

*J. Symbolic Comput.*

44 (2009), no. 8, 960-971

## The $q$ -WZ Method for Infinite Series

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**Abstract.** Motivated by the telescoping proofs of two identities of Andrews and Warnaar, we find that infinite  $q$ -shifted factorials can be incorporated into the implementation of the  $q$ -Zeilberger algorithm in the approach of Chen, Hou and Mu to prove nonterminating basic hypergeometric series identities. This observation enables us to extend the  $q$ -WZ method to identities on infinite series. We give the  $q$ -WZ pairs for some classical identities such as the  $q$ -Gauss sum, the  ${}_6\phi_5$  sum, the Ramanujan's  ${}_1\psi_1$  sum and Bailey's  ${}_6\psi_6$  sum.

**Keywords:** basic hypergeometric series, the  $q$ -Gosper algorithm, the  $q$ -Zeilberger algorithm, the  $q$ -WZ method.

**AMS Classification:** 33D15, 33F10.

## 1 Introduction

The objective of this paper is to give an extension of the  $q$ -WZ method to nonterminating basic hypergeometric series identities. We will follow the standard notation on  $q$ -series [9] and always assume  $|q| < 1$ . The  $q$ -shifted factorials  $(a; q)_n$  and  $(a; q)_\infty$  are

defined by

$$\begin{aligned}
(a; q)_n &= \begin{cases} 1, & \text{if } n = 0, \\ (1-a)(1-aq)\cdots(1-aq^{n-1}), & \text{if } n \geq 1, \end{cases} \\
(a; q)_{-n} &= \frac{1}{(aq^{-n}; q)_n}, \\
(a; q)_\infty &= (1-a)(1-aq)(1-aq^2)\cdots, \\
(a_1, a_2, \dots, a_k; q)_n &= (a_1; q)_n (a_2; q)_n \cdots (a_k; q)_n.
\end{aligned}$$

An  ${}_r\phi_s$  basic hypergeometric series is defined by

$${}_r\phi_s \left[ \begin{matrix} a_1, a_2, \dots, a_r \\ b_1, b_2, \dots, b_s \end{matrix}; q, z \right] := \sum_{n=0}^{\infty} \frac{(a_1, a_2, \dots, a_r; q)_n}{(q, b_1, \dots, b_s; q)_n} \left[ (-1)^n q^{\binom{n}{2}} \right]^{1+s-r} z^n, \quad (1.1)$$

where  $q \neq 0$  when  $r > s + 1$ . Further, an  ${}_r\psi_s$  bilateral basic hypergeometric series is defined by

$${}_r\psi_s \left[ \begin{matrix} a_1, a_2, \dots, a_r \\ b_1, b_2, \dots, b_s \end{matrix}; q, z \right] := \sum_{n=-\infty}^{\infty} \frac{(a_1, a_2, \dots, a_r; q)_n}{(b_1, b_2, \dots, b_s; q)_n} \left[ (-1)^n q^{\binom{n}{2}} \right]^{s-r} z^n. \quad (1.2)$$

It is assumed that  $q$ ,  $z$  and the parameters are such that each term of the series is well-defined. We say that an  ${}_r\phi_s$  series terminates if only a finite number of terms contribute. Otherwise, we say that the series  ${}_r\phi_s$  is nonterminating.

For the ordinary nonterminating hypergeometric identities, Gessel [10] and Koornwinder [14] provided computer proofs of Gauss' summation formula and Saalschütz' summation formula by means of a combination of Zeilberger's algorithm and asymptotic estimates. Vidunas [19] (see also Koepf [12] and Koornwinder [15]) presented a method to evaluate  ${}_2F_1\left(\begin{smallmatrix} a, b \\ c \end{smallmatrix} \middle| -1\right)$  when  $c - a + b$  is an integer. Recently, Chen, Hou and Mu [8] developed an approach to proving nonterminating basic hypergeometric identities based on the  $q$ -Zeilberger algorithm [13]. In this paper we will show how to apply the  $q$ -WZ method to prove nonterminating basic hypergeometric summation formulas by finding the  $q$ -WZ pairs. We will give some examples including the  $q$ -Gauss sum, the very-well-poised  ${}_6\phi_5$  sum, the Ramanujan's  ${}_1\psi_1$  sum and Bailey's very-well-poised series  ${}_6\psi_6$  sum [9].

## 2 The Andrews-Warnaar Identities

In this paper, we give telescoping proofs of the following two identities on partial theta functions:

$$\left( \sum_{n=0}^{\infty} (-1)^n a^n q^{\binom{n}{2}} \right) \left( \sum_{n=0}^{\infty} (-1)^n b^n q^{\binom{n}{2}} \right) = (q, a, b; q)_{\infty} \sum_{n=0}^{\infty} \frac{(abq^{n-1}; q)_n}{(q, a, b; q)_n} q^n, \quad (2.1)$$

$$1 + \sum_{n=1}^{\infty} (-1)^n q^{\binom{n}{2}} (a^n + b^n) = (a, b, q; q)_{\infty} \sum_{n=0}^{\infty} \frac{(ab/q; q)_{2n}}{(q, a, b, ab; q)_n} q^n. \quad (2.2)$$

The identity (2.2) was first proved by Warnaar in [20]. Andrews and Warnaar [6] derived the identity (2.1) and used it to prove (2.2).

As will be seen, the telescoping proofs suggest that the approach developed Chen, Hou and Mu [8] for proving nonterminating basic hypergeometric identities can be extended so that infinite  $q$ -shifted factorials can be allowed in a  $q$ -hypergeometric term. This idea immediately leads to an extension of the  $q$ -WZ method to identities on infinite series.

Note that the formula (2.2) is a generalization of the well-known Jacobi's triple product identity. When  $b = q/a$ , we get the Jacobi's triple product identity

$$\sum_{n=-\infty}^{\infty} (-1)^n a^n q^{\binom{n}{2}} = (a, q/a, q; q)_{\infty}, \quad (2.3)$$

where  $|q| < 1$  and  $a \neq 0$ .

We now describe how to prove the identities (2.1) and (2.2) by the telescoping method. Let us consider (2.1) first. Put

$$f(a) = \left( \sum_{n=0}^{\infty} (-1)^n q^{\binom{n}{2}} a^n \right) \left( \sum_{n=0}^{\infty} (-1)^n q^{\binom{n}{2}} b^n \right).$$

Note that the second factor does not contain the parameter  $a$ . It is easily verified that

$$f(a) = (1 - a)f(aq) + aqf(aq^2). \quad (2.4)$$

We proceed to show that the right hand side of (2.1) satisfies the same recurrence relation. Of course, we still need to verify the boundary conditions. Let

$$g(a) = \sum_{n=0}^{\infty} D_n(a), \quad \text{where} \quad D_n(a) = (q, a, b; q)_{\infty} \frac{(abq^{n-1}; q)_n q^n}{(q, a, b; q)_n}.$$

Then it is necessary to show that

$$g(a) - (1 - a)g(aq) - aqg(aq^2) = 0. \quad (2.5)$$

Here comes the key step of finding a telescoping relation for  $D_n(a)$ . Note that, for any  $n \geq 0$ , we have

$$\begin{aligned} & D_n(a) - (1 - a)D_n(aq) - aqD_n(aq^2) \\ &= \frac{(abq^n; q)_n (q, a, b; q)_\infty q^n}{(q, a, b; q)_n} \left( \frac{1 - abq^{n-1}}{1 - abq^{2n-1}} - \frac{1 - a}{1 - aq^n} - \frac{aq(1 - abq^{2n})}{(1 - aq^{n+1})(1 - aq^n)(1 - abq^n)} \right) \\ &= \frac{(abq^n; q)_n (q, a, b; q)_\infty q^n}{(q, a, b; q)_n} \left( \frac{a(1 - q^n)(1 - bq^{n-1})}{(1 - aq^n)(1 - abq^{2n-1})} - \frac{aq(1 - abq^{2n})}{(1 - aq^{n+1})(1 - aq^n)(1 - abq^n)} \right) \\ &= z_{n+1} - z_n, \end{aligned} \quad (2.6)$$

where

$$z_n = - \frac{(1 - q^n)(1 - bq^{n-1})(abq^n; q)_n (q, a, b; q)_\infty aq^n}{(1 - aq^n)(1 - abq^{2n-1})(q, a, b; q)_n}.$$

The above relation reveals that the infinite  $q$ -shifted factorial  $(q, a, b; q)_\infty$  can be incorporated into the telescoping relation and this step can be automated by the  $q$ -Gosper algorithm. Moreover, one sees that infinite  $q$ -shifted factorials can be incorporated into the  $q$ -Zeilberger algorithm so that the approach of Chen, Hou and Mu [8] can be extended to terms containing infinite  $q$ -shifted factorials. In particular, one can make the  $q$ -WZ method work for nonterminating hypergeometric series.

Now, let us return our attention to the proof of (2.1). Clearly,  $z_0 = 0$ . It is also easily seen that  $\lim_{n \rightarrow +\infty} z_n = 0$ . Summing (2.6) over the non-negative integers, we obtain the recurrence relation (2.5). In order to show that  $f(a) = g(a)$ , we will use the recurrence relation of  $f(a) - g(a)$  to reach this goal.

Let  $H(a) = f(a) - g(a)$ . From the recurrence relations for  $f(a)$  and  $g(a)$ , it follows that  $H(a)$  satisfies the recurrence relation

$$H(a) = (1 - a)H(aq) + aqH(aq^2). \quad (2.7)$$

Iterating the above relation yields that

$$H(a) = A_n H(aq^{n+1}) + B_n H(aq^{n+2}), \quad (2.8)$$

where  $A_n$  and  $B_n$  are given by

$$A_0 = (1 - a), \quad B_0 = aq, \quad A_1 = (1 - a)(1 - aq) + aq, \quad B_1 = (1 - a)aq^2,$$

and

$$A_{n+1} = (1 - aq^{n+1})A_n + aq^{n+1}A_{n-1}, \quad B_{n+1} = aq^{n+2}A_n, \quad n \geq 1.$$

Hence we have

$$A_{n+1} - A_n = -aq^{n+1}(A_n - A_{n-1}),$$

which implies that

$$\begin{aligned} |A_{n+1} - A_n| &= |(-1)^n a^n q^{\binom{n+2}{2}-1}| |(A_1 - A_0)| \\ &\leq |a^n q^{\binom{n+2}{2}-1}| (|A_1| + |A_0|). \end{aligned}$$

So, for fixed  $a$  and  $|q| < 1$ , the limit  $\lim_{n \rightarrow +\infty} A_n$  exists. Since  $B_{n+1} = aq^{n+2}A_n$ , the limit  $\lim_{n \rightarrow +\infty} B_n$  also exists. Again, by the relation (2.8), we find

$$H(a) = H(0) \left( \lim_{n \rightarrow +\infty} A_n + \lim_{n \rightarrow +\infty} B_n \right).$$

It remains to show that  $H(0) = 0$ , that is,

$$\sum_{n=0}^{\infty} (-1)^n b^n q^{\binom{n}{2}} = (q, b; q)_{\infty} \sum_{n=0}^{\infty} \frac{q^n}{(q, b; q)_n}. \quad (2.9)$$

We can use the telescoping method to prove (2.9). Let

$$G(b) = \sum_{n=0}^{\infty} (-1)^n b^n q^{\binom{n}{2}} - (q, b; q)_{\infty} \sum_{n=0}^{\infty} \frac{q^n}{(q, b; q)_n}.$$

It is easy to check that

$$G(b) = (1 - b)G(bq) + bqG(bq^2).$$

We aim to show that  $G(b) = 0$ . Since  $G(b)$  satisfies the same recurrence relation as  $H(a)$ , it suffices to confirm  $G(0) = 0$ , that is,

$$(q; q)_{\infty} \sum_{n=0}^{\infty} \frac{q^n}{(q; q)_n} = 1,$$

which is special case of Euler's identity [9, P. 354]

$$\sum_{n=0}^{\infty} \frac{z^n}{(q; q)_n} = \frac{1}{(z; q)_{\infty}}, \quad |z| < 1.$$

Indeed, the relation (2.9) is a limiting case of Heine's transformation of  ${}_2\phi_1$ . For completeness, we give a proof based on Euler's identities:

$$\begin{aligned}
(q, b; q)_\infty \sum_{n=0}^{\infty} \frac{q^n}{(q, b; q)_n} &= (q; q)_\infty \sum_{m=0}^{\infty} \frac{q^m}{(q; q)_m} \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}} (-bq^m)^n}{(q; q)_n} \\
&= (q; q)_\infty \sum_{n=0}^{\infty} \frac{(-1)^n b^n q^{\binom{n}{2}}}{(q; q)_n} \sum_{m=0}^{\infty} \frac{(q^{n+1})^m}{(q; q)_m} \\
&= \sum_{n=0}^{\infty} (-1)^n b^n q^{\binom{n}{2}}. \tag{2.10}
\end{aligned}$$

Thus, we have verified that  $H(a) = 0$ . This completes the proof.  $\blacksquare$

We remark that once the recurrence relation (2.7) is derived, one can also use the theorem of Chen, Hou and Mu [8, Theorem 3.1] to prove the existence of the limits of  $A_n$  and  $B_n$ .

We next present a telescoping proof of (2.2). Let

$$f(a) = 1 + \sum_{n=1}^{\infty} (-1)^n q^{\binom{n}{2}} (a^n + b^n).$$

It is easily seen that

$$(1 + aq)f(a) - (1 - a^2q)f(aq) - (aq + a^2q)f(aq^2) = (q - 1)a. \tag{2.11}$$

Let

$$g(a) = \sum_{n=0}^{\infty} D_n(a), \quad \text{where} \quad D_n(a) = (q, a, b; q)_\infty \frac{(ab/q; q)_{2n} q^n}{(q, a, b, ab; q)_n}.$$

It will be shown that

$$(1 + aq)g(a) - (1 - a^2q)g(aq) - (aq + a^2q)g(aq^2) = (q - 1)a. \tag{2.12}$$

Since

$$\begin{aligned}
&\frac{q^n - abq^{n-1}}{1 - abq^{2n-1}} - \frac{(1 - a^2q)(1 - ab)q^n}{(1 + aq)(1 - aq^n)(1 - abq^n)} \\
&\quad - \frac{(a^2q + aq)(1 - abq^{2n})(1 - abq)q^n}{(1 + aq)(1 - aq^n)(1 - aq^{n+1})(1 - abq^n)(1 - abq^{n+1})} \\
&= \frac{(1 - abq^{2n})(-1 + q + abq^{n+1} + a^2bq^{n+2} - aq^{n+2} - q^{n+1} - a^2bq^{2n+2} + a^2bq^{2n+3})a}{(1 - aq^{n+1})(1 - aq^n)(1 - abq^n)(1 + aq)(1 - abq^{n+1})}
\end{aligned}$$

$$- \frac{(-1 + q + abq^n + a^2bq^{n+1} - aq^{n+1} - q^n - a^2bq^{2n} + a^2bq^{2n+1})a(1 - bq^{n-1})(1 - q^n)}{(1 - aq^n)(1 - abq^{2n-1})(1 + aq)(1 - abq^n)},$$

multiplying both sides by

$$\frac{(ab; q)_{2n}(a, b, q; q)_\infty}{(q, a, b, ab; q)_n},$$

we deduce that

$$D_n(a) - \frac{(1 - a^2q)}{1 + aq} D_n(aq) - \frac{(a^2q + aq)}{1 + aq} D_n(aq^2) = z_{n+1} - z_n, \quad (2.13)$$

where

$$z_n = \frac{(-1 + q - aq^{n+1} - q^n + a^2bq^{n+1} + abq^n - a^2bq^{2n} + a^2bq^{2n+1})a}{(1 - aq^n)(1 - abq^{2n-1})} \\ \times \frac{(1 - bq^{n-1})(1 - q^n)(ab; q)_{2n}(q, a, b; q)_\infty}{(1 + aq)(1 - abq^n)(ab; q)_n(q, a, b; q)_n}.$$

Clearly,  $z_0 = 0$  and  $\lim_{n \rightarrow +\infty} z_n = \frac{(q-1)a}{1+aq}$ . Summing (2.13) over nonnegative integers, we obtain the recurrence relation (2.12).

Let  $H(a) = f(a) - g(a)$ . Then  $H(a)$  satisfies the following recurrence relation

$$H(a) = \frac{1 - a^2q}{1 + aq} H(aq) + \frac{aq + a^2q}{1 + aq} H(aq^2). \quad (2.14)$$

By iteration, we obtain

$$H(a) = A_n H(aq^{n+1}) + B_n H(aq^{n+2}), \quad (2.15)$$

where  $A_n$  and  $B_n$  are given by

$$A_0 = \frac{1 - a^2q}{1 + aq}, \quad A_1 = \frac{1 + a^3q^3}{1 + aq^2}, \\ B_0 = \frac{aq + a^2q}{1 + aq}, \quad B_1 = \frac{aq^2(1 - a^2q)}{(1 + aq^2)},$$

and for  $n \geq 1$ ,

$$A_{n+1} = \frac{1 - a^2q^{2n+3}}{1 + aq^{n+2}} A_n + \frac{aq^{n+1} + a^2q^{2n+1}}{1 + aq^{n+1}} A_{n-1}, \quad (2.16)$$

$$B_{n+1} = \frac{aq^{n+2} + a^2q^{2n+3}}{1 + aq^{n+2}} A_n. \quad (2.17)$$

Based on the above recurrence relations, one can deduce that both  $\lim_{n \rightarrow +\infty} A_n$  and  $\lim_{n \rightarrow +\infty} B_n$  exist. We note that Zeilberger [26] has shown that

$$A_n = \frac{1 + (-1)^{n+1} a^{n+2} q^{\binom{n+2}{2}}}{1 + aq^{n+1}}$$

and

$$B_n = \frac{aq^{n+1} \left(1 + (-1)^n a^{n+1} q^{\binom{n+1}{2}}\right)}{1 + aq^{n+1}}.$$

Now we see that the limits  $\lim_{n \rightarrow +\infty} A_n$  and  $\lim_{n \rightarrow +\infty} B_n$  exist. By the relation (2.15), we deduce that

$$H(a) = H(0) \left( \lim_{n \rightarrow +\infty} A_n + \lim_{n \rightarrow +\infty} B_n \right).$$

The identity (2.10) implies that  $f(0) = g(0)$ . So we have  $H(a) = 0$ . This completes the proof.  $\blacksquare$

We also note that once the recurrence relation (2.14) is established, one may assume that  $|a| < 1$  and may use the theorem in Chen, Hou and Mu [8, Theorem 3.1] to the existence of the limits of  $A_n$  and  $B_n$ . Moreover, we may drop the assumption  $|a| < 1$  by analytic continuation.

### 3 The $q$ -WZ Pairs for Infinite Series

Our approach to the  $q$ -WZ method for infinite series can be described as follows. The key step is to construct  $q$ -WZ pairs for infinite sums. Suppose that we aim to prove an identity of the form:

$$\sum_{k=N_0}^{\infty} F_k(a_1, a_2, \dots, a_t) = R(a_1, a_2, \dots, a_t), \quad (3.1)$$

where  $t$  is a positive integer, and the sum is either a unilateral or bilateral basic hypergeometric series, namely,  $N_0 = 0$  or  $N_0 = -\infty$ ,  $R(a_1, a_2, \dots, a_t)$  is either zero or a quotient of two products of infinite  $q$ -shifted factorials.

First, we set some parameters, say,  $a_1, \dots, a_p$ , ( $1 \leq p \leq t$ ) to  $a_1 q^n, \dots, a_p q^n$ , so that we get

$$\sum_{k=N_0}^{\infty} F_k(a_1 q^n, \dots, a_p q^n, a_{p+1}, \dots, a_t) = R(a_1 q^n, \dots, a_p q^n, a_{p+1}, \dots, a_t). \quad (3.2)$$



If  $R(a_1q^n, \dots, a_pq^n, a_{p+1}, \dots, a_t) \neq 0$ , set

$$F(n, k) = \frac{F_k(a_1q^n, \dots, a_pq^n, a_{p+1}, \dots, a_t)}{R(a_1q^n, \dots, a_pq^n, a_{p+1}, \dots, a_t)}.$$

Otherwise, set

$$F(n, k) = F_k(a_1q^n, \dots, a_pq^n, a_{p+1}, \dots, a_t).$$

Our goal is to show that

$$\sum_{k=N_0}^{\infty} F(n, k) = \text{constant}, \quad n = 0, 1, 2, \dots \quad (3.3)$$

The constant can be determined by setting  $n = 0$  and setting  $a_1, a_2, \dots, a_t$  to special values. We claim that the above goal can be achieved by adopting the  $q$ -WZ method for finite sums.

Let us recall the boundary and limit conditions for the  $q$ -WZ-method. Let  $f(n)$  denote the left hand side of (3.3), i.e.,

$$f(n) = \sum_{k=N_0}^{\infty} F(n, k)$$

and we aim to show that

$$f(n) = \text{constant}$$

for every nonnegative integer  $n$ . To this end, it suffices to show that  $f(n+1) - f(n) = 0$  for every nonnegative integer  $n$ . This can be done by finding  $G(n, k)$  such that

$$F(n+1, k) - F(n, k) = G(n, k+1) - G(n, k). \quad (3.4)$$

A pair of functions  $(F(n, k), G(n, k))$  that satisfy (3.4) is called a  $q$ -WZ pair. Once a  $q$ -WZ pair is found, one can check the boundary and limit conditions to ensure that  $f(n)$  equals the claimed constant. Here are the conditions:

(C1) For each integer  $n \geq 0$ ,  $\lim_{k \rightarrow \pm\infty} G(n, k) = 0$ .

(C2) For each integer  $k$ , the limit

$$f_k = \lim_{n \rightarrow \infty} F(n, k) \quad (3.5)$$

exists and is finite.

(C3) We have  $\lim_{L \rightarrow \infty} \sum_{n \geq 0} G(n, -L) = 0$ .

The WZ method can be formally stated as follows.

**Theorem 3.1** (Wilf and Zeilberger [22]). *Assume that  $(F(n, k), G(n, k))$  is a WZ pair (3.4). If (C1) holds, then we have*

$$\sum_k F(n, k) = \text{constant}, \quad n = 0, 1, 2, \dots \quad (3.6)$$

If (C2) and (C3) hold, then we have the companion identity

$$\sum_{n=0}^{\infty} G(n, k) = \sum_{j \leq k-1} (f_j - F(0, j)), \quad (3.7)$$

where  $f_j$  is defined by (3.5).

We now explain how to compute the desired  $q$ -WZ pair for the identity (3.1). In fact, it can be produced by applying the  $q$ -Gasper algorithm to  $F(n+1, k) - F(n, k)$ . It should be noted that  $F(n+1, k) - F(n, k)$  is a  $q$ -hypergeometric term with respect to  $q^k$ , even if  $F(n, k)$  contains infinite  $q$ -shifted factorials such as  $(aq^n; q)_{\infty}$ . Obviously,  $F(n+1, k) - F(n, k)$  is a  $q$ -hypergeometric term when  $R(a_1, \dots, a_t) = 0$ . Assume that  $R(a_1, \dots, a_t) \neq 0$ . Let

$$\begin{aligned} M_1 &= \frac{R(a_1 q^{n+1}, \dots, a_p q^{n+1}, a_{p+1}, \dots, a_t)}{R(a_1 q^n, \dots, a_p q^n, a_{p+1}, \dots, a_t)}, \\ M_2 &= \frac{F_{k+1}(a_1 q^{n+1}, \dots, a_p q^{n+1}, a_{p+1}, \dots, a_t)}{F_k(a_1 q^{n+1}, \dots, a_p q^{n+1}, a_{p+1}, \dots, a_t)}, \\ M_3 &= \frac{F_{k+1}(a_1 q^n, \dots, a_p q^n, a_{p+1}, \dots, a_t)}{F_k(a_1 q^{n+1}, \dots, a_p q^{n+1}, a_{p+1}, \dots, a_t)}, \\ M_4 &= \frac{F_k(a_1 q^n, \dots, a_p q^n, a_{p+1}, \dots, a_t)}{F_k(a_1 q^{n+1}, \dots, a_p q^{n+1}, a_{p+1}, \dots, a_t)}. \end{aligned}$$

Since  $M_1$  is a rational function in  $q^n$  and is independent of  $k$ ,  $M_2, M_3, M_4$  are all rational functions in  $q^k$ . Observe that

$$\frac{F(n+1, k+1) - F(n, k+1)}{F(n+1, k) - F(n, k)} = \frac{M_2 - M_1 M_3}{1 - M_1 M_4} \quad (3.8)$$

is a rational function in  $q^k$ , i.e.,  $F(n+1, k) - F(n, k)$  is a  $q$ -hypergeometric term with respect to  $q^k$ . It is necessary to mention that even if  $F(n, k)$  contains infinite  $q$ -shifted factorials of the form  $(aq^n; q)_{\infty}$ , the quotient (3.8) no longer contains the  $q$ -shifted

factorial  $(aq^n; q)_\infty$  and it is still a rational function in  $q^k$ . Consequently, we can employ the  $q$ -Gosper algorithm to determine whether  $G(n, k)$  exists. Nevertheless, it is also necessary to note that  $G(n, k)$  contains infinite  $q$ -shifted factorials if  $F(n, k)$  does.

There is another way to look at the above procedure. Suppose that  $F(n, k)$  contains an infinite  $q$ -shifted factorial  $(a; q)_\infty$ , where  $a$  is a chosen parameter for the substitution  $a \rightarrow aq^n$ . If we set  $G'(n, k) = R(aq^n)G(n, k)$ . Then the equation (3.4) becomes

$$F(n+1, k)R(aq^n) - F(n, k)R(aq^n) = G'(n, k+1) - G'(n, k).$$

It is evident that the infinite  $q$ -shifted factorial  $(aq^n; q)_\infty$  will disappear in the above equation, and one can use the  $q$ -Gosper algorithm to find a  $q$ -WZ pair if it exists.

We now take the  $q$ -binomial theorem [9, P. 354] as an example to explain the above steps:

$$\sum_{k=0}^{\infty} \frac{(a; q)_k}{(q; q)_k} z^k = \frac{(az; q)_\infty}{(z; q)_\infty}, \quad |z| < 1. \quad (3.9)$$

In this case, we have

$$F_k(a) = \frac{(a; q)_k}{(q; q)_k} z^k, \quad R(a) = \frac{(az; q)_\infty}{(z; q)_\infty}.$$

We choose the parameter  $a$ , and substitute  $a$  with  $aq^n$ . Then we set

$$F(n, k) = \frac{F_k(aq^n)}{R(aq^n)} = \frac{(aq^n; q)_k (z; q)_\infty}{(q; q)_k (azq^n; q)_\infty} z^k.$$

In order to find  $G(n, k)$  such that (3.4) holds, it is easily checked that  $F(n+1, k) - F(n, k)$  is a  $q$ -hypergeometric term. By examining the  $q$ -Gosper algorithm, one sees that it is capable to deal with the input  $F(n+1, k) - F(n, k)$ , or we can set

$$G'(n, k) = R(aq^n)G(n, k)$$

and find a solution of the equation

$$(1 - azq^n) \frac{(aq^{n+1}; q)_k}{(q; q)_k} z^k - \frac{(aq^n; q)_k}{(q; q)_k} z^k = G'(n, k+1) - G'(n, k). \quad (3.10)$$

Finally, we obtain the  $q$ -WZ pair

$$F(n, k) = \frac{(aq^n; q)_k (z; q)_\infty}{(q; q)_k (azq^n; q)_\infty} z^k,$$

$$G(n, k) = -\frac{(aq^n; q)_k (z; q)_\infty (a - aq^k)}{(q; q)_k (azq^n; q)_\infty (1 - aq^n)} q^n z^k.$$

If  $|z| < 1$ , it is easy to see that  $F(n, k)$  and  $G(n, k)$  satisfy the conditions (C1), (C2) and (C3). By (3.6),

$$\sum_{k=-\infty}^{\infty} F(n, k) = \sum_{k=0}^{\infty} F(n, k) = \text{constant}, \quad n = 0, 1, 2, \dots$$

Setting  $z = 0$  yields that the constant equals 1. Setting  $n = 0$ , we have

$$\sum_{k=0}^{\infty} F(0, k) = \text{constant} = 1.$$

By (3.7), we get the companion identity of (3.9)

$$\sum_{j=0}^k \frac{(a; q)_j}{(q; q)_j} z^j = (az; q)_{\infty} \sum_{j=0}^k \frac{z^j}{(q; q)_j} + \frac{az^{k+1}(a; q)_{k+1}}{(q; q)_k} \sum_{n=0}^{\infty} \frac{(az; q)_n (aq^{k+1}; q)_n}{(a; q)_{n+1}} q^n.$$

We remark that our algorithm depends on the choice of parameters  $a_{i_1}, \dots, a_{i_k}$ . For a given choice of parameters, it is not guaranteed that one can find a  $q$ -WZ pair. Nevertheless, this approach applies to many classical identities.

We now give a few more examples.

**Example 3.1.** The  $q$ -Gauss sum [9, P. 354]:

$$\sum_{k=0}^{\infty} \frac{(a, b; q)_k}{(q, c; q)_k} \left( \frac{c}{ab} \right)^k = \frac{(c/a, c/b; q)_{\infty}}{(c, c/ab; q)_{\infty}}, \quad |c/ab| < 1. \quad (3.11)$$

By computation we get the  $q$ -WZ pair

$$F(n, k) = \frac{(b, aq^n; q)_k (c/ab, cq^n; q)_{\infty}}{(q, cq^n; q)_k (c/a, cq^n/b; q)_{\infty}} \left( \frac{c}{ab} \right)^k,$$

$$G(n, k) = -\frac{(a - aq^k)(b, aq^n; q)_k (c/ab, cq^n; q)_{\infty}}{(1 - aq^n)(q, cq^n; q)_k (c/a, cq^n/b; q)_{\infty}} \left( \frac{c}{ab} \right)^k q^n.$$

If  $|c/ab| < 1$ , it is easy to verify that the two functions  $(F(n, k), G(n, k))$  satisfy the relation (3.4) and conditions (C1), (C2) and (C3). By (3.6), we have

$$\sum_{k=-\infty}^{\infty} F(n, k) = \sum_{k=0}^{\infty} F(n, k) = \text{constant}, \quad n = 0, 1, 2, \dots$$

Setting  $c = 0$  and  $n = 0$ , we find that the constant equals 1, and we have

$$\sum_{k=0}^{\infty} F(0, k) = \text{constant} = 1.$$

After simplification, we obtain the identity (3.11).

By (3.7), we obtain the companion identity of (3.11)

$$-\sum_{n=0}^{\infty} \frac{(a - aq^k)(b, aq^n; q)_k (c/ab, cq^n; q)_{\infty}}{(1 - aq^n)(q, cq^n; q)_k (c/a, cq^n/b; q)_{\infty}} \left(\frac{c}{ab}\right)^k q^n = \sum_{j \leq k-1} (f_j - F(0, j)), \quad (3.12)$$

where

$$f_j = \lim_{n \rightarrow \infty} F(n, j) = \frac{(b; q)_j (c/ab; q)_{\infty}}{(q; q)_j (c/a; q)_{\infty}} \left(\frac{c}{ab}\right)^j,$$

which can be restated as

$$\begin{aligned} \sum_{j=0}^k \frac{(a, b; q)_j}{(q, c; q)_j} \left(\frac{c}{ab}\right)^j &= \frac{(c/b; q)_{\infty}}{(c; q)_{\infty}} \sum_{j=0}^k \frac{(b; q)_j}{(q; q)_j} \left(\frac{c}{ab}\right)^j \\ &+ \frac{(a, b; q)_{k+1} c^{k+1}}{(q; q)_k (c; q)_{k+1} a^k b^{k+1}} \sum_{n=0}^{\infty} \frac{(aq^{k+1}, c/b; q)_n}{(a; q)_{n+1} (cq^{k+1}; q)_n} q^n. \end{aligned}$$

**Example 3.2.** The sum of a very-well-poised  ${}_6\phi_5$  series [9, P. 356]:

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{(1 - aq^{2k})(a, b, c, d; q)_k}{(1 - a)(q, aq/b, aq/c, aq/d; q)_k} \left(\frac{aq}{bcd}\right)^k \\ = \frac{(aq, aq/bc, aq/bd, aq/cd; q)_{\infty}}{(aq/b, aq/c, aq/d, aq/bcd; q)_{\infty}}, \quad |aq/bcd| < 1. \end{aligned} \quad (3.13)$$

By computation we get the following  $q$ -WZ pair:

$$\begin{aligned} F(n, k) &= \frac{(1 - aq^{n+2k})(c, d, aq^n, bq^n; q)_k}{(1 - aq^n)(q, aq/b, aq^{n+1}/c, aq^{n+1}/d; q)_k} \\ &\quad \times \frac{(aq/b, aq/bcd, aq^{n+1}/c, aq^{n+1}/d; q)_{\infty}}{(aq/bc, aq/bd, aq^{n+1}, aq^{n+1}/cd; q)_{\infty}} \left(\frac{aq}{bcd}\right)^k, \\ G(n, k) &= \frac{(c, d; q)_k (a/b, a/bcd; q)_{\infty}}{(q, a/b; q)_k (aq^n, aq^n/cd; q)_{\infty}} \end{aligned}$$

$$\begin{aligned} & \times \frac{(aq^n, bq^n; q)_k (aq^n/c, aq^n/d; q)_\infty}{(aq^n/c, aq^n/d; q)_k (a/bd, a/bc; q)_\infty} \\ & \times \frac{(a-bc)(a-bd)(aq^n-cd)(1-q^k)}{(a-bcd)(bq^n-1)(aq^{n+k}-c)(aq^{n+k}-d)} \left(\frac{aq}{bcd}\right)^k q^n. \end{aligned}$$

It is easily seen that  $F(n, k)$  and  $G(n, k)$  satisfy the conditions (C1), (C2) and (C3). Therefore, by (3.6), we have  $\sum_{k=0}^{\infty} F(n, k)$  is a constant. Setting  $n = 0$  and  $a = 0$ , we find that the constant equals 1. Thus we have

$$\sum_{k=0}^{\infty} F(0, k) = \text{constant} = 1,$$

which is nothing but (3.13). Since

$$f_k = \frac{(c, d; q)_k (aq/b, aq/bcd; q)_\infty}{(q, aq/b; q)_k (aq/bc, aq/bd; q)_\infty} \left(\frac{aq}{bcd}\right)^k$$

and

$$F(0, j) = \frac{(1-aq^{2j})(a, b, c, d; q)_j (aq/b, aq/c, aq/d, aq/bcd; q)_\infty}{(1-a)(q, aq/b, aq/c, aq/d; q)_j (aq, aq/bc, aq/bd, aq/cd; q)_\infty} \left(\frac{aq}{bcd}\right)^j,$$

by (3.7), we obtain the companion identity

$$\begin{aligned} & \sum_{j=0}^k \frac{(1-aq^{2j})(a, b, c, d; q)_j}{(1-a)(q, aq/b, aq/c, aq/d; q)_j} \left(\frac{aq}{bcd}\right)^k \\ & = \frac{(aq, aq/cd; q)_\infty}{(aq/c, aq/d; q)_\infty} \sum_{j=0}^k \frac{(c, d; q)_j}{(q, aq/b; q)_j} \left(\frac{aq}{bcd}\right)^j \\ & + \frac{b(aq; q)_k (b, c, d; q)_{k+1}}{(q, aq/b; q)_k (aq/c, aq/d; q)_{k+1}} \left(\frac{aq}{bcd}\right)^{k+1} \\ & \times \sum_{n=0}^{\infty} \frac{(aq/cd; q)_n (aq^{k+1}, bq^{k+1}; q)_n}{(b; q)_{n+1} (aq^{k+2}/c, aq^{k+2}/d; q)_n} q^n. \end{aligned}$$

**Example 3.3.** The Ramanujan's  ${}_1\psi_1$  sum [9, P. 357]

$${}_1\psi_1(a; b; q, z) = \frac{(q, b/a, az, q/az; q)_\infty}{(b, q/a, z, b/az; q)_\infty}, \quad |b/a| < |z| < 1. \quad (3.14)$$

In this case, we find that

$$F(n, k) = \frac{(aq^n; q)_k (z, b/az, bq^n, q^{1-n}/a; q)_\infty}{(bq^n; q)_k (q, b/a, azq^n, q^{1-n}/az; q)_\infty} z^k,$$

$$G(n, k) = \frac{(z, b/az, bq^n, q^{-n}/a; q)_\infty (aq^n; q)_k (1 - azq^n)}{(q, b/a, azq^n, q^{-n}/az; q)_\infty (bq^n; q)_k (z - azq^n)} z^k.$$

If  $|b/a| < |z| < 1$ , utilizing the following identity

$$(a; q)_{-n} = \frac{(-q/a)^n q^{\binom{n}{2}}}{(q/a; q)_n}, \quad n = 0, 1, 2, \dots, \quad (3.15)$$

we can verify that  $G(n, k)$  satisfies the condition (C1). It follows that (3.6),

$$\sum_{k=-\infty}^{\infty} F(n, k) = \text{constant}, \quad n = 0, 1, 2, \dots \quad (3.16)$$

Setting  $n = 0$ ,  $b = q$  and utilizing  $q$ -binomial theorem (3.9), we see that the constant equals 1. Setting  $n = 0$ , we obtain the identity (3.14). However, we note that the conditions for the companion identity do not hold in this case.

**Example 3.4.** The sum of a very-well-poised  ${}_6\psi_6$  series [9, P. 357]:

$$\sum_{k=-\infty}^{\infty} \frac{(1 - aq^{2k})(b, c, d, e; q)_k}{(1 - a)(aq/b, aq/c, aq/d, aq/e; q)_k} \left( \frac{a^2 q}{bcde} \right)^k$$

$$= \frac{(aq, aq/bc, aq/bd, aq/be, aq/cd, aq/ce, aq/de, q, q/a; q)_\infty}{(aq/b, aq/c, aq/d, aq/e, q/b, q/c, q/d, q/e, a^2 q/bcde; q)_\infty}. \quad (3.17)$$

We obtain the following  $q$ -WZ pair:

$$F(n, k) = \frac{(1 - aq^{n+2k})(d, e, bq^n, cq^n; q)_k (aq/b, aq/c; q)_\infty}{(1 - aq^n)(aq/b, aq/c, aq^{n+1}/d, aq^{n+1}/e; q)_k (aq/bd, aq/be; q)_\infty}$$

$$\times \frac{(q/d, q/e, a^2 q/bcde, aq^{n+1}/d, aq^{n+1}/e, q^{1-n}/b, q^{1-n}/c; q)_\infty}{(q, aq/cd, aq/ce, aq^{n+1}, aq^{n+1}/de, q^{1-n}/a, aq^{1-n}/bc; q)_\infty} \left( \frac{a^2 q}{bcde} \right)^k,$$

$$G(n, k) = \frac{(d, e, bq^n, cq^n; q)_k (a/b, a/c, 1/e, a^2/bcde, 1/d; q)_\infty}{(a/b, a/c, aq^n/d, aq^n/e; q)_k (q, a/bd, a/be, a/cd, a/ce; q)_\infty}$$

$$\times \frac{(aq^n/d, aq^n/e, q^{-n}/b, q^{-n}/c; q)_\infty (-1 + aq^n)}{(aq^n, aq^n/de, aq^{-n}/bc, q^{-n}/a; q)_\infty (1 - bq^n)(1 - cq^n)}$$

$$\times \frac{(a-bd)(a-be)(a-cd)(a-ce)(aq^n-de)q^n}{(aq^{n+k}-d)(aq^{n+k}-e)(a-ad)(1-e)(a^2-bcde)} \left(\frac{a^2q}{bcde}\right)^k.$$

Since  $|a^2q/bcde| < 1$ , from the identity (3.15) it follows that  $G(n, k)$  satisfies the condition (C1). By (3.6), we find

$$\sum_{k=-\infty}^{\infty} F(n, k) = \text{constant}, \quad n = 0, 1, 2, \dots \quad (3.18)$$

In order to determine the constant, we set  $n = 0$  and  $b = a$ . From the  ${}_6\phi_5$  summation formula (3.13), we see that the constant equals

$$\begin{aligned} \sum_{k=-\infty}^{\infty} F(0, k) &= \sum_{k=0}^{\infty} \frac{(1-aq^{2k})(a, c, d, e; q)_k}{(1-a)(aq/c, aq/d, aq/e; q)_k} \\ &\quad \times \frac{(aq, aq/cd, aq/ce, aq/de; q)_{\infty}}{(aq/c, aq/d, aq/e, aq/cde; q)_{\infty}} \left(\frac{aq}{cde}\right)^k = 1, \end{aligned}$$

which can be restated as (3.17). Nevertheless, we note that the conditions for the companion identity do not hold in this case.

**Acknowledgments.** The authors would like to thank George E. Andrews, Q.H. Hou and Doron Zeilberger for valuable comments. This work was supported by the 973 Project, the PCSIRT Project of the Ministry of Education, the Ministry of Science and Technology, and the National Science Foundation of China.

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