{m^2}{2}\right)^{1/3}$. In this case, we show that $\Delta_2(m,i) < 0$.

Case 3: $\left(\frac{m^2}{2}\right)^{1/3} < i < m^{2/3}$. In this case, $\Delta_2(m,i)$ can be either nonnegative or negative. We establish Theorem 4.3 when $\Delta_2(m,i)$ is nonnegative.

Case 4: $m^{2/3} \le i \le m-4$. We show that $\Delta_2(m,i) > 0$ and give a new lower bound on the ratio $d_i(m+1)/d_i(m)$ which implies that (4.2) is positive.

Case 5: $m-3 \le i \le m-1$. It can be verified (4.2) is positive.

The following notation will be used in the statement of Theorem 4.1. Let

$$U(m,i) = \frac{(m+1)^2(m+1-i)R(m,i)}{i(m+i)^2}, \tag{4.3}$$

$$V(m,i) = \frac{(m+1)S(m,i)}{i(m+i-1)(m+i)^2}, \tag{4.4}$$

$$W(m,i) = \frac{T(m,i)}{4i(m+i-1)(m+i)^2}, \tag{4.5}$$

$$\Delta_2(m,i) = V^2(m,i) - 4U(m,i)W(m,i) = \frac{(m+1)^2X(m,i)}{i(m+i)^2(m+i-1)^2}, \tag{4.6}$$

where $R(m,i)$, $S(m,i)$, $T(m,i)$ and $X(m,i)$ are given in the Appendix. Obviously, $U(m,i)$ is positive for $1 \le i \le m-1$.

In Case 1, we obtain the following inequality.

Theorem 4.1. *If $\Delta_2(m,i) \geq 0$, we have for $1 \leq i \leq \left(\frac{m^2}{2}\right)^{1/3} - m^{1/3}$ and $m \geq 15$,*

$$\frac{d_i(m+1)}{d_i(m)} < \frac{-V(m,i) - \sqrt{\Delta_2(m,i)}}{2U(m,i)}. \tag{4.7}$$

Proof. From the inequality (2.6) of Chen and Gu [11], we see that (4.7) can be deduced from the following relation

$$\frac{4m^2 + 7m + i\sqrt{4i^2+4m+1} + 3 - 2i^2}{2(m+1)(m+1-i)} < \frac{-V(m,i) - \sqrt{\Delta_2(m,i)}}{2U(m,i)}. \tag{4.8}$$

To prove (4.8), let

$$\begin{aligned} A_1(m,i) &= 2(m+1)(m+1-i),\\ B_1(m,i) &= 4m^2+7m+3-2i^2,\\ C_1(m,i) &= 4i^2+4m+1. \end{aligned}$$

Clearly, (4.8) can be restated as

$$D_1(m,i) > A_1(m,i)\sqrt{\Delta_2(m,i)} + 2iU(m,i)\sqrt{C_1(m,i)}, \tag{4.9}$$

where $D_1(m,i)$ is given by

$$\begin{aligned} D_1(m,i) =& -V(m,i)A_1(m,i) - 2U(m,i)B_1(m,i)\\ =& \frac{2(m+1)^2(m+1-i)(2m+1)(i^2-i+m+m^2)(m+2+i)^2}{(i+m)^2(i+m-1)}. \end{aligned}$$

Hence $D_1(m,i)$ is positive for $1 \leq i \leq m$. Since $D_1(m,i)$ is positive, the inequality (4.9) follows from the inequality

$$D_1^2(m,i) > \left(A_1(m,i)\sqrt{\Delta_2(m,i)} + 2iU(m,i)\sqrt{C_1(m,i)}\right)^2, \tag{4.10}$$

which can be rewritten as

$$E_1(m,i) > 4iA_1(m,i)U(m,i)\sqrt{\Delta_2(m,i)C_1(m,i)}, \tag{4.11}$$

where $E_1(m,i)$ is given by

$$E_1(m,i) = D_1^2(m,i) - A_1^2(m,i)\Delta_2(m,i) - 4i^2U^2(m,i)C_1(m,i). \tag{4.12}$$

It can be seen that (4.11) is valid if $E_1(m,i)$ is positive and the following inequality holds,

$$E_1^2(m,i) > 16i^2 A_1^2(m,i) U^2(m,i) \Delta_2(m,i) C_1(m,i). \tag{4.13}$$

Given the definition (4.12) of $E_1(m,i)$, it is easily checked that

$$E_1(m,i) = -\frac{8(m+1-i)^2(m+1)^4 R_1(m,i) S_1(m,i)}{i(m+i-1)(m+i)^3}, \tag{4.14}$$

where $R_1(m,i)$ and $S_1(m,i)$ are given in the Appendix. Using the expression (4.7) of $E_1(m,i)$, we see that the positivity of $E_1(m,i)$ can be derived from the fact that $S_1(m,i)$ is negative for $1 \le i \le \left(\frac{m^2}{2}\right)^{1/3} - m^{1/3}$ and $m \ge 15$. We now proceed to show that $S_1(m,i)$ is negative. For $15 \le m \le 728$, the claim can be directly verified. Therefore, we may assume that $m \ge 729$. By putting the terms of $S_1(m,i)$ into groups as given in the Appendix, it can be seen that the sum in every pair of parentheses is negative for $1 \le i \le \left(\frac{m^2}{2}\right)^{1/3} - m^{1/3}$ and $m \ge 729$. Moreover, it is easily checked that

$$8i^5m^2 - 4i^2m^4 < -15m^{11/3}i^2 + 20m^{10/3}i^2 - 8m^3i^2.$$

It follows that

$$\begin{aligned} S_1(m,i) &< -15m^{11/3}i^2 + 20m^{10/3}i^2 - 8m^3i^2 + 36i^4m^2 + 12i^3m^3 \\ &< (-5m^{5/3} + 43m^{4/3})m^2i^2, \end{aligned}$$

which is negative when $m \ge 729$. So we conclude that $E_1(m,i) > 0$ for $1 \le i \le \left(\frac{m^2}{2}\right)^{1/3} - m^{1/3}$ and $m \ge 15$.

We now turn to the proof of (4.13). Consider the difference of the squares of both sides. It is routine to check that

$$\begin{aligned} F_1(m,i) &= E_1^2(m,i) - 16i^2U^2(m,i)A_1^2(m,i)\Delta_2(m,i)C_1(m,i) \\ &= \frac{-256(m+1-i)^4(m+1)^8 M_1^2(m,i) N_1(m,i)}{i^2(i+m-1)^2(i+m)^6}, \end{aligned} \tag{4.15}$$

where $M_1(m,i)$ and $N_1(m,i)$ are given in the Appendix. It is now easy to see that $N_1(m,i) < 0$ for $1 \le i < \left(\frac{m^2}{2}\right)^{1/3} - m^{1/3}$ and $m \ge 15$. So we have $F_1(m,i) > 0$ for $1 \le i < \left(\frac{m^2}{2}\right)^{1/3} - m^{1/3}$ and $m \ge 15$. Hence the inequality (4.13) holds. This completes the proof. ∎

For Case 2, the following lemma asserts that $\Delta_2(m,i)$ is negative.

Lemma 4.2. *For* $\left(\frac{m^2}{2}\right)^{1/3} - m^{1/3} \le i \le \left(\frac{m^2}{2}\right)^{1/3}$ *and* $m \ge 50$*, we have* $\Delta_2(m,i) < 0$.

Proof. By the definition (4.6) of $\Delta_2(m,i)$, it suffices to show that $X(m,i)$ is negative for $\left(\frac{m^2}{2}\right)^{1/3} - m^{1/3} \le i \le \left(\frac{m^2}{2}\right)^{1/3}$ and $m \ge 50$. For $50 \le m \le 2743$, the lemma can be directly verified. Hence we may assume that $m \ge 2744$. Note that the expression in every pair of parentheses is negative for $\left(\frac{m^2}{2}\right)^{1/3} - m^{1/3} \le i \le \left(\frac{m^2}{2}\right)^{1/3}$ and $m \ge 2744$. On the other hand, it can be checked that

$$16i^7m^4 - 16i^4m^6 + 4im^8 = 4im^4(2i^3 - m^2)^2 < 58im^{22/3} \le 47m^8,$$

$$64i^8m^3 - 24i^2m^7 + 16i^{11} + 64i^{10}m + 96i^9m^2 \le -8i^2m^7 + 176i^9m^2 \le -5m^{25/3} + 22m^8.$$

This yields

$$X(m,i) < -5m^{25/3} + 69m^8,$$

where $X(m,i)$ is given in the Appendix. But the right-hand side of the above inequality is negative when $m \ge 2744$. This completes the proof. ∎

As will be seen, Theorems 4.2 and 4.3 have the same expression of the lower bound for $d_i(m+1)/d_i(m)$. This expression will be needed in the proof of Theorem 1.6. It should be noted that for the case of Theorem 4.2, we shall show that this lower bound can be derived from the lower bound of Kauers and Paule [21]. Numerical evidence shows that the bound in Theorem 4.3 seems sharper than the bound of Kauers and Paule when i is large.

For Case 3, we have the following inequality. It should remarked that in this case $\Delta_2(m,i)$ can be either positive or negative, and there is no need to specify the range of i for which $\Delta_2(m,i)$ is positive.

Theorem 4.3. *If* $\Delta_2(m,i) \ge 0$*, we have for* $\left(\frac{m^2}{2}\right)^{1/3} \le i \le m^{2/3}$ *and* $m \ge 2$,

$$\frac{d_i(m+1)}{d_i(m)} > \frac{-V(m,i) + \sqrt{\Delta_2(m,i)}}{2U(m,i)}. \tag{4.16}$$

Proof. By the lower bound of $d_i(m+1)/d_i(m)$, as given in (2.5), we see that (4.16) can be obtained from the following relation

$$\frac{4m^2 + 7m + i + 3}{2(m+1)(m+1-i)} > \frac{-V(m,i) + \sqrt{\Delta_2(m,i)}}{2U(m,i)}, \tag{4.17}$$

which can be rewritten as

$$U(m,i)\frac{4m^2 + 7m + i + 3}{(m+1)(m+1-i)} + V(m,i) > \sqrt{\Delta_2(m,i)}. \tag{4.18}$$

In order to prove (4.18), we shall show that for $\left(\frac{m^2}{2}\right)^{1/3} \leq i \leq m^{2/3}$ and $m \geq 2$,

$$U(m,i)\frac{4m^2+7m+i+3}{(m+1)(m+1-i)} + V(m,i) > 0. \tag{4.19}$$

and

$$\left(U(m,i)\frac{4m^2+7m+i+3}{(m+1)(m+1-i)} + V(m,i)\right)^2 - \Delta_2(m,i) > 0 \tag{4.20}$$

We first deal with inequality (4.19). It is easily checked that

$$U(m,i)\frac{4m^2+7m+i+3}{(m+1)(m+1-i)} + V(m,i) = \frac{(m+1)P(m,i)}{(m+i)^2(m+i-1)},$$

where $P(m,i)$ is given in the Appendix. Since the sum in every pair of parentheses in the expression of $P(m,i)$ is nonnegative for $\left(\frac{m^2}{2}\right)^{1/3} \leq i \leq m^{2/3}$ and $m \geq 2$, it follows that $P(m,i) > 0$. Thus, we obtain (4.19).

We still need to consider the inequality (4.20). Clearly,

$$\left(U(m,i)\frac{4m^2+7m+i+3}{(m+1)(m+1-i)} + V(m,i)\right)^2 - \Delta_2(m,i) = \frac{4(m+1)^2 G_1(m,i) H_1(m,i)}{(m+i)^4(i+m-1)i},$$

where $G_1(m,i)$ and $H_1(m,i)$ are given in the Appendix. We see that $G_1(m,i) > 0$ and $H_1(m,i) > 0$ for $\left(\frac{m^2}{2}\right)^{1/3} \leq i \leq m^{2/3}$ and $m \geq 2$. Hence the inequality (4.20) holds. This completes the proof. ∎

For Case 4, we give a lower bound for $d_i(m+1)/d_i(m)$ that takes the same form as the lower bound in Case 3.

Theorem 4.4. *For $m \geq 273$ and $m^{2/3} \leq i \leq m-4$, we have*

$$\frac{d_i(m+1)}{d_i(m)} > \frac{-V(m,i) + \sqrt{\Delta_2(m,i)}}{2U(m,i)}. \tag{4.21}$$

For the clarity of presentation, we establish two lemmas for the proof of Theorem 4.4. First, we prove that $\Delta_2(m,i)$ is positive.

Lemma 4.5. *For $m^{2/3} \leq i \leq m-1$ and $m \geq 19$, we have $\Delta_2(m,i) > 0$.*

Proof. By the definition (4.6) of $\Delta_2(m,i)$, it suffices to show that $X(m,i)$ is positive for $m^{2/3} \leq i \leq m-1$ and $m \geq 19$. By direct computation we find that the lemma holds for $19 \leq m \leq 132$. Moreover, for $m \geq 133$ and $m^{2/3} \leq i \leq m-1$, it can be seen that $X(m,i) > 0$, see the Appendix. This completes the proof. ∎

The proof of Theorem 4.4 is by induction on m. The inductive argument requires an inequality concerning the desired lower bound. We present this inequality in Lemma 4.6. Let

$$\begin{aligned}
Y_1(m,i) &= \frac{(m+i+1)(4m+3)(4m+5)}{4(m+2-i)(m+1)(m+2)},\\
Y_2(m,i) &= \frac{-4i^2+8m^2+24m+19}{2(m+2-i)(m+2)},\\
Y_3(m,i) &= 2U(m+1,i)Y_2(m,i)+V(m+1,i) = \frac{(m+2)Y_5(m,i)}{(m+i)i(m+i+1)},\\
Y_4(m,i) &= Y_3^2(m,i) - \Delta_2(m+1,i) = \frac{(m+2)^2 Y_6(m,i)}{(m+1+i)^2 i^2 (m+i)},
\end{aligned}$$

where the explicit expressions for $Y_5(m,i)$ and $Y_6(m,i)$ are given in the Appendix. It is easily seen that $Y_1(m,i)$, $Y_2(m,i)$, $Y_3(m,i)$ and $Y_4(m,i)$ are all positive for $1 \leq i \leq m-1$ and $m \geq 2$.

Lemma 4.6. *For $m^{2/3} \leq i \leq m-4$ and $m \geq 273$, we have*

$$\frac{-V(m,i)+\sqrt{\Delta_2(m,i)}}{2U(m,i)} > \frac{Y_1(m,i)}{Y_2(m,i) - \frac{-V(m+1,i)+\sqrt{\Delta_2(m+1,i)}}{2U(m+1,i)}}. \tag{4.22}$$

Proof. Let us rewrite (4.22) as

$$\frac{-V(m,i)+\sqrt{\Delta_2(m,i)}}{2U(m,i)} > \frac{2U(m+1,i)Y_1(m,i)}{Y_3(m,i)-\sqrt{\Delta_2(m+1,i)}}. \tag{4.23}$$

Since $Y_3(m,i) > 0$ and $Y_4(m,i) > 0$ for $m^{2/3} \leq i \leq m-4$ and $m \geq 273$, the inequality (4.23) follows from the inequality

$$V(m,i)\sqrt{\Delta_2(m+1,i)} + Y_3\sqrt{\Delta_2(m,i)} > Z_1(m,i) + \sqrt{\Delta_2(m,i)\Delta_2(m+1,i)}, \tag{4.24}$$

where $Z_1(m,i)$ is given by

$$Z_1(m,i) = 4U(m,i)U(m+1,i)Y_1(m,i) + V(m,i)Y_3(m,i). \tag{4.25}$$

Clearly, $Z_1(m,i) < 0$ for $m^{2/3} \leq i \leq m-4$ and $m \geq 273$. To prove (4.24), we shall show that the following three inequalities hold,

$$Z_1(m,i) + \sqrt{\Delta_2(m,i)\Delta_2(m+1,i)} < 0, \tag{4.26}$$

$$V(m,i)\sqrt{\Delta_2(m+1,i)} + Y_3(m,i)\sqrt{\Delta_2(m,i)} < 0 \tag{4.27}$$

and

$$\Big(V(m,i)\sqrt{\Delta_2(m+1,i)} + Y_3(m,i)\sqrt{\Delta_2(m,i)}\Big)^2 < \Big(Z_1(m,i) + \sqrt{\Delta_2(m,i)\Delta_2(m+1,i)}\Big)^2. \tag{4.28}$$

We first consider inequality (4.26). Let

$$Z_2(m,i) = \Delta_2(m,i)\Delta_2(m+1,i) - Z_1^2(m,i). \tag{4.29}$$

Employing the same argument as in the proofs of Lemmas 3.2, 4.2 and 4.5, we find that $Z_2(m,i) < 0$ for $m^{2/3} \le i \le m-4$ and $m \ge 273$. The detailed verification is omitted since the expansion of $Z_2(m,i)$ is a little lengthy. Thus we obtain (4.26) since both $Z_1(m,i)$ and $Z_2(m,i)$ are negative for $m^{2/3} \le i \le m-4$ and $m \ge 273$.

We now turn to the proof of (4.27). Note that $V(m,i) < 0$ for $1 \le i \le m-1$. Let

$$Z_3(m,i) = Y_3^2(m,i)\Delta_2(m,i) - V^2(m,i)\Delta_2(m+1,i). \tag{4.30}$$

It is not difficult to show that $Z_3(m,i) < 0$ for $m^{2/3} \le i \le m-4$ and $m \ge 273$. The detailed proof is omitted as before. Since $Z_3(m,i)$ and $V(m,i)$ are negative and $Y_3(m,i)$ and $\Delta_2(m,i)$ are positive for $m^{2/3} \le i \le m-4$ and $m \ge 273$, we arrive at (4.27).

It remains to prove (4.28), which can be restated as

$$Z_4(m,i) > Z_5(m,i)\sqrt{\Delta_2(m,i)\Delta_2(m+1,i)}, \tag{4.31}$$

where $Z_4(m,i)$ and $Z_5(m,i)$ are given by

$$Z_4(m,i) = V^2(m,i)\Delta_2(m+1,i) + Y_3^2(m,i)\Delta_2(m,i) - Z_1^2(m,i) - \Delta_2(m,i)\Delta_2(m+1,i), \tag{4.32}$$

$$Z_5(m,i) = 2Z_1(m,i) - 2V(m,i)Y_3(m,i). \tag{4.33}$$

Using the same argument as in the proofs of Lemmas 3.2, 4.2 and 4.5, we can deduce that $Z_4(m,i)$ and $Z_5(m,i)$ are positive for $m^{2/3} \le i \le m-4$ and $m \ge 273$. Therefore, (4.31) is a consequence of the fact that

$$Z_6(m,i) = Z_5^2(m,i)\Delta_2(m,i)\Delta_2(m+1,i) - Z_4^2(m,i) \tag{4.34}$$

is positive for $m^{2/3} \le i \le m-4$ and $m \ge 273$, which is not difficult to prove although $Z_6(m,i)$ is rather tedious. This completes the proof. ∎

We are now in a position to prove Theorem 4.4.

Proof of Theorem 4.4. We proceed by induction on m. It is easy to check that the theorem holds for $m = 273$. We assume that the theorem is true for $n \geq 273$, that is,

$$d_i(n+1) \geq \frac{-V(n,i) + \sqrt{\Delta_2(n,i)}}{2U(n,i)} d_i(n), \qquad n^{2/3} \leq i \leq n-4. \tag{4.35}$$

We aim to show that (4.21) holds for $m = n+1$, that is,

$$d_i(n+2) \geq \frac{-V(n+1,i) + \sqrt{\Delta_2(n+1,i)}}{2U(n+1,i)} d_i(n+1), \quad (n+1)^{2/3} \leq i \leq n-3. \tag{4.36}$$

In view of Lemma 4.6 and inequality (4.35), we find

$$d_i(n+1) > \frac{Y_1(n,i)}{Y_2(n,i) - \frac{-V(n+1,i)+\sqrt{\Delta_2(n+1,i)}}{2U(n+1,i)}} d_i(n).$$

It follows that for $n^{2/3} \leq i \leq n-4$,

$$Y_2(n,i)d_i(n+1) - Y_1(n,i)d_i(n) > \frac{-V(n+1,i) + \sqrt{\Delta_2(n+1,i)}}{2U(n+1,i)} d_i(n+1). \tag{4.37}$$

By the recurrence relation (2.3), the left hand side of (4.37) equals $d_i(n+2)$. Thus we have verified (4.36) for $(n+1)^{2/3} \leq i \leq n-4$. It is still necessary to show that (4.36) is true for $i = n-3$, that is,

$$d_{n-3}(n+2) > \frac{-V(n+1,n-3) + \sqrt{\Delta_2(n+1,n-3)}}{2U(n+1,n-3)} d_{n-3}(n+1). \tag{4.38}$$

Let

$$\begin{aligned} f(n) = {} & 256n^{11} - 4608n^{10} + 36544n^9 - 177920n^8 + 572592n^7 - 1218432n^6 \\ & + 1573768n^5 - 940352n^4 - 66903n^3 - 65525n^2 - 3657n - 963. \end{aligned}$$

By the expression (1.5) of $d_i(m)$, we have

$$\begin{aligned} \frac{d_{n-3}(n+2)}{d_{n-3}(n+1)} &= \frac{(2n+5)(16n^4+80n^3+180n^2+240n+189)(2n-1)}{10(n+2)(45+72n+68n^2+48n^3+16n^4)} \\ &> \frac{12-65n+14n^2+3108n^4-3041n^3-1020n^5+136n^6+16n^7}{10(n+2)(2n-3)(1+2n+33n^2+4n^4-16n^3)} \\ &\quad + \frac{(n-1)\sqrt{(n-3)f(n)}}{10(n+2)(2n-3)(1+2n+33n^2+4n^4-16n^3)} \end{aligned}$$

$$=\frac{-V(n+1,n-3)+\sqrt{\Delta_2(n+1,n-3)}}{2U(n+1,n-3)}.$$

Hence the proof is complete by induction. ∎

Finally, we are ready to complete the proof of Theorem 1.6.

Proof of Theorem 1.6. For $2 \le m \le 272$, the theorem can be easily verified. So we may assume that $m \ge 273$. The difference (4.1) can be represented in terms of $d_i(m+1)$ and $d_i(m)$. From (2.4) it follows that

$$d_{i-2}(m) = \frac{(i-1)(2m+1)}{(m+2-i)(m+i-1)} d_{i-1}(m) - \frac{i(i-1)}{(m+2-i)(m+i-1)} d_i(m). \quad (4.39)$$

Using recurrence relations (3.16), (3.18) and (4.39), we find that

$$\begin{aligned}&(i+1)(i+2)(m+i+3)^2\left(d_i^2(m)-d_{i-1}(m)d_{i+1}(m)\right)\\&\quad-(m+1-i)(m+2-i)(m+i+2)^2\left(d_{i-1}^2(m)-d_{i-2}(m)d_i(m)\right)\\&\qquad=U(m,i)d_i^2(m+1)+V(m,i)d_i(m+1)d_i(m)+W(m,i)d_i^2(m).\end{aligned} \quad (4.40)$$

Hence the theorem says that (4.2) is positive. If $\Delta_2(m,i) < 0$, it is obvious that (4.2) is positive since $U(m,i) > 0$ for $1 \le i \le m-1$. We now assume that $\Delta_2(m,i) \ge 0$.

Recall the five cases for the range of i as given before. Case 1: $1 \le i < \left(\frac{m^2}{2}\right)^{1/3} - m^{1/3}$. By Theorem 4.1, we see that (4.2) is positive. Case 2: $\left(\frac{m^2}{2}\right)^{1/3} - m^{1/3} \le i \le \left(\frac{m^2}{2}\right)^{1/3}$. Note that in this case, by Lemma 4.2, we have $\Delta_2(m,i) < 0$, which belongs to the case that we have already considered before. Case 3: $\left(\frac{m^2}{2}\right)^{1/3} < i < m^{2/3}$. It follows from Theorem 4.3 that (4.2) is positive. Case 4: $m^{2/3} \le i \le m-4$. The lower bound given in Theorem 4.4 ensures that (4.2) is positive. It remains to consider the case when $i = m-3,\ m-2,\ m-1$. Here we only verify the statement for $i = m-3$. The other two cases can be justified analogously. By (1.5), we see that

$$\begin{aligned}&U(m,m-3)d_{m-3}^2(m+1)+V(m,m-3)d_{m-3}(m+1)d_{m-3}(m)\\&\quad+W(m,m-3)d_{m-3}^2(m)=\frac{(m+1)^2(m-2)g(m)}{9216(2m+1)^2(2m-1)^2(2m-3)^2}2^{-2m}\binom{2m+2}{m+1}^2,\end{aligned}$$

where $g(m)$ is given by

$$g(m) = 2048m^{12} - 10240m^{11} + 16512m^{10} - 3456m^9 - 35232m^8 + 99120m^7 + 44488m^6$$

$$- 375620m^5 + 431652m^4 - 182601m^3 + 7362m^2 + 13797m - 2430,$$

which is positive for $m \geq 273$. This completes the proof. ∎

To conclude this paper, we show that the 2-log-concavity of the Boros-Moll polynomials implies the log-concavity of the sequence $\{i(i+1)(d_i^2(m)-d_{i-1}(m)d_{i+1}(m))\}_{1\leq i\leq m}$, as stated in Theorem 1.7.

Clearly, for $i \geq 2$, we have

$$\frac{i(i+1)}{(i-1)(i+2)} > 1. \tag{4.41}$$

By Theorem 1.4 and the inequality (4.41), we obtain that for $2 \leq i \leq m-1$,

$$\frac{d_{i-1}^2(m) - d_{i-2}(m)d_i(m)}{d_i^2(m) - d_{i-1}(m)d_{i+1}(m)} < \frac{i(i+1)}{(i-1)(i+2)} \frac{d_i^2(m) - d_{i-1}(m)d_{i+1}(m)}{d_{i+1}^2(m) - d_i(m)d_{i+2}(m)}.$$

Replacing i by $i+1$, we find that for $1 \leq i \leq m-2$,

$$\frac{d_i^2(m) - d_{i-1}(m)d_{i+1}(m)}{d_{i+1}^2(m) - d_i(m)d_{i+2}(m)} < \frac{(i+1)(i+2)}{i(i+3)} \frac{\left(d_{i+1}^2(m) - d_i(m)d_{i+2}(m)\right)}{\left(d_{i+2}^2(m) - d_{i+1}(m)d_{i+3}(m)\right)},$$

which can be written as

$$\frac{i(i+1)\left(d_i^2(m) - d_{i-1}(m)d_{i+1}(m)\right)}{(i+1)(i+2)\left(d_{i+1}^2(m) - d_i(m)d_{i+2}(m)\right)} < \frac{(i+1)(i+2)\left(d_{i+1}^2(m) - d_i(m)d_{i+2}(m)\right)}{(i+2)(i+3)\left(d_{i+2}^2(m) - d_{i+1}(m)d_{i+3}(m)\right)}.$$

Thus the sequence $\{i(i+1)(d_i^2(m) - d_{i-1}(m)d_{i+1}(m))\}_{1\leq i\leq m}$ is log-concave.

Acknowledgments. We wish to thank the referee for helpful suggestions. This work was supported by the 973 Project, the PCSIRT Project of the Ministry of Education, and the National Science Foundation of China.

References

[1] T. Amdeberhan, D. Manna and V.H. Moll, The 2-adic valuation of a sequence arising from a rational integral, J. Combin. Theory Ser. A, 115 (2008) 1474–1486.

[2] T. Amdeberhan and V.H. Moll, A formula for a quartic integral: a survey of old proofs and some new ones, Ramanujan J. 18 (2009) 91–102.

[3] T. Amdeberhan and V.H. Moll, A by-product of an integral evaluation, Ramanujan J. to appear.

[4] G. Boros and V.H. Moll, An integral hidden in Gradshteyn and Ryzhik, J. Comput. Appl. Math. 106 (1999) 361–368.

[5] G. Boros and V.H. Moll, A sequence of unimodal polynomials, J. Math. Anal. Appl. 237 (1999) 272–285.

[6] G. Boros and V.H. Moll, A criterion for unimodality, Electron. J. Combin. 6 (1999) R3.

[7] G. Boros and V.H. Moll, The double square root, Jacobi polynomials and Ramanujan's Master Theorem, J. Comput. Appl. Math. 130 (2001) 337–344.

[8] G. Boros and V.H. Moll, Irresistible Integrals, Cambridge University Press, Cambridge, 2004.

[9] P. Brändén, Iterated sequences and the geometry of zeros, J. Reine Angew. Math. 658 (2011) 115–131.

[10] W.Y.C. Chen, D.Q.J. Dou and A.L.B. Yang, Brändén's conjectures on the Boros-Moll polynomials, arXiv:Math.CO/1205.0305.

[11] W.Y.C. Chen and C.C.Y. Gu, The reverse ultra log-concavity of the Boros-Moll polynomials, Proc. Amer. Math. Soc. 137 (2009) 3991–3998.

[12] W.Y.C. Chen, S.X.M. Pang and E.X.Y. Qu, On the combinatorics of the Boros-Moll polynomials, Ramanujan J. 21 (2010) 41–51.

[13] W.Y.C. Chen, S.X.M. Pang and E.X.Y. Qu, Partially 2-colored permutations and the Boros-Moll polynomials, Ramanujan J. 27 (2012), 297–304.

[14] William Y.C. Chen, Larry X.W. Wang, and Ernest X.W. Xia, The interlacing log-concavity of the Boros-Moll polynomials, Pacific J. Math. 254 (2011) 89-99.

[15] W.Y.C. Chen, L.X.W. Wang and A.L.B. Yang, Schur positivity and the q-log-convexity of the Narayana polynomials, J. Alg. Combin. 32 (2010) 303–338.

[16] W.Y.C. Chen and E.X.W. Xia, The ratio monotonicity of the Boros-Moll polynomials, Math. Comput. 78 (2009) 2269–2282.

[17] W.Y.C. Chen and E.X.W. Xia, Proof of Moll's minimum conjecture, European J. Combin., to appear.

[18] W.Y.C. Chen, A.L.B. Yang, and E.L.F. Zhou, Ratio monotonicity of polynomials derived from nondecreasing sequences, Electron. J. Combin. 17 (2010), N37.

[19] T. Craven and G. Csordas, Iterated Laguerre and Turán inequalities, J. Inequal. Pure Appl. Math. 3 (2002) Article 39, 14pp.

[20] S. Fisk, Questions about determinants and polynomials, arXiv:Math.CA/0808.1850.

[21] M. Kauers and P. Paule, A computer proof of Moll's log-concavity conjecture, Proc. Amer. Math. Soc. 135 (2007) 3847–3856.

[22] D.V. Manna and V.H. Moll, A remarkable sequence of integers, Expo. Math. 27 (2009) 289–312.

[23] P.R. McNamara and B. Sagan, Infinite log-concavity: developments and conjectures, Adv. Appl. Math. 44 (2010) 1–15.

[24] V.H. Moll, The evaluation of integrals: A personal story, Notices Amer. Math. Soc. 49 (2002) 311–317.

[25] V.H. Moll, Combinatorial sequences arising from a rational integral, Online J. Anal. Combin. 2 (2007), # 4.

[26] H.S. Wilf and D. Zeilberger, An algorithmic proof theory for hypergeometric (ordinary and "q") multisum/integral identities, Invent. Math. 108 (1992) 575–633.

Appendix

In the statement of Theorem 3.1, the polynomials $D(m,i)$, $E(m,i)$, $F(m,i)$, $G(m,i)$ and $H(m,i)$ are are given by

$$\begin{aligned}
D(m,i) &= 6m^2i + 2m^2i^2 + 21mi + 14mi^2 + 4mi^3 + 10i \\
&\quad + 17i^2 + 10i^3 + 2i^4 + 2m^3 + 12m^2 + 18m, \\
E(m,i) &= 4i^2(i^2 - 2m^2)(i+m)^2 + 2(i+m)(10i^4 - 4m^4 - 9im^3 - 27i^2m^2 - 4i^3m) \\
&\quad + 27i^4 - 55i^3m - 175i^2m^2 - 139im^3 - 62m^4 - 16i^3 - 155i^2m \\
&\quad - 229im^2 - 162m^3 - 60i^2 - 142im - 162m^2 - 30i - 54m, \\
F(m,i) &= 32i^2m^2(i-m)(i+m)^3 + 16m(4i^4 + 10i^3m - 14i^2m^2 - 3im^3 - 2m^4)(i+m)^2 \\
&\quad + 2(i+m)(-152m^5 - 250im^4 - 377i^2m^3 + 111i^3m^2 + 181i^4m + 15i^5) \\
&\quad + 168i^5 + 694i^4m - 280i^3m^2 - 2052i^2m^3 - 2160im^4 - 1106m^5 + 273i^4 \\
&\quad - i^3m - 1809i^2m^2 - 2831im^3 - 1968m^4 + 18i^3 - 898i^2m - 1936im^2 \\
&\quad - 1836m^3 - 207i^2 - 663im - 864m^2 - 90i - 162m, \\
G(m,i) &= m^2(2i^3 - m^2)^2 + (56i^6m - 24i^3m^3) + (20i^5m^2 - 2i^2m^4) \\
&\quad + 4i^8 + 8i^7m + 40i^7 + 169i^6 + 166i^5m + 70i^4m^2, \\
H(m,i) &= 1588i^7 + 4440i^6m + 4768i^5m^2 + 2148i^4m^3 + 324i^3m^4 + 144i^2m^5 \\
&\quad + 104im^6 + 52m^7 + 2345i^6 + 6666i^5m + 6991i^4m^2 + 3624i^3m^3 + 1567i^2m^4 \\
&\quad + 646im^5 + 289m^6 + 2418i^5 + 7232i^4m + 8044i^3m^2 + 5340i^2m^3 + 2234im^4 \\
&\quad + 892m^5 + 1903i^4 + 5810i^3m + 7225i^2m^2 + 4104im^3 + 1618m^4 + 1086i^3 \\
&\quad + 3332i^2m + 3470im^2 + 1608m^3 + 321i^2 + 914im + 657m^2.
\end{aligned}$$

In the proof of Theorem 3.1, the polynomials $K(m,i)$ and $L(m,i)$ are given by

$$\begin{aligned}
K(m,i) &= 4(2i^4 + 4i^3m + 2i^2m^2 + 10i^3 + 14i^2m + 6im^2 + 2m^3 + 17i^2 + 21im + 12m^2 \\
&\qquad + 10i + 18m)(2i^3m^2 + 2i^2m^3 - 2i^5 - 2i^4m - 9i^4 + 2i^3m + 16i^2m^2 + 6im^3 \\
&\qquad + m^4 - 7i^3 + 23i^2m + 23im^2 + 9m^3 + 12i^2 + 16im + 20m^2 + 8i + 8m), \\
L(m,i) &= 2(2i^4 + 4i^3m + 2i^2m^2 + 10i^3 + 14i^2m + 6im^2 + 2m^3 + 17i^2
\end{aligned}$$

$$
\begin{aligned}
&+ 21im + 12m^2 + 10i + 18m)\big(- 4i^3m - 8i^2m^2 - 4im^3 - 20i^2m \\
&- 24im^2 - 4m^3 + 7i^2 - 28im - 19m^2 + 20i - 20m + 7 \\
&+ (2i^2m + 4im^2 + 2m^3 + i^2 + 24m + 14im + 13m^2 + 6i + 9)\sqrt{4i^2 + 4m + 1}\big).
\end{aligned}
$$

In the statement of Theorem 4.1, the polynomials $R(m,i)$, $S(m,i)$, $T(m,i)$ and $X(m,i)$ are given by

$$
\begin{aligned}
R(m,i) &= 2i^2m^2 + 4mi^3 + 6im^2 + 14mi^2 + 2i^4 + 10i^3 \\
&\quad + 21mi + 17i^2 + 2m^3 + 12m^2 + 18m + 10i, \\
S(m,i) &= 4i^2(i^2 - 2m^2)(i+m)^3 + 2(8i^4 - 4i^3m - 21i^2m^2 - 9im^3 - 4m^4)(i+m)^2 \\
&\quad + (i+m)(-54m^4 - 121im^3 - 99i^2m^2 - 41i^3m + 7i^4) - 41i^4 \\
&\quad - 98i^3m - 187i^2m^2 - 262im^3 - 100m^4 - 41i^3 - 51i^2m \\
&\quad - 106im^2 + 25i^2 + 45im + 108m^2 + 30i + 54m, \\
T(m,i) &= 32i^2m^2(i+m)^4 + 16m(4i^4 + 18i^3m + 18i^2m^2 + 7im^3 + 2m^4)(i+m)^2 \\
&\quad + 2(i+m)(120m^5 + 414im^4 + 601i^2m^3 + 523i^3m^2 + 199i^4m + 15i^5) \\
&\quad + 132i^5 + 850i^4m + 1912i^3m^2 + 2652i^2m^3 + 2084im^4 + 562m^5 + 153i^4 \\
&\quad + 417i^3m + 983i^2m^2 + 1307im^3 + 300m^4 - 48i^3 - 328i^2m \\
&\quad - 248im^2 - 432m^3 - 177i^2 - 405im - 540m^2 - 90i - 162m, \\
X(m,i) &= 16i^7m^4 - 16i^4m^6 + 4im^8 + 64i^8m^3 - 24i^2m^7 + 16i^{11} + 64i^{10}m + 96i^9m^2 \\
&\quad + (128i^{10} + 448i^9m + 624i^8m^2 + 448i^7m^3 + 160i^6m^4 - 100i^3m^6) \\
&\quad + (372i^9 + 1280i^8m + 1868i^7m^2 + 1256i^6m^3 + 128i^5m^4 - 240i^4m^5) \\
&\quad + (340i^8 + 1712i^7m + 2520i^6m^2 + 620i^5m^3 - 1132i^4m^4 - 1096i^3m^5 \\
&\quad - 528i^2m^6) + (3692i^2m - 52im^7 - 16m^8 - 523i^7 - 2i^6m - 509i^5m^2 \\
&\quad - 2584i^4m^3 - 3749i^3m^4 - 2910i^2m^5 - 635im^6 - 176m^7 - 1416i^6 \\
&\quad - 5048i^3m^3 - 5940i^2m^4 - 1810im^5 - 656m^6 - 586i^5 - 3890i^4m \\
&\quad - 3588i^2m^3 - 667im^4 - 688m^5 + 1240i^4 + 1054i^3m + 2274i^2m^2 \\
&\quad + 3216im^3 + 1104m^4 + 1221i^3 + 2896im^2 + 2160m^3 - 3550i^5m \\
&\quad - 4508i^4m^2 - 268i^2 - 2525i^3m^2 + 488im - 432m^2 - 524i - 1296m).
\end{aligned}
$$

In the proof of Theorem 4.1, the polynomials $R_1(m,i)$, $S_1(m,i)$, $M_1(m,i)$ and $N_1(m,i)$ are given by

$$\begin{aligned}R_1(m,i) =&2i^2m^2+4mi^3+6im^2+14mi^2+2i^4\\&+10i^3+21mi+17i^2+2m^3+12m^2+18m+10i,\\S_1(m,i) =&8i^5m^2-4i^2m^4+36i^4m^2+12i^3m^3+(16i^6m-4m^5)+(8i^7-2im^4)\\&+(32i^6+52i^5m+30i^5+88i^4m+66i^3m^2-28i^2m^3-6im^4)\\&+(36m-27i^4+55i^3m-65i^2m^2-23im^3-24m^4-56i^3\\&-101i^2m-9im^2-32m^3-9i^2-20im+24m^2+22i),\\M_1(m,i) =&2i^4+4i^3m+2i^2m^2+10i^3+14i^2m+6im^2\\&+2m^3+17i^2+21im+12m^2+10i+18m,\\N_1(m,i) =&4i^{10}m-40i^8m^3-96i^7m^4-128i^6m^5-128i^5m^6-88i^4m^7-32i^3m^8-4i^2m^9\\&+i^{10}+12i^9m-92i^8m^2-400i^7m^3-774i^6m^4-1100i^5m^5-1072i^4m^6\\&-592i^3m^7-171i^2m^8-32im^9-4m^{10}+6i^9-58i^8m-556i^7m^2-1602i^6m^3\\&-3236i^5m^4-4334i^4m^5-3204i^3m^6-1270i^2m^7-322im^8-48m^9-3i^8\\&-351i^7m-1487i^6m^2-4194i^5m^3-7663i^4m^4-7213i^3m^5-3519i^2m^6\\&-1122im^7-208m^8-87i^7-695i^6m-2422i^5m^2-5984i^4m^3-6495i^3m^4\\&-3165i^2m^5-1272im^6-336m^7-161i^6-399i^5m-1212i^4m^2-107i^3m^3\\&+2447i^2m^4+1012im^5+104m^6+87i^5+839i^4m+3175i^3m^2+6101i^2m^3\\&+2902im^4+816m^5+377i^4+1388i^3m+3137i^2m^2+862im^3+432m^4\\&+32i^3-20i^2m-1308im^2-432m^3-252i^2-720im-324m^2.\end{aligned}$$

In the proof of Theorem 4.3, the polynomials $P(m,i)$, $G_1(m,i)$ and $H_1(m,i)$ are given by

$$\begin{aligned}P(m,i) =& 4i^6+(4i^3m^3-2m^5)+(38i^3m^2-9m^4)+(14i^2m^3-11m^3)+12i^5m\\&+18i^5+44i^4m+(21i^4-10i^3)+60i^3m+(35i^2m-21im)+12i^4m^2\\&+(64i^2m^2-10m^2-22m)+16im^3+(34im^2-27i^2-6i),\\G_1(m,i) =& 2i^4+4i^3m+2i^2m^2+10i^3+14i^2m+6im^2\end{aligned}$$

$$+ 2m^3 + 17i^2 + 21im + 12m^2 + 10i + 18m,$$

$$\begin{aligned} H_1(m,i) = {} & 2i^7 + 4i^6m + 7i^6 + 11i^5m + 8i^4m^2 + 14i^3m^3 + 15i^2m^4 + 3i^4 \\ & + (7im^5 - 4i^4m^3) + (2m^6 - 2i^3m^4) + 7i^5 + 34i^4m + 68i^3m^2 + 58i^2m^3 \\ & + (29im^4 - 10i^2m) + (12m^5 - 12m^3) + (61i^3m - 14i^2 - 40im) \\ & + (63i^2m^2 - 25im^2 - 18m^2) + 21im^3 + 16m^4 - 5i^3. \end{aligned}$$

In the proof of Lemma 4.6, the polynomials $Y_5(m,i)$ and $Y_6(m,i)$ are given by

$$\begin{aligned} Y_5(m,i) = {} & 4i^2(2m^2 - i^2)(i+m)^2 + 2(i+m)(4m^4 + 7im^3 + 31i^2m^2 + 4i^3m - 12i^4) \\ & - 35i^4 + 59i^3m + 199i^2m^2 + 151im^3 + 82m^4 + 16i^3 + 181i^2m + 321im^2 \\ & + 282m^3 + 70i^2 + 294im + 368m^2 + 106i + 160m, \\ Y_6(m,i) = {} & (2i^4 + 4i^3m + 2i^2m^2 + 14i^3 + 18i^2m + 6im^2 + 2m^3 + 33i^2 + 33im + 18m^2 \\ & + 37i + 48m + 32)\big(32i^2m^2(m-i)(i+m)^2 + 16m(i+m)(2m^4 + im^3 \\ & + 16i^2m^2 - 11i^3m - 4i^4) - 30i^5 - 394i^4m - 110i^3m^2 + 762i^2m^3 + 300im^4 \\ & + 368m^5 - 168i^4 - 338i^3m + 1154i^2m^2 + 558im^3 + 1538m^4 + 1028i^2m \\ & - 141i^3 + 631im^2 + 2882m^3 + 391i^2 + 639im + 2480m^2 + 260i + 800m\big). \end{aligned}$$

In the proof of Lemma 4.5, we need to check that for $m \geq 133$ and $m^{2/3} \leq i \leq m-1$, $X(m,i) > 0$. Indeed, we have

$$\begin{aligned} X(m,i) \geq {} & 16i^{11} + 64i^{10}m + 96i^9m^2 + 40i^8m^3 + 24(i^8m^3 - i^2m^7) + 16(i^7m^4 - i^4m^6) \\ & + 128i^{10} + (448i^9m - 176m^7) + 624i^8m^2 + (292i^7m^3 - 240i^4m^5 - 52im^7) \\ & + (116i^6m^4 - 100i^3m^6 - 16m^8) + (1868i^7m^2 - 1132i^4m^4 - 635im^6) \\ & + 1096(i^6m^3 - i^3m^5) + (160m^{2/3} - 2910)i^2m^5 + (620m^{2/3} - 2584)i^5m^3 \\ & + (128m^{4/3} - 3749)i^3m^4 + (4m - 528)m^6i^2 + (340m^{2/3} - 523)i^7 + 1712i^7m \\ & + (2520i^6m^2 - 2i^6m - 509i^5m^2) + 372i^9 + 1280i^8m - 22928m^6 - 11944m^5 \\ \geq {} & 96m^8 - 22928m^6 - 11944m^5, \end{aligned}$$

which is positive for $m \geq 133$.