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On the Number of Partitions with Designated Summands

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Abstract. Andrews, Lewis and Lovejoy introduced the partition function $PD(n)$ as the number of partitions of n with designated summands, where we assume that among parts with equal size, exactly one is designated. They proved that $PD(3n+2)$ is divisible by 3. We obtain a Ramanujan type identity for the generating function of $PD(3n+2)$ which implies the congruence of Andrews, Lewis and Lovejoy. For $PD(3n)$, Andrews, Lewis and Lovejoy showed that the generating function can be expressed as an infinite product of powers of $(1-q^{2n+1})$ times a function $F(q^2)$. We find an explicit formula for $F(q^2)$, which leads to a formula for the generating function of $PD(3n)$. We also obtain a formula for the generating function of $PD(3n+1)$. Our proofs rely on Chan's identity on Ramanujan's cubic continued fraction and some identities on cubic theta functions. By introducing a rank for partitions with designated summands, we give a combinatorial interpretation of the congruence of Andrews, Lewis and Lovejoy for $PD(3n+2)$.

Keywords: partition with designated summands, Ramanujan type identity, Ramanujan's cubic continued fraction, cubic theta function.

AMS Classifications: 05A17, 11P83, 05A30.

1 Introduction

Andrews, Lewis and Lovejoy [2] investigated the number of partitions with designated summands which are defined on ordinary partitions by designating exactly one part among parts with equal size. For example, there are ten partitions of 4 with designated summands:

$$4', \quad 3'+1', \quad 2'+2, \quad 2+2', \quad 2'+1'+1,$$
$$2'+1+1', \quad 1'+1+1+1, \quad 1+1'+1+1, \quad 1+1+1'+1, \quad 1+1+1+1'.$$

Just for comparison, let us recall the notion of overpartitions. An overpartition of n is a partition of n in which the first occurrence of each part can be overlined. For example, there are fourteen overpartitions of 4:

$$4 \quad 4' \quad 3+1 \quad 3'+1 \quad 3+1' \quad 3'+1', \quad 2+2$$
$$2'+2 \quad 2+1+1 \quad 2'+1+1 \quad 2+1'+1 \quad 2'+1'+1, \quad 1+1+1+1 \quad 1'+1+1+1.$$

Overpartitions have been extensively studied, and they possess many analogous properties to ordinary partitions, see, for example, [8, 9, 11, 13].

The concept of partitions with designated summands goes back to MacMahon [14]. He considered partitions with designated summands and with exactly k different sizes, see also Andrews and Rose [5]. Let $PD(n)$ denote the number of partitions of n with designated summands. Andrews, Lewis and Lovejoy [2] derived the following generating function of $PD(n)$.

Theorem 1.1. *We have*

$$\sum_{n=0}^{\infty} PD(n)q^n = \frac{(q^6;q^6)_\infty}{(q;q)_\infty(q^2;q^2)_\infty(q^3;q^3)_\infty}, \tag{1.1}$$

where $|q|<1$ and $(a;q)_\infty$ stands for the q-shifted factorial

$$(a;q)_\infty = \prod_{n=1}^{\infty}(1-aq^{n-1}).$$

By using modular forms and q-series identities, Andrews, Lewis and Lovejoy showed that the partition function $PD(n)$ has many interesting divisibility properties. In particular, they obtained the following Ramanujan type congruence.

Theorem 1.2. ([2, Corollary 7]) *For $n \geq 0$, we have*

$$PD(3n+2) \equiv 0 \pmod{3}. \tag{1.2}$$

In this paper, we obtain the following Ramanujan type identity for the generating function of $PD(3n+2)$ which implies the above congruence.

Theorem 1.3. *We have*

$$\sum_{n=0}^{\infty} PD(3n+2)q^n = 3\frac{(q^3;q^6)_\infty^3(q^6;q^6)_\infty^6}{(q;q^2)_\infty^5(q^2;q^2)_\infty^8}. \tag{1.3}$$

Andrews, Lewis and Lovejoy also obtained explicit formulas for the generating functions for $PD(2n)$ and $PD(2n+1)$ by using Euler's algorithm for infinite products [1, P. 98] and Sturm's criterion [16]. As for $PD(3n)$, they showed that the generating function permits the following form.

Theorem 1.4. ([2, Theorem 23]) *Define $c(n)$ by*

$$\sum_{n=0}^{\infty} PD(3n)q^n = \prod_{n=1}^{\infty}(1-q^n)^{-c(n)}, \tag{1.4}$$

then for any positive integer n,

$$c(6n+1) = 5,$$
$$c(6n+3) = 2,$$
$$c(6n+5) = 5.$$

Equivalently, the above theorem says that there exists a series $F(q^2)$ such that

$$\sum_{n=0}^{\infty} PD(3n)q^n = \frac{F(q^2)}{(q;q^6)_\infty^5(q^3;q^6)_\infty^2(q^5;q^6)_\infty^5}. \tag{1.5}$$

We find an explicit formula for $F(q^2)$, that is,

$$F(q^2) = \frac{(q^4;q^4)_\infty^6(q^6;q^6)_\infty^4}{(q^2;q^2)_\infty^{10}(q^{12};q^{12})_\infty^2} + 3q^2\frac{(q^{12};q^{12})_\infty^6}{(q^2;q^2)_\infty^6(q^4;q^4)_\infty^2}, \tag{1.6}$$

which leads to the following generating function of $PD(3n)$.

Theorem 1.5. *We have*

$$\sum_{n=0}^{\infty} PD(3n)q^n = \frac{1}{(q;q^6)_\infty^5(q^3;q^6)_\infty^2(q^5;q^6)_\infty^5}\times \left(\frac{(q^4;q^4)_\infty^6(q^6;q^6)_\infty^4}{(q^2;q^2)_\infty^{10}(q^{12};q^{12})_\infty^2} + 3q^2\frac{(q^{12};q^{12})_\infty^6}{(q^2;q^2)_\infty^6(q^4;q^4)_\infty^2}\right). \tag{1.7}$$

We also obtain the generating function for $PD(3n+1)$.

Theorem 1.6. *We have*

$$\sum_{n=0}^{\infty} PD(3n+1)q^n = \frac{(q^3;q^6)_\infty^3 (q^6;q^6)_\infty^6}{(q;q^2)_\infty^5 (q^2;q^2)_\infty^8} \left(4q \frac{(q;q^2)_\infty^2}{(q^3;q^6)_\infty^6} + \frac{(q^3;q^6)_\infty^3}{(q;q^2)_\infty} \right). \tag{1.8}$$

The proofs of the generating function formulas (1.3), (1.7) and (1.8) rely on Chan's identity on Ramanujan's cubic continued fraction [10] and cubic theta functions [6,12]. In Section 3, we shall give a combinatorial interpretation of the congruence $PD(3n+2) \equiv 0 \pmod 3$ by introducing a rank for partitions with designated summands.

2 Proofs

In this section, we give proofs of the generating functions for $PD(3n)$, $PD(3n+1)$ and $PD(3n+2)$. It should be noted that the generating function of $PD(3n)$ derived this way does not directly lead to a formula for $F(q^2)$. To compute $F(q^2)$, we shall make use of some identities on cubic theta functions.

Recall that Ramanujan's cubic continued fraction $v(q)$ is given by

$$v(q) := \frac{q^{\frac{1}{3}}}{1} \, {}_{+} \, \frac{q+q^2}{1} \, {}_{+} \, \frac{q^2+q^4}{1} \, {}_{+ \cdots}.$$

It is known that

$$v(q) = q^{\frac{1}{3}} \frac{(q;q^2)_\infty}{(q^3;q^6)_\infty^3},$$

see Andrews and Berndt [3, P. 94]. The following identity is due to Chan [10, Eq. (13)].

Theorem 2.1. *We have*

$$\frac{1}{(q;q)_\infty (q^2;q^2)_\infty} = \frac{(q^9;q^9)_\infty^3 (q^{18};q^{18})_\infty^3}{(q^3;q^3)_\infty^4 (q^6;q^6)_\infty^4} \times \left\{ \left(\frac{1}{x^2(q^3)} - 2q^3 x(q^3) \right) + q \left(\frac{1}{x(q^3)} + 4q^3 x^2(q^3) \right) + 3q^2 \right\}, \tag{2.1}$$

where

$$x(q) = q^{-\frac{1}{3}} v(q) = \frac{(q;q^2)_\infty}{(q^3;q^6)_\infty^3}. \tag{2.2}$$

Proof of Theorems 1.3 and 1.6. Multiplying both sides of (2.1) by

$$\frac{(q^6;q^6)_\infty}{(q^3;q^3)_\infty},$$

we find

$$\frac{(q^6;q^6)_\infty}{(q;q)_\infty(q^2;q^2)_\infty(q^3;q^3)_\infty} = \frac{(q^9;q^9)_\infty^3(q^{18};q^{18})_\infty^3}{(q^3;q^3)_\infty^5(q^6;q^6)_\infty^3}\left\{\left(\frac{1}{x^2(q^3)} - 2q^3x(q^3)\right)\right.$$
$$\left. +q\left(\frac{1}{x(q^3)} + 4q^3x^2(q^3)\right) + 3q^2\right\}. \tag{2.3}$$

Observe that the left-hand side of (2.3) is the generating function for $PD(n)$. Extracting those terms involving the powers q^{3n+1} and q^{3n+2} respectively, we deduce that

$$\sum_{n=0}^{\infty} PD(3n+1)q^{3n+1} = q\frac{(q^9;q^9)_\infty^3(q^{18};q^{18})_\infty^3}{(q^3;q^3)_\infty^5(q^6;q^6)_\infty^3}\left(4q^3x^2(q^3) + \frac{1}{x(q^3)}\right), \tag{2.4}$$

$$\sum_{n=0}^{\infty} PD(3n+2)q^{3n+2} = 3q^2\frac{(q^9;q^9)_\infty^3(q^{18};q^{18})_\infty^3}{(q^3;q^3)_\infty^5(q^6;q^6)_\infty^3}. \tag{2.5}$$

Thus Theorem 1.3 can be deduced from (2.5) by dividing both sides by q^2 and substituting q^3 by q. Similarly, Theorem 1.6 can be deduced from (2.4) by dividing both sides by q and substituting q^3 by q. This completes the proof. ∎

If we extract the terms involving the powers q^{3n} in (2.3), and substitute q^3 by q, we get

$$\sum_{n=0}^{\infty} PD(3n)q^n = \frac{(q^3;q^3)_\infty^3(q^6;q^6)_\infty^3}{(q;q)_\infty^5(q^2;q^2)_\infty^3}\left(-2qx(q) + \frac{1}{x^2(q)}\right). \tag{2.6}$$

It turns out that $F(q^2)$ in the generating function formula for $PD(3n)$ can be computed from (2.6) with the aid of some identities for cubic theta functions. These cubic theta functions are introduced by Borwein, Borwein and Garvan [7] and are defined by

$$a(q) = \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2},$$

$$b(q) = \sum_{m,n=-\infty}^{\infty} \omega^{m-n}q^{m^2+mn+n^2}, \quad \omega = e^{2\pi i/3},$$

$$c(q) = \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2+m+n}.$$

Recall that

$$c(q) = 3\frac{(q^3;q^3)_\infty^3}{(q;q)_\infty}, \tag{2.7}$$

see Berndt, Bhargava and Garvan [6, Eq. (5.5)]. We shall also use the following identities for $a(q)$ and $c(q)$

$$\begin{aligned} a(q) &= a(q^4) + 6q\frac{(q^4;q^4)_\infty^2(q^{12};q^{12})_\infty^2}{(q^2;q^2)_\infty(q^6;q^6)_\infty}, &(2.8)\\ c(q) &= qc(q^4) + 3\frac{(q^4;q^4)_\infty^3(q^6;q^6)_\infty^2}{(q^2;q^2)_\infty^2(q^{12};q^{12})_\infty}, &(2.9)\\ a(q) &= a(q^2) + 2q\frac{c^2(q^2)}{c(q)}. &(2.10)\end{aligned}$$

Identity (2.8) for $a(q)$ and identity (2.9) for $c(q)$ are due to Hirschhorn, Garvan, and Borwein [12, Eqs.(1.36) and (1.34)]. Identity (2.10) for $a(q)$ and $c(q)$ is obtained by Berndt, Bhargava, Garvan [6, Eq. (6.3)].

We obtain the following identity on Ramanujan's cubic continued fraction $v(q)$, which is stated in terms of $x(q)$ as given by (2.2).

Theorem 2.2. *We have*

$$\frac{1}{x^2(q)} - 2qx(q) = 3q^2\frac{(q^2;q^2)_\infty^2(q^{12};q^{12})_\infty^6}{(q^4;q^4)_\infty^2(q^6;q^6)_\infty^6} + \frac{(q^4;q^4)_\infty^6}{(q^2;q^2)_\infty^2(q^6;q^6)_\infty^2(q^{12};q^{12})_\infty^2}. \tag{2.11}$$

Proof. We first establish a connection between Ramanujan's cubic continued fraction $v(q)$ and the cubic theta function $c(q)$. It is easy to check that

$$\frac{1}{x^2(q)} = \frac{(q^3;q^6)_\infty^6}{(q;q^2)_\infty^2} = \frac{(q^2;q^2)_\infty^2}{(q^6;q^6)_\infty^6} \times \left(\frac{(q^3;q^3)_\infty^3}{(q;q)_\infty}\right)^2 = \frac{(q^2;q^2)_\infty^2}{9(q^6;q^6)_\infty^6} \times c^2(q), \tag{2.12}$$

$$2qx(q) = 2q\frac{(q;q^2)_\infty}{(q^3;q^6)_\infty^3} = 2q\frac{(q^6;q^6)_\infty^3}{(q^2;q^2)_\infty} \times \left(\frac{(q;q)_\infty}{(q^3;q^3)_\infty^3}\right) = 6q\frac{(q^6;q^6)_\infty^3}{(q^2;q^2)_\infty} \times \frac{1}{c(q)}. \tag{2.13}$$

We now consider the 2-dissection of $1/x^2(q)$. Identity (2.9) can be viewed as the 2-dissection of $c(q)$. Hence we deduce that

$$c^2(q) = \left(q^2c^2(q^4) + 9\frac{(q^4;q^4)_\infty^6(q^6;q^6)_\infty^4}{(q^2;q^2)_\infty^4(q^{12};q^{12})_\infty^2}\right) + q\left(6c(q^4)\frac{(q^4;q^4)_\infty^3(q^6;q^6)_\infty^2}{(q^2;q^2)_\infty^2(q^{12};q^{12})_\infty}\right).$$

This yields the 2-dissection of $1/x^2(q)$,

$$\begin{aligned}\frac{1}{x^2(q)} &= \left(q^2c^2(q^4)\frac{(q^2;q^2)_\infty^2}{9(q^6;q^6)_\infty^6} + \frac{(q^2;q^2)_\infty^2}{(q^6;q^6)_\infty^6}\frac{(q^4;q^4)_\infty^6(q^6;q^6)_\infty^4}{(q^2;q^2)_\infty^4(q^{12};q^{12})_\infty^2}\right)\\ &\quad + q\left(6c(q^4)\frac{(q^2;q^2)_\infty^2}{9(q^6;q^6)_\infty^6}\frac{(q^4;q^4)_\infty^3(q^6;q^6)_\infty^2}{(q^2;q^2)_\infty^2(q^{12};q^{12})_\infty}\right)\\ &= \left(q^2\frac{(q^2;q^2)_\infty^2(q^{12};q^{12})_\infty^6}{(q^4;q^4)_\infty^2(q^6;q^6)_\infty^6} + \frac{(q^4;q^4)_\infty^6}{(q^2;q^2)_\infty^2(q^6;q^6)_\infty^2(q^{12};q^{12})_\infty^2}\right)\\ &\quad + 2q\left(\frac{(q^4;q^4)_\infty^2(q^{12};q^{12})_\infty^2}{(q^6;q^6)_\infty^4}\right). \qquad (2.14)\end{aligned}$$

Next, we aim to derive the 2-dissection of $q/c(q)$. By (2.10), we find

$$\frac{q}{c(q)} = \frac{a(q)-a(q^2)}{2c^2(q^2)}. \qquad (2.15)$$

Substituting (2.8) into (2.15), we arrive at

$$\frac{q}{c(q)} = \frac{1}{2c^2(q^2)}\left(a(q^4) + 6q\frac{(q^4;q^4)_\infty^2(q^{12};q^{12})_\infty^2}{(q^2;q^2)_\infty(q^6;q^6)_\infty} - a(q^2)\right). \qquad (2.16)$$

Using (2.10) with q replaced by q^2, we get

$$a(q^2) - a(q^4) = 2q^2\frac{c^2(q^4)}{c(q^2)}.$$

Hence (2.16) can be written as

$$\begin{aligned}\frac{q}{c(q)} &= \frac{1}{2c^2(q^2)}\left(-2q^2\frac{c^2(q^4)}{c(q^2)} + 6q\frac{(q^4;q^4)_\infty^2(q^{12};q^{12})_\infty^2}{(q^2;q^2)_\infty(q^6;q^6)_\infty}\right)\\ &= -q^2\frac{c^2(q^4)}{c^3(q^2)} + 3q\frac{(q^4;q^4)_\infty^2(q^{12};q^{12})_\infty^2}{c^2(q^2)(q^2;q^2)_\infty(q^6;q^6)_\infty}.\end{aligned}$$

Thus, we obtain the following 2-dissection of $2qx(q)$,

$$\begin{aligned}2qx(q) &= -6q^2\frac{(q^6;q^6)_\infty^3c^2(q^4)}{(q^2;q^2)_\infty c^3(q^2)} + 18q\frac{(q^6;q^6)_\infty^3(q^4;q^4)_\infty^2(q^{12};q^{12})_\infty^2}{c^2(q^2)(q^2;q^2)_\infty^2(q^6;q^6)_\infty}\\ &= -2q^2\frac{(q^2;q^2)_\infty^2(q^{12};q^{12})_\infty^6}{(q^6;q^6)_\infty^6(q^4;q^4)_\infty^2} + 2q\frac{(q^4;q^4)_\infty^2(q^{12};q^{12})_\infty^2}{(q^6;q^6)_\infty^4}. \qquad (2.17)\end{aligned}$$

Subtracting (2.17) from (2.14), we obtain (2.11). This completes the proof. ∎

Proof of Theorem 1.5. By (2.6), we have

$$\begin{aligned}\sum_{n=0}^{\infty} PD(3n)q^n &= \frac{(q^3;q^3)_\infty^3(q^6;q^6)_\infty^3}{(q;q)_\infty^5(q^2;q^2)_\infty^3}\left(-2qx(q)+\frac{1}{x^2(q)}\right)\\ &= \frac{(q^3;q^6)_\infty^3(q^6;q^6)_\infty^6}{(q;q^2)_\infty^5(q^2;q^2)_\infty^8}\left(-2qx(q)+\frac{1}{x^2(q)}\right)\\ &= \frac{1}{(q;q^6)_\infty^5(q^3;q^6)_\infty^2(q^5;q^6)_\infty^5}\times\frac{(q^6;q^6)_\infty^6}{(q^2;q^2)_\infty^8}\left(-2qx(q)+\frac{1}{x^2(q)}\right). \quad (2.18)\end{aligned}$$

Applying (2.11) to (2.18), we are led to the generating function for $PD(3n)$ as given by (1.7). This completes the proof. ∎

3 A combinatorial interpretation

In this section, we give a combinatorial interpretation of the congruence $PD(3n+2) \equiv 0 \pmod 3$. In doing so, we introduce the *pd*-rank of a partition with designated summands. This rank function enables us to divide the set of partitions of $3n+2$ with designated summands into three equinumerous classes. The definition of the *pd*-rank is based on the following representation of a partition with designated summands by a pair of partitions.

Theorem 3.1. *There is a bijection Δ between the set of partitions of n with designated summands and the set of pairs of partitions (α, β) with $|\alpha| + |\beta| = n$, where α is an ordinary partition and β is a partition into parts $\not\equiv \pm 1 \pmod 6$.*

It is clear that the above theorem is a consequence of formula (1.1) for the generating function of partitions with designated summands. We shall give a combinatorial proof of this theorem which give rise to the notion of the *pd*-rank. Our construction is based on the bijective proof of MacMahon's theorem given by Andrews, Eriksson, Petrov and Romik [4].

Combinatorial proof of Theorem 3.1. Let λ be a partition of n with designated summands. We wish to construct a pair of partitions (α, β) such that $|\alpha| + |\beta| = n$, where α is an ordinary partition and β is a partition into parts $\not\equiv \pm 1 \pmod 6$.

Suppose that t is a part of λ, and suppose that t appears m_t times with the i-th part being designated. There are two cases.

- If $i = 1$, then move all the parts equal to t (including the designated part) in λ to the partition α.

- If $i \neq 1$, then move i parts equal to t in λ to γ and $(m_t - i)$ parts equal to t in λ to α.

It can be seen that each part occurs at least twice in γ . The partition β with parts $\not\equiv \pm 1 \pmod 6$ can be obtained from the partition γ with the aid of the following bijection of Andrews, Eriksson, Petrov and Romik.

First, write γ as in the form of $1^{m_1}2^{m_2}\cdots l^{m_l}$, where m_k is the multiplicity of k. Since $m_k \neq 1$ for any k, there is a unique way to write m_k in the form $m_k = s_k + t_k$, where $s_k = 0$ or 3, and $t_k \in \{0, 2, 4, 6, 8, \dots\}$. Now, the partition $\beta = 1^{b_1}2^{b_2}\dots$ is determined as follows:

$$\begin{aligned} b_{6k+1} &= 0, & b_{6k+5} &= 0, \\ b_{6k+2} &= \frac{1}{2}t_{3k+1}, & b_{6k+4} &= \frac{1}{2}t_{3k+2}, \\ b_{6k+3} &= \frac{1}{3}s_{2k+1} + t_{6k+3}, & b_{6k+6} &= \frac{1}{3}s_{2k+2} + t_{6k+6}. \end{aligned}$$

It is clear that β is a partition into parts $\not\equiv \pm 1 \pmod 6$ and the above procedure is reversible. Hence Δ is a bijection. This completes the proof. ∎

The *pd*-rank of a partition λ with designated summands can be defined in terms of the pair of partitions (α, β) under the map Δ.

Definition 3.2. *Let λ be a partition with designated summands and let $(\alpha, \beta) = \Delta(\lambda)$. Then the pd-rank of λ, denoted $r_d(\lambda)$, is defined by*

$$r_d(\lambda) = l_e(\alpha) - l_e(\beta), \tag{3.1}$$

where $l_e(\alpha)$ is the number of even parts of α and $l_e(\beta)$ is the number of even parts of β.

The following theorem shows that the *pd*-rank can be used to divide the set of partitions of $3n + 2$ with designated summands into three equinumerous classes.

Theorem 3.3. *For $i = 0, 1, 2$, let $N_d(i, 3; n)$ denote the number of partitions of n with designated summands with pd-rank congruent to $i \pmod 3$. Then we have*

$$N_d(0, 3; 3n + 2) = N_d(1, 3; 3n + 2) = N_d(2, 3; 3n + 2). \tag{3.2}$$

Proof. Let $N_d(m; n)$ denote the number of partitions of n with designated summands with *pd*-rank m. By the definition of the *pd*-rank, we see that

$$\sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} N_d(m; n) z^m q^n = \frac{1}{(zq^2; q^2)_\infty (q; q^2)_\infty} \times \frac{1}{(z^{-1}q^2; q^2)_\infty (q^3; q^6)_\infty}. \tag{3.3}$$

Setting $z=\zeta=e^{\frac{2\pi i}{3}}$, we find that

$$\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty}N_d(m;n)\zeta^m q^n=\sum_{n=0}^{\infty}\sum_{i=0}^{2}N_d(i,3;n)\zeta^i q^n$$

$$=\frac{1}{(\zeta q^2;q^2)_\infty(q;q^2)_\infty(\zeta^{-1}q^2;q^2)_\infty(q^3;q^6)_\infty}$$

$$=\frac{(-q^3;q^3)_\infty}{(q;q^2)_\infty(\zeta q^2;q^2)_\infty(\zeta^{-1}q^2;q^2)_\infty}. \tag{3.4}$$

Multiplying the right hand side of (3.4) by

$$\frac{(q^2;q^2)_\infty}{(q^2;q^2)_\infty},$$

and noting that

$$(1-x)(1-x\zeta)(1-x\zeta^2)=1-x^3,$$

we deduce that

$$\sum_{n=0}^{\infty}\sum_{i=0}^{2}N_d(i,3;n)\zeta^i q^n=\frac{(-q^3;q^3)_\infty}{(q;q^2)_\infty(\zeta q^2;q^2)_\infty(\zeta^{-1}q^2;q^2)_\infty}\times\frac{(q^2;q^2)_\infty}{(q^2;q^2)_\infty}$$

$$=\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}\times\frac{(-q^3;q^3)_\infty}{(q^6;q^6)_\infty}.$$

By Gauss's identity [1, P. 23]

$$\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}=\sum_{n=0}^{\infty}q^{\binom{n+1}{2}},$$

we get

$$\sum_{n=0}^{\infty}\sum_{i=0}^{2}N_d(i,3;n)\zeta^i q^n=\frac{(-q^3;q^3)_\infty}{(q^6;q^6)_\infty}\sum_{n=0}^{\infty}q^{\binom{n+1}{2}}. \tag{3.5}$$

Since

$$\binom{n+1}{2}\equiv 0 \text{ or } 1 \pmod 3,$$

the coefficient of q^{3n+2} in (3.5) is zero. It follows that

$$N_d(0,3;3n+2)+N_d(1,3;3n+2)\zeta+N_d(1,3;3n+2)\zeta^2=0.$$

Since the minimal polynomial of ζ is $1+x+x^2$, we conclude that

$$N_d(0,3;3n+2)=N_d(1,3;3n+2)=N_d(2,3;3n+2).$$

This completes the proof. ∎

For example, for $n = 5$, we have $PD(5) = 15$. The fifteen partitions of 5 with designated summands, the corresponding pairs of partitions, and the *pd*-ranks modulo 3 are listed in Table 3.1. It can be checked that

$$N_d(0, 3; 5) = N_d(1, 3; 5) = N_d(2, 3; 5) = 5.$$

λ	$(\alpha, \beta) = \Delta(\lambda)$	$r_d(\lambda) \pmod 3$
$5'$	$(5,\ \emptyset)$	0
$4' + 1'$	$(4 + 1,\ \emptyset)$	1
$3' + 2'$	$(3 + 2,\ \emptyset)$	1
$3' + 1' + 1$	$(3 + 1 + 1,\ \emptyset)$	0
$3' + 1 + 1'$	$(3,\ 2)$	2
$2' + 2 + 1'$	$(2 + 2 + 1,\ \emptyset)$	2
$2 + 2' + 1'$	$(1,\ 4)$	2
$2' + 1' + 1 + 1$	$(2 + 1 + 1 + 1,\ \emptyset)$	1
$2' + 1 + 1' + 1$	$(2 + 1,\ 2)$	0
$2' + 1 + 1 + 1'$	$(2,\ 3)$	1
$1' + 1 + 1 + 1 + 1$	$(1 + 1 + 1 + 1 + 1,\ \emptyset)$	0
$1 + 1' + 1 + 1 + 1$	$(1 + 1 + 1,\ 2)$	2
$1 + 1 + 1' + 1 + 1$	$(1 + 1,\ 3)$	0
$1 + 1 + 1 + 1' + 1$	$(1,\ 2 + 2)$	1
$1 + 1 + 1 + 1 + 1'$	$(\emptyset,\ 3 + 2)$	2

Table 3.1: The case for $n = 5$.

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References

[1] G.E. Andrews, The Theory of Partitions, Encycl. Math. and Its Applications, Vol. 2, Addison-Wesley, Reading, 1976.

[2] G.E. Andrews, R.P. Lewis, and J. Lovejoy, Partitions with designated summands, Acta Arith. 105 (2002) 51–66.

[3] G.E. Andrews and B.C. Berndt, Ramanujan's Lost Notebook. Part I, Springer, New York, 2005.

[4] G.E. Andrews, H. Eriksson, F. Petrov, and D. Romik, Integrals, partitions and MacMahon's theorem, J. Comb. Theory A 114 (2007) 545–554.

[5] G.E. Andrews and S.C.F. Rose, MacMahon's sum-of-divisors functions, Chebyshev polynomials, and quasi-modular forms, J. Reine Angew. Math, to appear.

[6] B.C. Berndt, S. Bhargava, and F.G. Garvan, Ramanujan's theories of elliptic functions to alternative bases, Trans. Amer. Math. Soc. 347 (1995) 4163–4244.

[7] J.M. Borwein, P.B. Borwein, and F.G. Garvan, Some cubic modular identities of Ramanujan, Trans. Amer. Math. Soc. 343 (1994) 35–47.

[8] K. Bringmann and J. Lovejoy, Overpartitions and class numbers of binary quadratic forms, Proc. Nat. Acad. Sci. USA 106 (2009) 5513–5516.

[9] K. Bringmann, J. Lovejoy and R. Osburn, Rank and crank moments for overpartitions, J. Number Theory 129 (2009) 1758–1772.

[10] H.-C. Chan, Ramanujan's cubic continued fraction and a generalization of his "most beautiful identity", Int. J. Number Theory 6 (2010) 673–680.

[11] S. Corteel and J. Lovejoy, Overpartitions, Trans. Amer. Math. Soc. 356 (2004) 1623–1635.

[12] M.D. Hirschhorn, F. Garvan, and J. Borwein, Cubic analogues of the Jacobian theta function $\theta(z, q)$, Canad. J. Math. 45 (1993) 673–694.

[13] J. Lovejoy, Overpartition theorems of the Rogers-Ramaujan type, J. London Math. Soc. 69 (2004) 562–574.

[14] P.A. MacMahon, Divisors of numbers and their continuations in the theory of partitions, Proc. London Math. Soc. Ser. 2 19 (1919) 75–113.

[15] P.A. MacMahon, Combinatory Analysis, vols. I and II, Cambridge Univ. Press, Cambridge, 1915–1916, reissued, Chelsea, 1960.

[16] J. Sturm, On the congruence properties of modular forms, Springer Lect. Notes in Math. Vol.1240, pp. 275-280, Springer-Verlag, Berlin/New York, 1984.