Combinatorial Telescoping for an Identity of Andrews on Parity in Partitions

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Abstract

Recently Andrews proposed a problem of finding a combinatorial proof of an identity on the q-little Jacobi polynomials. We give a classification of certain triples of partitions and find bijections based on this classification. By the method of combinatorial telescoping for identities on sums of positive terms, we establish a recurrence relation that leads to the identity of Andrews.

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1 Introduction

In the study of parities in partition identities, Andrews [2] obtained the following identity on the little q-Jacobi polynomials [5, p. 27]:

$$ {}_2\phi_1\left(\begin{matrix} q^{-n}, q^{n+1} \\ -q \end{matrix}; q, -q\right) = (-1)^n q^{\binom{n+1}{2}} \sum_{j=-n}^{n} (-1)^j q^{-j^2}. \tag{1.1}$$

The basic hypergeometric series ${}_2\phi_1$ is defined as follows,

$$ {}_2\phi_1\left(\begin{matrix} a, b \\ c \end{matrix}; q, z\right) := \sum_{n=0}^{\infty} \frac{(a;q)_n (b;q)_n}{(c;q)_n (q;q)_n} z^n,$$

where $|z| < 1$, $|q| < 1$ and

$$\begin{aligned} (a;q)_n &= (1-a)(1-aq)\cdots(1-aq^{n-1}), \\ (a;q)_\infty &= \prod_{i=0}^{\infty}(1-aq^i), \end{aligned}$$

see Gasper and Rahman [5].

Let $G_n(q)$ denote the sum on the left hand side of (1.1). Andrews [2] established the following recurrence relation for $n \geq 1$,

$$G_n(q) + q^n G_{n-1}(q) = 2q^{-\binom{n}{2}}, \tag{1.2}$$

from which (1.1) can be easily deduced. As one of the fifteen open problems, Andrews asked for a combinatorial proof of identity (1.1).

In this paper, we give a combinatorial interpretation of a homogeneous recurrence relation for the sum

$$F_n(q) = q^{\binom{n}{2}} {}_2\phi_1 \left(\begin{matrix} q^{-n}, q^{n+1} \\ -q \end{matrix} ; q, -q \right),$$

that is,

$$F_n(q) + (q^{2n-1} - 1)F_{n-1}(q) - q^{2n-3}F_{n-2}(q) = 0, \tag{1.3}$$

for $n \geq 2$. It is readily seen that (1.3) is a consequence of (1.2) and identity (1.1) can be easily derived from (1.3).

To be more specific, we shall present the method of combinatorial telescoping for sums of positive terms, which is a variant of the method of combinatorial telescoping for alternating sums. In this framework, we find a classification of certain triples of partitions and a sequence of bijections, leading to a combinatorial explanation of recurrence relation (1.3).

The method of combinatorial telescoping for alternating sums was proposed by Chen, Hou and Sun [3], which can be used to show that an alternating sum satisfies certain recurrence relation. It applies to many q-series identities on alternating sums such as Watson's identity [8]

$$\sum_{k=0}^{\infty} (-1)^k \frac{1 - aq^{2k}}{(q;q)_k (aq^k;q)_\infty} a^{2k} q^{k(5k-1)/2} = \sum_{n=0}^{\infty} \frac{a^n q^{n^2}}{(q;q)_n}, \tag{1.4}$$

and Sylvester's identity [9]

$$\sum_{k=0}^{\infty} (-1)^k q^{k(3k+1)/2} x^k \frac{1 - xq^{2k+1}}{(q;q)_k (xq^{k+1};q)_\infty} = 1. \tag{1.5}$$

For the purpose of this paper, we consider a sum of positive terms

$$\sum_{k=0}^{\infty} f(n,k). \tag{1.6}$$

Suppose that $f(n,k)$ is a weighted count of a set $A_{n,k}$, namely,

$$f(n,k) = \sum_{\alpha \in A_{n,k}} w(\alpha),$$

where w is a weight function. We wish to find sets $B_{n,k}$, $H_{n,k}$ and $H'_{n,k}$ with a weight assignment w such that there exists a weight preserving bijection

$$\phi_{n,k} \colon A_{n,k} \cup H_{n,k} \cup H'_{n,k+1} \longrightarrow B_{n,k} \cup H_{n,k+1} \cup H'_{n,k}, \tag{1.7}$$

where $\cup$ stands for disjoint union. Let

$$\begin{aligned} g(n,k) &= \sum_{\alpha \in B_{n,k}} w(\alpha), \\ h(n,k) &= \sum_{\alpha \in H_{n,k}} w(\alpha), \\ h'(n,k) &= \sum_{\alpha \in H'_{n,k}} w(\alpha). \end{aligned}$$

Then the bijection $\phi_{n,k}$ in (1.7) implies that

$$f(n,k) + h(n,k) + h'(n,k+1) = g(n,k) + h(n,k+1) + h'(n,k). \tag{1.8}$$

Like the conditions for creative telescoping [6, 7, 10], we assume that $H_{n,0} = H'_{n,0} = \emptyset$ and $H_{n,k}$, $H'_{n,k}$ vanishes for sufficiently large k. Summing (1.8) over k yields the following relation

$$\sum_{k=0}^{\infty} f(n,k) = \sum_{k=0}^{\infty} g(n,k). \tag{1.9}$$

It is often the case that relation (1.9) can be expressed as a recurrence relation.

For example, to derive the recurrence relation (1.3) for $F_n(q)$, we let

$$F_{n,k} = \frac{(q^{n-k+1};q)_{2k}}{(q^2;q^2)_k} q^{\binom{n-k}{2}}. \tag{1.10}$$

Then $F_n(q)$ can be written as

$$F_n(q) = \sum_{k=0}^{\infty} F_{n,k}. \tag{1.11}$$

Let

$$\begin{aligned} f(n,k) &= F_{n,k} + q^{2n-1} F_{n-1,k}, \\ g(n,k) &= F_{n-1,k} + q^{2n-3} F_{n-2,k}. \end{aligned}$$

By using the method of combinatorial telescoping, one can establish relation (1.9), which can be rewritten as the recurrence relation (1.3) of $F_n(q)$.

Indeed, once we have bijections $\phi_{n,k}$ in (1.7), combining all these bijections, we are led to a correspondence

$$\phi_n \colon A_n \cup H_n \longrightarrow B_n \cup H_n, \tag{1.12}$$

given by $\phi_n(\alpha) = \phi_{n,k}(\alpha)$ if $\alpha \in A_{n,k} \cup H_{n,k} \cup H'_{n,k+1}$, where

$$A_n = \bigcup_{k=0}^{\infty} A_{n,k}, \quad B_n = \bigcup_{k=0}^{\infty} B_{n,k} \quad \text{and} \quad H_n = \bigcup_{k=0}^{\infty} \left(H_{n,k} \cup H'_{n,k}\right).$$

By the method of cancelation, see Feldman and Propp [4], the above bijection ϕ_n implies a bijection

$$\psi_n \colon A_n \longrightarrow B_n.$$

More precisely, we can define the bijection $\psi_n \colon A_n \to B_n$ by setting $\psi_n(a)$ to be the first element b that falls into B_n while iterating the action of ϕ_n on $a \in A_n$.

In the next section, we shall give explicit constructions of the bijections for the recurrence relation (1.3) which implies the following identity:

$$\sum_{k=0}^{n} \frac{(q^{n-k+1};q)_{2k}}{(q^2;q^2)_k} q^{\binom{n-k}{2}} = (-1)^n q^{n^2} \sum_{j=-n}^{n} (-1)^j q^{-j^2}. \tag{1.13}$$

Notice that (1.13) is obtained from (1.1) by multiplying both sides by $q^{\binom{n}{2}}$. As will be seen, the summand $F_{n,k}$ of the left hand side of (1.13) can be viewed as a weighted count of some set $P_{n,k}$ of triples of partitions. So we may write

$$F_{n,k} = \sum_{\alpha \in P_{n,k}} w(\alpha).$$

We shall construct bijections

$$\phi_{n,k} \colon P_{n,k} \cup \{2n-1\} \times P_{n-1,k-1} \to P_{n-1,k-1} \cup \{2n-3\} \times P_{n-2,k}$$

for $k = 1, 2, \ldots, n-2$. Moreover, for $k = n-1$ or n, we provide an involution $I_{n,k}$ on

$$P_{n,k} \cup \{2n-1\} \times P_{n-1,k-1}$$

with the invariant set $P_{n-1,k-1}$. Furthermore, one can verify that the bijections $\phi_{n,k}$ and the involutions $I_{n,k}$ are weight preserving. This yields recurrence relation (1.3), which leads to the identity of Andrews.

2 The Combinatorial Telescoping

The objective of this section is to construct the bijections $\phi_{n,k}$ and the involutions $I_{n,k}$ as mentioned in the introduction so that we can use the combinatorial telescoping argument to establish recurrence relation (1.3).

Let us recall some notation and definitions on partition as used in Andrews [1]. A *partition* is a nonincreasing finite sequence of nonnegative integers $\lambda = (\lambda_1, \ldots, \lambda_\ell)$. The integers λ_i are called the parts of λ. The sum of parts and the number of parts are denoted by $|\lambda| = \lambda_1 + \cdots + \lambda_\ell$ and $\ell(\lambda) = l$, respectively. The partition with no parts is denoted by $\emptyset$. Denote by D the set of partitions of distinct parts, and denote by E the set of partitions of even parts. We shall use diagrams to represent partitions and use rows to represent parts.

Define $P_{n,k}$ to be the set of triples (τ, λ, μ), where

$$\tau = (n-k-1, n-k-2, \ldots, 2, 1, 0)$$

is a triangular partition, λ is a partition of distinct parts such that $n-k+1 \leq \lambda_i \leq n+k$ and μ is a partition of even parts not exceeding $2k$, see Figure 2.1. As will be seen, there is a reason to include the zero part in a triangular partition.

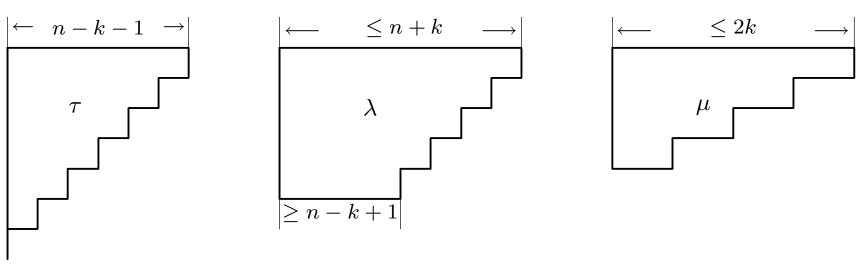

Figure 2.1: Illustration of an element $(\tau, \lambda, \mu) \in P_{n,k}$.

For $k = 0$, we have $P_{n,0} = \{(\tau, \emptyset, \emptyset)\}$, where $\tau = (n-1, n-2, \ldots, 2, 1, 0)$, and for $k > n$, we set $P_{n,k} = \emptyset$. For $k = n-1$ and $k = n$, we have

$$\begin{aligned} P_{n,n-1} &= \{(\tau, \lambda, \mu) \colon \tau = (0),\ 2 \le \lambda_i \le 2n-1,\ \lambda \in D,\ \mu_1 \le 2n-2,\ \mu \in E\}, \\ P_{n,n} &= \{(\tau, \lambda, \mu) \colon \tau = \emptyset,\ 1 \le \lambda_i \le 2n,\ \lambda \in D,\ \mu_1 \le 2n,\ \mu \in E\}. \end{aligned}$$

It should be mentioned that we have imposed the distinction between the partition of with only a zero part and the empty partition. Under this convention, one sees that $\bigcup_{k\ge 0} P_{n,k}$ is a disjoint union. Moreover, the k-th summand $F_{n,k}$ of $F_n(q)$ as given in (1.10) can be viewed as a weighted count of $P_{n,k}$, that is,

$$F_{n,k} = \sum_{(\tau,\lambda,\mu)\in P_{n,k}} (-1)^{\ell(\lambda)} q^{\,|\tau|+|\lambda|+|\mu|}.$$

We now proceed to construct the bijections $\phi_{n,k}$ in (1.7). Let

$$\begin{aligned} A_{n,k} &= P_{n,k} \cup \{2n-1\} \times P_{n-1,k}, \\ B_{n,k} &= P_{n-1,k} \cup \{2n-3\} \times P_{n-2,k}, \\ H_{n,k} &= \{2n-1\} \times P_{n-1,k-1}, \\ H'_{n,k} &= P_{n-1,k-1}. \end{aligned}$$

The following theorem gives a combinatorial telescoping relation for $P_{n,k}$.

Theorem 2.1 *For $n \ge 2$ and $0 \le k \le n-2$, there is a bijection*

$$\phi_{n,k} \colon P_{n,k} \cup \{2n-1\} \times P_{n-1,k-1} \to P_{n-1,k-1} \cup \{2n-3\} \times P_{n-2,k}. \tag{2.1}$$

Proof. For $k = 0$, as $P_{n-1,k-1}$ is the empty set, and the bijection $\phi_{n,0}$ is defined by

$$\phi_{n,0} \colon (\tau, \emptyset, \emptyset) \mapsto (2n-3, (\tau', \emptyset, \emptyset)),$$

where τ' is obtained from τ by removing the first two parts. For example, when $n = 2$, we have $\tau = (1, 0)$ and the triple of partitions $((1,0), \emptyset, \emptyset)$ is mapped to $(1, (\emptyset, \emptyset, \emptyset))$, which belongs to $\{2n-3\} \times P_{n-2,k}$. Because of the zero part, it is always possible to remove first two parts of τ.

For $k > 0$, the bijection $\phi_{n,k}$ is essentially a classification of the set $P_{n,k}$ into four classes, that is,

$$P_{n,k} = A_{n,k} \cup B_{n,k} \cup C_{n,k} \cup P_{n-1,k-1},$$

where

$$\begin{aligned} A_{n,k} &= \{(\tau, \lambda, \mu) \in P_{n,k} \colon \lambda_1 \le n+k-2,\ \mu_1 = 2k\}, \\ B_{n,k} &= \{(\tau, \lambda, \mu) \in P_{n,k} \colon \text{either } n+k \text{ or } n+k-1 \text{ appears in } \lambda, \text{ but not both}\}, \\ C_{n,k} &= \{(\tau, \lambda, \mu) \in P_{n,k} \colon \lambda_1 = n+k,\ \lambda_2 = n+k-1\}. \end{aligned}$$

In other words, for the triple of partitions $(\tau, \lambda, \mu) \in P_{n,k}$, if neither $n+k$ nor $n+k-1$ appears in λ and $2k$ does not appear in μ, then (τ, λ, μ) falls into $P_{n-1,k-1}$. If neither $n+k$ nor $n+k-1$ appears in λ and $2k$ appears in μ, then (τ, λ, μ) falls into $A_{n,k}$. If exactly one of $n+k$ and $n+k-1$ appears in λ, then

(τ, λ, μ) falls into $B_{n,k}$. If both $n+k$ and $n+k-1$ appear in λ, then (τ, λ, μ) falls into $C_{n,k}$.

For $P_{n-2,k}$, we need the following classification

$$P_{n-2,k} = A'_{n,k} \cup B'_{n,k} \cup C'_{n,k} \cup D_{n,k},$$

where

$$A'_{n,k} = \{(\tau, \lambda, \mu) \in P_{n-2,k} \colon \lambda_\ell \geq n-k+1\},$$

$$B'_{n,k} = \{(\tau, \lambda, \mu) \in P_{n-2,k} \colon n-k \text{ or } n-k-1 \text{ appears in } \lambda, \text{ but not both}\},$$

$$C'_{n,k} = \{(\tau, \lambda, \mu) \in P_{n-2,k} \colon \lambda_\ell = n-k-1,\ \lambda_{\ell-1} = n-k,\ \mu_1 = 2k\},$$

$$D_{n,k} = \{(\tau, \lambda, \mu) \in P_{n-2,k} \colon \lambda_\ell = n-k-1,\ \lambda_{\ell-1} = n-k,\ \mu_1 < 2k\}.$$

In other words, for the triple of partitions $(\tau, \lambda, \mu) \in P_{n-2,k}$, if neither $n-k$ nor $n-k-1$ appears in λ , then (τ, λ, μ) falls into $A'_{n,k}$. If exactly one of $n-k$ and $n-k-1$ appears in λ, then (τ, λ, μ) falls into $B'_{n,k}$. If both $n-k$ and $n-k-1$ appear in λ and $2k$ appears in μ, then (τ, λ, μ) falls into $C'_{n,k}$. If both $n-k$ and $n-k-1$ appear in λ and $2k$ does not appear in μ, then (τ, λ, μ) falls into $D'_{n,k}$.

We are now ready to describe the bijection $\phi_{n,k}$. Assume that (τ, λ, μ) is a triple of partitions in $P_{n,k}$.

Case 1: $(\tau, \lambda, \mu) \in P_{n-1,k-1}$. Set $\phi_{n,k}(\tau, \lambda, \mu)$ to be (τ, λ, μ) itself.

Case 2: $(\tau, \lambda, \mu) \in A_{n,k}$. Removing the first two rows from τ and removing the first row from μ, we get τ' and μ', respectively. Let $\lambda' = \lambda$. Then we have $(\tau', \lambda', \mu') \in A'_{n,k}$ and

$$|\tau| + |\lambda| + |\mu| = 2n-3 + |\tau'| + |\lambda'| + |\mu'|.$$

So we obtain a bijection $\varphi_A \colon A_{n,k} \to \{2n-3\} \times A'_{n,k}$ as given by $(\tau, \lambda, \mu) \mapsto (2n-3, (\tau', \lambda', \mu'))$. Figure 2.2 gives an illustration of the correspondence.

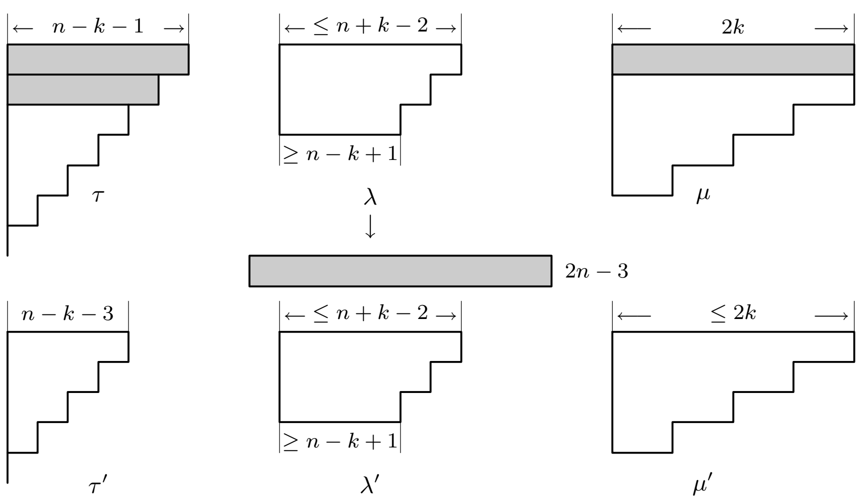

Figure 2.2: The bijection φ_A in Case 2.

Case 3: $(\tau, \lambda, \mu) \in B_{n,k}$. Removing the first two rows from τ, we get τ'. Subtracting $2k$ from the part λ_1 in λ, we get a partition λ'. Let $\mu' = \mu$. Then we have $(\tau', \lambda', \mu') \in B'_{n,k}$ and

$$|\tau| + |\lambda| + |\mu| = 2n - 3 + |\tau'| + |\lambda'| + |\mu'|.$$

Thus we obtain a bijection $\varphi_B \colon B_{n,k} \to \{2n-3\} \times B'_{n,k}$ defined by $(\tau, \lambda, \mu) \mapsto (2n-3, (\tau', \lambda', \mu'))$. See Figure 2.3 for an illustration.

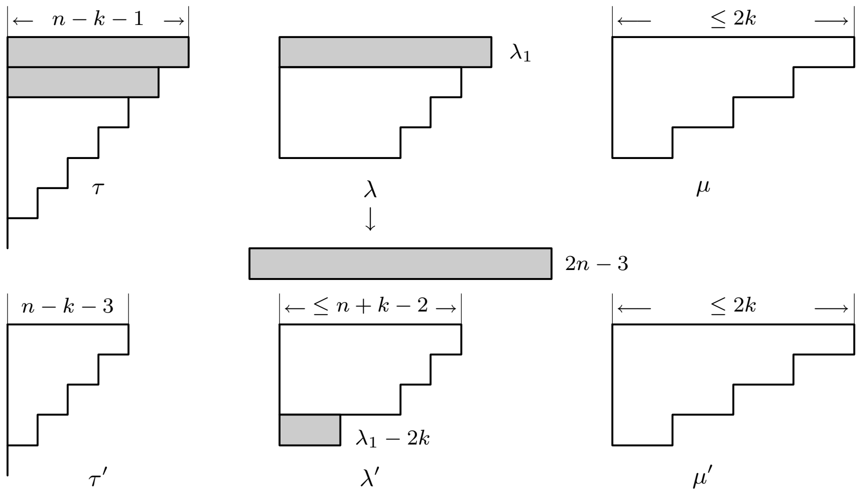

Figure 2.3: The bijection φ_B in Case 3.

Case 4: $(\tau, \lambda, \mu) \in C_{n,k}$. Removing the first two rows from τ, we get τ'. Subtracting $2k$ from the parts $n+k-1$ and $n+k$ in λ, we get a partition λ'. Adding $2k$ to μ as a new part, we get μ'. Then we have $(\tau', \lambda', \mu') \in C'_{n,k}$ and

$$|\tau| + |\lambda| + |\mu| = 2n - 3 + |\tau'| + |\lambda'| + |\mu'|.$$

Thus we obtain a bijection $\varphi_C \colon C_{n,k} \to \{2n-3\} \times C'_{n,k}$ as given by $(\tau, \lambda, \mu) \mapsto (2n-3, (\tau', \lambda', \mu'))$. This case is illustrated in Figure 2.4.

We now consider the quadruples $(2n-1, (\tau, \lambda, \mu))$ in $\{2n-1\} \times P_{n-1,k-1}$. For any $(\tau, \lambda, \mu) \in P_{n-1,k-1}$, remove the first two rows of τ and add two parts $n-k$ and $n-k-1$ to λ to get τ' and λ'. Let $\mu' = \mu$. Then we see that $(\tau', \lambda', \mu') \in D_{n,k}$ and

$$2n - 1 + |\tau| + |\lambda| + |\mu| = 2n - 3 + |\tau'| + |\lambda'| + |\mu'|.$$

Thus we obtain a bijection

$$\varphi_D \colon \{2n-1\} \times P_{n-1,k-1} \to \{2n-3\} \times D_{n,k}$$

as given by $(2n-1, (\tau, \lambda, \mu)) \mapsto (2n-3, (\tau', \lambda', \mu'))$. This case is illustrated by Figure 2.5.

Combining the bijections φ_A, φ_B, φ_C and φ_D, we complete the proof. ∎

In the following theorem, we provide involutions $I_{n,k}$ for $k = n-1$ and $k = n$, where $n \geq 1$.

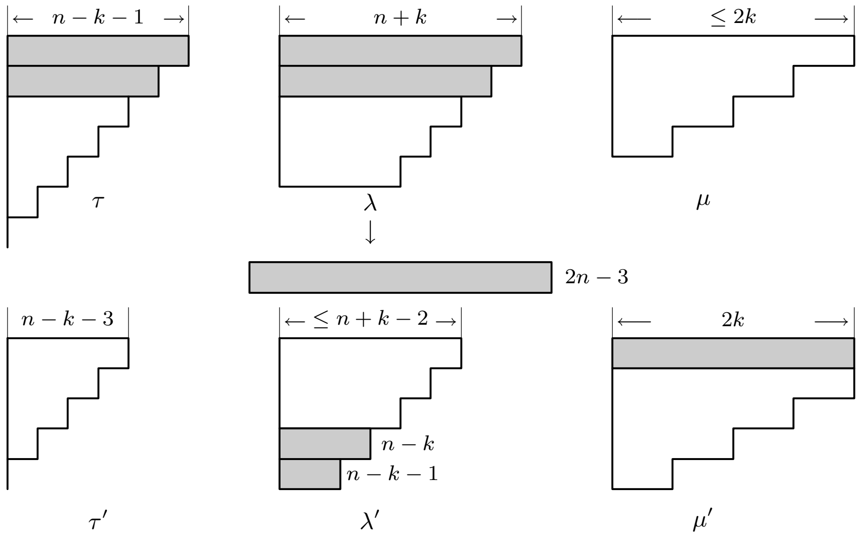

Figure 2.4: The bijection φ_C in Case 4.

Theorem 2.2 *For $n \geq 1$ and for $k = n-1$ or n, there is an involution $I_{n,k}$ on*

$$P_{n,k} \cup \{2n-1\} \times P_{n-1,k-1}$$

with the invariant set $P_{n-1,k-1}$.

Proof. We only give the description of the involution $I_{n,n}$ since $I_{n,n-1}$ can be constructed in the same manner.
Case 1. For $(\emptyset, \lambda, \mu) \in P_{n,n}$, if the first part of λ is $2n$, then move it to μ. Conversely, if μ contains a part $2n$ but λ does not, then move this part from μ back to λ.
Case 2. For $(\emptyset, \lambda, \mu) \in P_{n,n}$ with $\lambda_1 = 2n-1$ and $\mu_1 < 2n$, remove the first part $2n-1$ of λ to get λ', and set

$$I_{n,n}(\emptyset, \lambda, \mu) = (2n-1, (\emptyset, \lambda', \mu),$$

which belongs to $\{2n-1\} \times P_{n-1,n-1}$. Conversely, for

$$(2n-1, (\emptyset, \lambda, \mu)) \in \{2n-1\} \times P_{n-1,n-1},$$

adding a part $2n-1$ to λ, we get λ' and set

$$I_{n,n}(2n-1, (\emptyset, \lambda, \mu)) = (\emptyset, \lambda', \mu),$$

which belongs to $P_{n,n}$.
Case 3. It can be seen that the set of triples $(\emptyset, \lambda, \mu) \in P_{n,n}$ with $\lambda_1 < 2n-1$ and $\mu_1 < 2n$ is exactly $P_{n-1,n-1}$. So we set $P_{n-1,n-1}$ to be the invariant set of the involution.

In summary, we obtain an involution on $P_{n,n} \cup \{2n-1\} \times P_{n-1,n-1}$ with the invariant set $P_{n-1,n-1}$. ∎

The weight functions w on $P_{n,k}$, $\{2n-1\} \times P_{n-1,k}$ and $\{2n-3\} \times P_{n-2,k}$

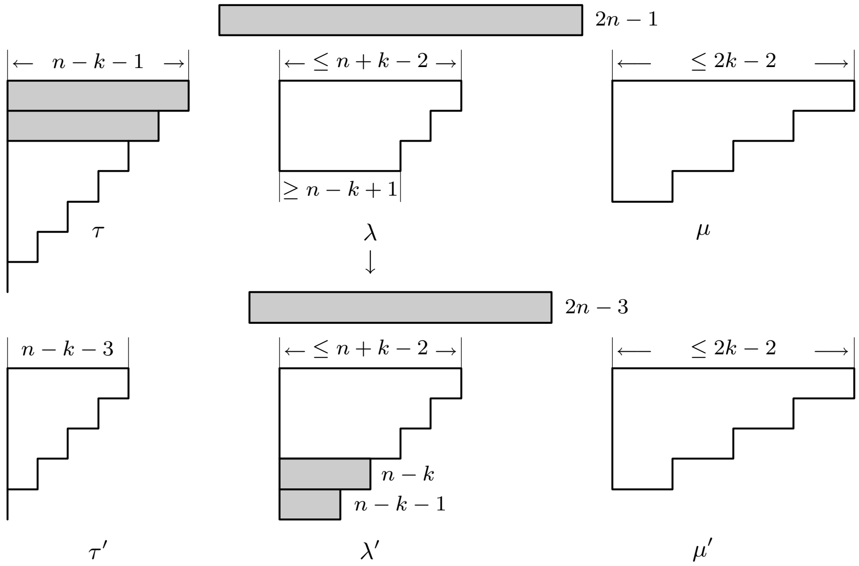

Figure 2.5: The bijection φ_D on $\{2n-1\} \times P_{n-1,k-1}$.

are defined by

$$\begin{aligned} w(\tau, \lambda, \mu) &= (-1)^{\ell(\lambda)}\, q^{\,|\,\tau\,|+|\,\lambda\,|+|\,\mu\,|}, \\ w(2n-1, (\tau, \lambda, \mu)) &= q^{2n-1}\, (-1)^{\ell(\lambda)}\, q^{\,|\,\tau\,|+|\,\lambda\,|+|\,\mu\,|}, \\ w(2n-3, (\tau, \lambda, \mu)) &= q^{2n-3}\, (-1)^{\ell(\lambda)}\, q^{\,|\,\tau\,|+|\,\lambda\,|+|\,\mu\,|}. \end{aligned}$$

One sees that the bijections and involutions in Theorems 2.1 and 2.2 are weight preserving. For example, for $n = 8$ and $k = 4$, let

$$\tau = (3, 2, 1, 0), \quad \lambda = (10, 9, 8) \quad \text{and} \quad \mu = (8, 8, 4).$$

It can be verified that $(\tau, \lambda, \mu) \in A_{8,4}$. Applying the bijection $\phi_{8,4}$ we get

$$\tau' = (1, 0), \quad \lambda' = (10, 9, 8) \quad \text{and} \quad \mu' = (8, 4).$$

Moreover, it can be checked that

$$w(\tau, \lambda, \mu) = w(13, (\tau', \lambda', \mu')) = -q^{53}.$$

Since $\phi_{n,k}$ and $I_{n,k}$ are weight preserving, we get the following recurrence relation for $F_n(q)$.

Corollary 2.3 *For $n \geq 2$, we have*

$$F_n(q) + (q^{2n-1} - 1)F_{n-1}(q) - q^{2n-3}F_{n-2}(q) = 0. \tag{2.2}$$

It is easy to verify that

$$(-1)^n q^{n^2} \sum_{j=-n}^{n} (-1)^j q^{-j^2} \tag{2.3}$$

also satisfies recurrence relation (2.2). Taking the initial values into consideration, we are led to the identity of Andrews.

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