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The Method of Combinatorial Telescoping

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Abstract. We present a method for proving q-series identities by combinatorial telescoping, in the sense that one can transform a bijection or a classification of combinatorial objects into a telescoping relation. We shall illustrate this method by giving a combinatorial proof of Watson's identity which implies the Rogers-Ramanujan identities.

Keywords. Watson's identity, Sylvester's identity, Rogers-Ramanujan identities, combinatorial telescoping

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1 Introduction

The main objective of this paper is to present the method of combinatorial telescoping for proving q-series identities. A benchmark of this approach is the classical identity of Watson which implies the Rogers-Ramanujan identities.

There have been many combinatorial proofs of the Rogers-Ramanujan identities. Schur [13] provided an involution for the following identity which is equivalent to the first Rogers-Ramanujan identity:

$$\prod_{k=1}^{\infty}(1-q^k)\left(1+\sum_{k=1}^{\infty}\frac{q^{k^2}}{(1-q)(1-q^2)\cdots(1-q^k)}\right)=\sum_{k=-\infty}^{\infty}(-1)^k q^{k(5k-1)/2}.$$

Andrews [1] proved the Rogers-Ramanujan identities by introducing the notion of k-partitions. Garsia and Milne [9] gave a bijection by using the involution principle.

Bressoud and Zeilberger [5,6] provided a different involution principle proof based on an algebraic proof due to Bressoud [4]. Boulet and Pak [3] found a combinatorial proof which relies on the symmetry properties of a generalization of Dyson's rank.

Let us consider a summation of the following form

$$\sum_{k=0}^{\infty}(-1)^k f(k). \tag{1.1}$$

Suppose that $f(k)$ is a weighted count of a set A_k, that is,

$$f(k) = \sum_{\alpha\in A_k} w(\alpha).$$

Motivated by the idea of creative telescoping of Zeilberger [16], we aim to find sets B_k and H_k with a weight assignment w such that there is a weight preserving bijection

$$\phi_k \colon A_k \longrightarrow B_k \cup H_k \cup H_{k+1}, \tag{1.2}$$

where $\cup$ stands for disjoint union. Since ϕ_k and ϕ_{k+1} are weight preserving, both $\phi_k^{-1}(H_{k+1})$ and $\phi_{k+1}^{-1}(H_{k+1})$ have the same weight as H_{k+1}. Realizing that $\phi_k^{-1}(H_{k+1}) \subseteq A_k$ and $\phi_{k+1}^{-1}(H_{k+1}) \subseteq A_{k+1}$, they cancel each other in the sum (1.1). More precisely, if we set

$$g(k) = \sum_{\alpha\in B_k} w(\alpha) \quad \text{and} \quad h(k) = \sum_{\alpha\in H_k} w(\alpha),$$

then the bijection (1.2) implies that

$$f(k) = g(k) + h(k) + h(k+1). \tag{1.3}$$

To see that the above equation is indeed a telescoping relation with respect to the sum (1.1), let

$$f'(k) = (-1)^k f(k), \quad g'(k) = (-1)^k g(k), \quad h'(k) = (-1)^k h(k).$$

Thus we have

$$f'(k) = g'(k) + h'(k) - h'(k+1). \tag{1.4}$$

Just like the conditions for the creative telescoping, we suppose that $H_0 = \emptyset$ and H_k vanishes for sufficiently large k. Summing (1.4) over k, we deduce the following relation

$$\sum_{k=0}^{\infty}(-1)^k f(k) = \sum_{k=0}^{\infty}(-1)^k g(k), \tag{1.5}$$

which is often an identity we wish to establish.

The above approach to proving an identity like (1.5) is called combinatorial telescoping. It can be seen that the bijections ϕ_k lead to a correspondence between $A = \bigcup_{k=0}^{\infty} A_k$ and $B = \bigcup_{k=0}^{\infty} B_k$ after the cancelations of H_k's. To be more specific, we can derive a bijection

$$\phi\colon A \setminus \bigcup_{k=0}^{\infty} \phi_k^{-1}(H_k \cup H_{k+1}) \longrightarrow B$$

and an involution

$$\psi\colon \bigcup_{k=0}^{\infty} \phi_k^{-1}(H_k \cup H_{k+1}) \longrightarrow \bigcup_{k=0}^{\infty} \phi_k^{-1}(H_k \cup H_{k+1}),$$

given by $\phi(\alpha) = \phi_k(\alpha)$ if $\alpha \in A_k$ and

$$\psi(\alpha) = \begin{cases} \phi_{k-1}^{-1}\phi_k(\alpha), & \text{if } \alpha \in \phi_k^{-1}(H_k), \\ \phi_{k+1}^{-1}\phi_k(\alpha), & \text{if } \alpha \in \phi_k^{-1}(H_{k+1}). \end{cases}$$

In the examples of this paper, the set A_k is of the following form

$$A_k = \bigcup_{n=0}^{\infty} A_{n,k}.$$

Fix an integer n, for any nonnegative integer k, we can establish a bijection $\phi_{n,k}$ such that the corresponding set $B_{n,k}$ is related to $A_{n,k}, A_{n-1,k}, \ldots, A_{n-r,k}$ for an integer r. Let

$$F_{n,k} = \sum_{\alpha \in A_{n,k}} w(\alpha)$$

be a weighted count of the set $A_{n,k}$, and let

$$F_n = \sum_{k=0}^{\infty} (-1)^k F_{n,k}.$$

By (1.5), the bijections $\{\phi_{n,k}\}_{k=0}^{\infty}$ imply a recurrence relation of F_n, which leads to an explicit expression $u(n)$ for F_n by iteration. Finally, we deduce the following identity

$$\sum_{k=0}^{\infty} (-1)^k f(k) = \sum_{k=0}^{\infty} (-1)^k \sum_{n=0}^{\infty} F_{n,k} = \sum_{n=0}^{\infty} F_n = \sum_{n=0}^{\infty} u(n). \tag{1.6}$$

As a simple example, one can easily give a combinatorial telescoping proof of the classical identity of Gauss, see also, [7, 11, 12]:

$$\sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix} = \begin{cases} 0, & n \text{ odd}, \\ (1-q)(1-q^3)\cdots(1-q^{n-1}), & n \text{ even}. \end{cases}$$

Let us consider the following reformulation

$$\sum_{k=0}^{n}(-1)^k \frac{1}{(q;q)_k(q;q)_{n-k}} = \begin{cases} 0, & n \text{ odd}, \\ \dfrac{1}{(1-q^2)(1-q^4)\cdots(1-q^n)}, & n \text{ even}. \end{cases} \tag{1.7}$$

Let

$$P_{n,k} = \{(\lambda,\mu)\colon \lambda_1 \le k,\ \mu_1 \le n-k\},$$

where λ and μ are partitions, and let

$$H_{n,k} = \{(\lambda,\mu) \in P_{n,k}\colon m_k(\lambda) < m_{n-k}(\mu)\},$$

where $m_k(\lambda)$ denotes the number of occurrences of the part k in λ and we adopt the convention that $m_0(\lambda) = +\infty$. By definition, $H_{n,k} = \emptyset$ for $k = 0$ or $k > n$. For any integers $n \ge 1$ and $k \ge 0$, we shall construct a bijection

$$\phi_{n,k}\colon P_{n,k} \longrightarrow \{0, n, 2n, \ldots\} \times P_{n-2,k} \cup H_{n,k} \cup H_{n,k+1}.$$

Let $(\lambda,\mu) \in P_{n,k}$. If $m_k(\lambda) < m_{n-k}(\mu)$, then $(\lambda,\mu) \in H_{n,k}$. In this case, $\phi_{n,k}\big((\lambda,\mu)\big) = (\lambda,\mu)$. If $m_k(\lambda) \ge m_{n-k}(\mu)$, we let $m_{n-k}(\mu) = t$. In this case, if $\mu_{t+1} = n-1-k$, we increase each of the first t parts of λ by one and decrease each of the first t parts of μ by one. It is easily seen that the resulting pair of partitions (λ',μ') belongs to $H_{n,k+1}$ and we set $\phi_{n,k}\big((\lambda,\mu)\big) = (\lambda',\mu')$. Finally, if $\mu_{t+1} \le n-2-k$, then we set

$$\phi_{n,k}\big((\lambda,\mu)\big) = \big(tn, (\hat{\lambda},\hat{\mu})\big) \in \{0, n, 2n, \ldots\} \times P_{n-2,k},$$

where $\hat{\lambda} = (\lambda_{t+1}, \lambda_{t+2}, \ldots)$ and $\hat{\mu} = (\mu_{t+1}, \mu_{t+2}, \ldots)$ are the partitions obtained from λ and μ by removing the first t parts. Define the weight function w on $P_{n,k}$ and $\{0, n, 2n, \ldots\} \times P_{n-2,k}$ as follows

$$w(\lambda,\mu) = q^{|\lambda|+|\mu|}, \quad \text{and} \quad w(tn,(\lambda,\mu)) = q^{tn+|\lambda|+|\mu|},$$

where $|\lambda| = \lambda_1 + \lambda_2 + \cdots$. It can be checked that $\phi_{n,k}$ is weight preserving. Hence we obtain the following recurrence relation

$$F_n(q) = \frac{1}{1-q^n} F_{n-2}(q), \tag{1.8}$$

where $F_n(q)$ denotes the sum on the left hand side of (1.7). By iteration of (1.8), we arrive at (1.7).

It should be noted that the bijections $\phi_{n,k}$ lead to an involution on $P_{n,k}$, which can be considered as a variation of the involution given by Chen, Hou and Lascoux [7].

In Section 2, we use the idea of combinatorial telescoping to give a proof of Watson's identity [15] in the following form, see also [10, Section 2.7],

$$\sum_{k=0}^{\infty}(-1)^k \frac{1-aq^{2k}}{(q;q)_k(aq^k;q)_\infty} a^{2k} q^{k(5k-1)/2} = \sum_{n=0}^{\infty} \frac{a^n q^{n^2}}{(q;q)_n}, \tag{1.9}$$

where

$$(a;q)_k = (1-a)(1-aq)\cdots(1-aq^{k-1}), \quad \text{and} \quad (a;q)_\infty = \prod_{i=0}^{\infty}(1-aq^i).$$

Setting $a=1$, Watson's identity reduces to Schur's identity [3]

$$\frac{1}{(q;q)_\infty} \sum_{k=-\infty}^{\infty} (-1)^k q^{k(5k-1)/2} = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n}.$$

Applying Jacobi's triple product identity to the left hand side, we are led to the first Rogers-Ramanujan identity. Similarly, setting $a=q$ in Watson's identity yields the second Rogers-Ramanujan identity.

Here is a sketch of the proof. Assume that the k-th summand regardless of the sign on the left hand side of (1.9) is the weight of a set P_k. We further divide P_k into a disjoint union of subsets $P_{n,k}, n=0,1,\ldots$, by considering the expansion of the summand in the parameter a. For a positive integer n and a nonnegative integer k, we can construct a bijection

$$\phi_{n,k}\colon P_{n,k} \to \{n\} \times P_{n,k} \cup \{2n-1\} \times P_{n-1,k} \cup H_{n,k} \cup H_{n,k+1}. \tag{1.10}$$

Let

$$F_n(a,q) = \sum_{k=0}^{\infty}(-1)^k \sum_{\alpha \in P_{n,k}} w(\alpha).$$

The bijections $\phi_{n,k}$ yield a recurrence relation

$$F_n(a,q) = q^n F_n(a,q) + aq^{2n-1} F_{n-1}(a,q), \quad n \geq 1.$$

By iteration, we find that $F_n(a,q) = a^n q^{n^2}/(q;q)_n$, and hence (1.9) holds.

As another example, it can be seen that the method of combinatorial telescoping also applies to Sylvester's identity [14]

$$\sum_{k=0}^{\infty}(-1)^k q^{k(3k+1)/2} x^k \frac{1-xq^{2k+1}}{(q;q)_k(xq^{k+1};q)_\infty} = 1. \tag{1.11}$$

This identity has been investigated by Andrews [1, 2].

2 Watson's identity

In this section, we shall use Watson's identity as an example to illustrate the idea of combinatorial telescoping. Let us recall some definitions concerning partitions. A *partition* is a non-increasing finite sequence of positive integers $\lambda = (\lambda_1, \ldots, \lambda_\ell)$. The integers λ_i are called the *parts* of λ. The sum of parts and the number of parts are denoted by $|\lambda| = \lambda_1 + \cdots + \lambda_\ell$ and $\ell(\lambda) = \ell$, respectively. The number of k-parts in λ is denoted by $m_k(\lambda)$. The special partition with no parts is denoted by $\varnothing$. We shall use diagrams to represent partitions and use columns instead of rows to represent parts.

Set

$$P_k = \{(\tau, \lambda, \mu)\colon \tau = (k^{2k}, k-1, \ldots, 2, 1),\ \lambda_{\ell(\lambda)} \geq k,\ \lambda_i \neq 2k,\ \mu_1 \leq k\}, \tag{2.1}$$

where k^{2k} denotes $2k$ occurrences of a part k. In other words, τ is a trapezoid partition with $|\tau| = k(5k-1)/2$, λ is a partition with parts at least k but not equal to $2k$, and μ is a partition with parts at most k. In particular, we have $P_0 = \{(\varnothing, \lambda, \varnothing)\}$. It is clear that the k-th summand of the left hand side of (1.9) without sign can be viewed as the weight of P_k, that is,

$$\sum_{(\tau,\lambda,\mu)\in P_k} a^{\ell(\lambda)+2k} q^{|\tau|+|\lambda|+|\mu|}.$$

According to the exponent of a in the above definition, we divide P_k into a disjoint union of subsets

$$P_{n,k} = \{(\tau, \lambda, \mu) \in P_k\colon \ell(\lambda) = n - 2k\}, \tag{2.2}$$

with $P_{n,0} = \{(\varnothing, \lambda, \varnothing) \in P_0\colon \ell(\lambda) = n\}$ and $P_{n,k} = \emptyset$ for $n < 2k$. The elements of $P_{n,k}$ are illustrated in Figure 2.1.

We have the following combinatorial telescoping relation for $P_{n,k}$.

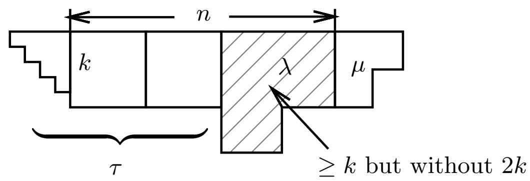

Figure 2.1: The diagram $(\tau, \lambda, \mu) \in P_{n,k}$

Theorem 2.1 *Let*

$$H_{n,k} = \{(\tau, \lambda, \mu) \in P_{n,k} \colon m_k(\lambda) + 2 > m_k(\mu)\}. \tag{2.3}$$

Then, for any positive integer n and any nonnegative integer k, there is a bijection

$$\phi_{n,k} \colon P_{n,k} \longrightarrow \{n\} \times P_{n,k} \cup \{2n-1\} \times P_{n-1,k} \cup H_{n,k} \cup H_{n,k+1}. \tag{2.4}$$

Proof. The bijection is essentially a classification of $P_{n,k}$ into four cases. Let (τ, λ, μ) be a 3-tuple of partitions in $P_{n,k}$.

Case 1. $m_k(\lambda) + 2 > m_k(\mu)$. In this case, $(\tau, \lambda, \mu) \in H_{n,k}$ and the image of (τ, λ, μ) is defined to be itself.

Case 2. $m_k(\lambda) + 2 \leq m_k(\mu)$ and $m_{2k+1}(\lambda) = 0$. Denote the set of 3-tuples (τ, λ, μ) in this case by $U_{n,k}$. Note that

$$U_{n,0} = \{(\varnothing, \lambda, \varnothing) \in P_{n,0} \colon m_1(\lambda) = 0\}.$$

Since $m_k(\mu) \geq m_k(\lambda) + 2$, we can remove $(m_k(\lambda) + 2)$ k-parts from μ to generate a partition μ'. In the meantime, we change each k-part of λ into a $2k$-part in order to obtain a partition λ' whose minimal part is strictly greater than k.

Next, we decrease each part of λ' by one in order to produce a partition λ'' whose minimal part is greater than or equal to k. Since λ contains no parts equal to $2k+1$, we see that λ'' contains no parts equal to $2k$. Thus we obtain a bijection $\varphi_1 \colon U_{n,k} \to \{n\} \times P_{n,k}$ defined by $(\tau, \lambda, \mu) \mapsto (n, (\tau, \lambda'', \mu'))$. This case is illustrated by Figure 2.2.

Case 3. $m_k(\lambda) + 2 \leq m_k(\mu)$, $m_{2k+1}(\lambda) > 0$ and $m_{k+1}(\lambda) + m_{2k+2}(\lambda) = 0$. Denote the set of 3-tuples (τ, λ, μ) in this case by $V_{n,k}$. We remark that when $k = 0$, one 1-part is regarded as a $(2k+1)$-part and the other 1-parts are regarded as $(k+1)$-parts so that

$$V_{n,0} = \{(\varnothing, \lambda, \varnothing) \in P_{n,0} \colon m_1(\lambda) = 1 \text{ and } m_2(\lambda) = 0\}.$$

Let λ', μ' be given as in Case 2. We can remove one $(2k+1)$-part from λ' and decrease each of the remaining parts by two in order to obtain λ''. This leads to a bijection

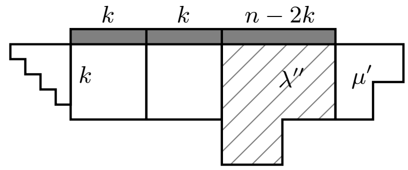

Figure 2.2: The resulting partition under the bijection φ_1.

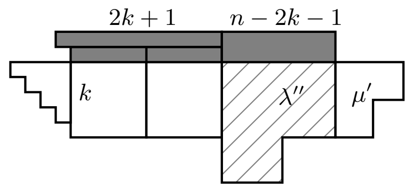

Figure 2.3: The resulting partition under the bijection φ_2.

$\varphi_2 \colon V_{n,k} \to \{2n-1\} \times P_{n-1,k}$ as given by $(\tau, \lambda, \mu) \mapsto (2n-1, (\tau, \lambda'', \mu'))$. See Figure 2.3 for an illustration.

Case 4. $m_k(\lambda) + 2 \le m_k(\mu)$, $m_{2k+1}(\lambda) > 0$ and $m_{k+1}(\lambda) + m_{2k+2}(\lambda) > 0$. Denote the set of 3-tuples (τ, λ, μ) in this case by $W_{n,k}$. As in Case 3, we have

$$W_{n,0} = \{(\varnothing, \lambda, \varnothing) \in P_{n,0} \colon m_1(\lambda) > 0 \text{ and } m_1(\lambda) + m_2(\lambda) > 1\}.$$

Let λ', μ' be given as in Case 2. We can change each $(2k+2)$-part of λ' to a $(k+1)$-part and add $m_{2k+2}(\lambda')$ $(k+1)$-parts to μ'. Denote the resulting partitions by λ'' and μ''. Then we have

$$m_{k+1}(\lambda'') = m_{k+1}(\lambda) + m_{2k+2}(\lambda) > 0, \quad m_{k+1}(\mu'') = m_{2k+2}(\lambda). \tag{2.5}$$

Now remove one $(k+1)$-part and one $(2k+1)$-part from λ'' to obtain λ'''. By (2.5), we find

$$m_{k+1}(\lambda''') = m_{k+1}(\lambda'') - 1 \ge m_{k+1}(\mu'') - 1.$$

Moreover, it is clear that

$$|\lambda| + |\mu| = 2k + (k+1) + (2k+1) + |\lambda'''| + |\mu''|.$$

Let τ' be the trapezoid partition of size $k+1$. So we obtain a bijection $\varphi_3 \colon W_{n,k} \to H_{n,k+1}$ defined by $(\tau, \lambda, \mu) \mapsto (\tau', \lambda''', \mu'')$. This case is illustrated in Figure 2.4. ∎

Assign a weight function w on $P_{n,k}$, $\{n\} \times P_{n,k}$ and $\{2n-1\} \times P_{n-1,k}$ as follows:

$$\begin{aligned} w\big(\tau, \lambda, \mu\big) &= a^n q^{|\tau|+|\lambda|+|\mu|}, \\ w\big(n, (\tau, \lambda, \mu)\big) &= q^n \cdot a^n q^{|\tau|+|\lambda|+|\mu|}, \\ w\big(2n-1, (\tau, \lambda, \mu)\big) &= aq^{2n-1} \cdot a^{n-1} q^{|\tau|+|\lambda|+|\mu|}. \end{aligned}$$

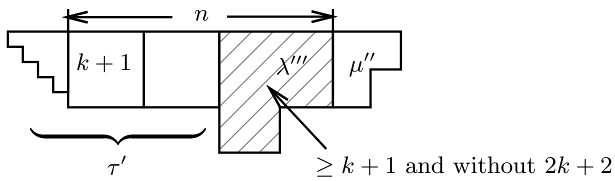

Figure 2.4: The resulting partition under the bijection φ_3.

Observe that the bijections φ_1, φ_2 and φ_3 are weight preserving. In addition, $H_{n,0} = \emptyset$ and $H_{n,k} = \emptyset$ for $k > \frac{n}{2}$. Thus the bijections $\phi_{n,k}$ immediately lead to a recurrence relation of $F_n(a,q)$ defined as follows.

Corollary 2.2 *Let*

$$F_n(a,q) = \sum_{k=0}^{\infty} (-1)^k \sum_{(\tau,\lambda,\mu)\in P_{n,k}} a^n q^{|\tau|+|\lambda|+|\mu|}. \tag{2.6}$$

Then, for any positive integer n, we have

$$F_n(a,q) = q^n F_n(a,q) + aq^{2n-1} F_{n-1}(a,q). \tag{2.7}$$

Since $F_0(a,q) = 1$, by iteration we find that

$$F_n(a,q) = \frac{aq^{2n-1}}{1-q^n} F_{n-1}(a,q) = \frac{a^2q^{4n-4}}{(1-q^n)(1-q^{n-1})} F_{n-2}(a,q) = \cdots = \frac{a^n q^{n^2}}{(q;q)_n}.$$

Summing over n, we arrive at Watson's identity (1.9).

3 Sylvester's identity

In this section, we describe the approach of combinatorial telescoping for Sylvester's identity (1.11). Define

$$Q_{n,k} = \{(\tau,\lambda)\colon \tau = (k^{k+1}, k-1, \ldots, 2, 1), \lambda_i \neq 2k+1, m_{>k}(\lambda) = n-k\},$$

where $m_{>k}(\lambda)$ denotes the number of parts of λ which are greater than k. See Figure 3.1 for an illustration. In particular, we have

$$Q_{n,0} = \{(\varnothing,\lambda)\colon \lambda_i \neq 1, \ell(\lambda) = n\}.$$

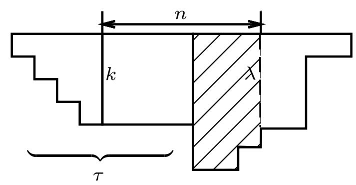

Figure 3.1: The diagram of $(\tau,\lambda)\in Q_{n,k}$.

Let

$$H_{n,k}=\{(\tau,\lambda)\in Q_{n,k}\colon m_{k+1}(\lambda)\geq m_k(\lambda)\}.$$

Then, for each positive integer n and each nonnegative integer k, we have a bijection

$$\phi_{n,k}:Q_{n,k}\longrightarrow\{n\}\times Q_{n,k}\cup H_{n,k}\cup H_{n,k+1},$$

which is a classification of $Q_{n,k}$ into three cases. Let $(\tau,\lambda)\in Q_{n,k}$.

Case 1. $m_{k+1}(\lambda)\geq m_k(\lambda)$. In this case, $(\tau,\lambda)\in H_{n,k}$ and the image of (τ,λ) under $\phi_{n,k}$ is defined to be itself.

Case 2. $m_{k+1}(\lambda)<m_k(\lambda)$ and $m_{2k+2}(\lambda)=0$. Denote the set of pairs (τ,λ) in this case by $U_{n,k}$. We remove one k-part from λ. Then, for each $(k+1)$-part of λ, we can add it to a k-part to form a $(2k+1)$-part. Finally, we decrease each part greater than $k+1$ by one to generate a partition λ'. Since $m_{2k+2}(\lambda)=0$, we see that $(\tau,\lambda')\in Q_{n,k}$. So we obtain a bijection $\varphi_1\colon U_{n,k}\to\{n\}\times Q_{n,k}$ given by $(\tau,\lambda)\mapsto(n,(\tau,\lambda'))$.

Case 3. $m_{k+1}(\lambda)<m_k(\lambda)$ and $m_{2k+2}(\lambda)>0$. Denote the set of pairs (τ,λ) in this case by $V_{n,k}$. We first remove one k-part and one $(2k+2)$-part from λ and add them to τ to form a partition τ'. Here τ' is a trapezoid partition of size $k+1$. Then for each $(k+1)$-part of λ we combine it with a k-part to form a $(2k+1)$-part. Finally we decompose each $(2k+3)$-part of λ into a $(k+1)$-part and a $(k+2)$-part to form a partition λ'. Since $m_{2k+3}(\lambda')=0$, we obtain a bijection $\varphi_2\colon V_{n,k}\to H_{n,k+1}$ defined by $(\tau,\lambda)\mapsto(\tau',\lambda')$.

It is not difficult to see that Sylvester's identity follows from the bijections $\phi_{n,k}$. Let

$$I_n(q)=\sum_{k=0}^{\infty}(-1)^k\sum_{(\tau,\lambda)\in Q_{n,k}}q^{|\tau|+|\lambda|}.$$

Noting that $H_{n,0}=\emptyset$ because of the definition $m_0(\lambda)=+\infty$, the bijections $\phi_{n,k}$ lead to the recurrence relation

$$I_n(q)=q^nI_n(q),$$

which implies that $I_n(q)=0$ for $n\geq 1$. Clearly $I_0(q)=1$, and hence Sylvester's identity holds.

To conclude this paper, we notice that both Watson's identity and Sylvester's identity can be verified by employing the q-Zeilberger algorithm for infinite q-series developed by Chen, Hou and Mu [8]. Let

$$f(a)=\sum_{k=0}^{\infty}(-1)^k\frac{(1-aq^{2k})}{(q;q)_k(aq^k;q)_\infty}a^{2k}q^{k(5k-1)/2}.$$

Denote the k-th summand of $f(a)$ by $F_k(a)$. The q-Zeilberger algorithm gives that

$$F_k(a)-F_k(aq)-aqF_k(aq^2)=H_{k+1}(a)-H_k(a), \tag{3.1}$$

where

$$H_k(a)=(-1)^k\frac{(-1-q^k+aq^{2k})}{(q;q)_{k-1}(aq^k;q)_\infty}a^{2k}q^{k(5k-1)/2}.$$

Summing (3.1) over k, we find that

$$f(a)=f(aq)+aqf(aq^2).$$

Extracting the coefficients of a^n leads to the same recurrence relation as (2.7). It is easily checked that the right hand side of (1.9) satisfies the same recursion. By Theorem 3.1 of Chen, Hou and Mu [8], one sees that (1.9) holds for any a provided that it is valid for the trivial case $a=0$. Similarly, let

$$f(x)=\sum_{k=0}^{\infty}(-1)^kq^{k(3k+1)/2}x^k\frac{1-xq^{2k+1}}{(q;q)_k(xq^{k+1};q)_\infty}.$$

The q-Zeilberger algorithm gives that

$$F_k(x)-F_k(xq)=H_{k+1}(x)-H_k(x), \tag{3.2}$$

where $F_k(x)$ is the k-th summand of $f(x)$ and

$$H_k(x)=(-1)^{k+1}\frac{q^{k(3k+1)/2}x^k}{(q;q)_{k-1}(xq^{k+1};q)_\infty}. \tag{3.3}$$

Summing (3.2) over k, we deduce that $f(x)=f(xq)$, which implies $f(x)=1$.

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