Discrete Math.
309 (2009), no. 21, 6235-6244

Labeled Partitions with Colored Permutations

William Y. C. Chen[1], Henry Y. Gao[2], Jia He[3]

Center for Combinatorics, LPMC-TJKLC
Nankai University, Tianjin 300071, P. R. China

emails: [1]chen@nankai.edu.cn, [2]gaoyong@cfc.nankai.edu.cn,
[3]hejia1@msu.edu

Abstract. In this paper, we extend the notion of labeled partitions with ordinary permutations to colored permutations. We use this structure to derive the generating function of the fmaj_k indices of colored permutations. We further give a combinatorial treatment of a relation on the q-derangement numbers with respect to colored permutations. Based on labeled partitions, we provide an involution that implies the generating function formula due to Gessel and Simon for signed q-counting of the major indices. This involution can be extended to signed permutations. This gives a combinatorial interpretation of a formula of Adin, Gessel and Roichman.

Keywords: labeled partition, flag major index, colored permutation, q-derangement number

AMS Mathematical Subject Classifications: 05A30, 05A19, 05A15

1 Introduction

In this paper, we introduce the notion of labeled partitions with colored permutations and use this structure to study the fmaj index and the q-derangement numbers. To be more specific, we will be concerned with the wreath product $S_n^k = C_k \wr S_n$ of the symmetric group on $[n] = \{1, 2, \ldots, n\}$ and the cyclic group C_k on $\{0, 1, \ldots, k-1\}$, see Adin and Roichman [2] and Wagner [18]. The elements in S_n^k are also called colored permutations, see Bagno, Butman and Garber [5]. Derangements with respect to the wreath product S_n^k have been studied by Chow and Shiu [9], and Faliharimalala and Zeng [10, 11].

A k-colored permutation is written in the form $\pi(1)_{c_1}\,\pi(2)_{c_2}\,\cdots\,\pi(n)_{c_n}$, where $\pi(1)\,\pi(2)\,\cdots\,\pi(n)$ is a permutation on $[n]$ and $c_i \in \{0, 1, \ldots, k-1\}$. For example, $4_2\,3_0\,1_2\,5_0\,2_1$ is a colored permutation in S_5^3. We define a total order on the colored letters as follows

$$1_{k-1} < 2_{k-1} < \cdots < n_{k-1} < 1_{k-2} < 2_{k-2} < \cdots < n_{k-2} < \cdots < 1_0 < 2_0 < \cdots < n_0. \quad (1.1)$$

Let us recall the following definitions:

$$\begin{aligned} D(\sigma) &:= \{i \in [n-1] \colon \sigma(i) > \sigma(i+1)\}, \\ \mathrm{maj}(\sigma) &:= \sum_{i \in D(\sigma)} i, \\ N_j(\sigma) &:= \#\{i \in [n] \colon \sigma(i) \text{ has subscript } j\}, \quad j = 1, \ldots, k-1, \\ \mathrm{fmaj_k}(\sigma) &:= k\mathrm{maj}(\sigma) + N_1(\sigma) + 2N_2(\sigma) + \cdots + (k-1)N_{k-1}(\sigma). \end{aligned} \quad (1.2)$$

The set $D(\sigma)$ is called the descent set of $\sigma \in S_n^k$, and an element in $D(\sigma)$ is called a descent of σ. It should be noted that Adin and Roichman [2] have given the definition of flag major index of an element in S_n^k by the unique factorization into Coxeter elements, and they have shown that fmaj_k has the above expression (1.2). In this paper, we will use the formula (1.2) as the definition of the fmaj_k index.

For $k = 1$, S_n^1 is usually written as S_n. For $k = 2$, S_n^2 becomes the group of signed permutations on $[n]$, often denoted by B_n, and the minus sign is often denoted by a bar. Moreover, setting $k = 2$, the fmaj_k index reduces to the fmaj index for signed permutations as defined by

$$\mathrm{fmaj}(\pi) = 2\mathrm{maj}(\pi) + N(\pi),$$

where $N(\pi)$ denotes the number of negative elements of π and $\mathrm{maj}(\pi)$ is defined with respect to the following order

$$\bar{1} < \bar{2} < \cdots < \bar{n} < 1 < 2 < \cdots < n.$$

Using labeled partitions with colored permutations, we give a combinatorial proof of the generating function formula for the $\mathrm{fmaj_k}$ indices on S_n^k,

$$\sum_{\pi \in S_n^k} q^{\mathrm{fmaj}_k(\pi)} = [k]_q [2k]_q \cdots [nk]_q, \quad (1.3)$$

where $[k]_q = 1 + q + q^2 + \cdots + q^{k-1}$. The above formula is a natural extension of the formulas for the generating functions for the major index and the fmaj index, see Faliharimalala and Zeng [11]. Bijective proofs have been given by Adin and Roichman [2], Haglund, Loehr and Remmel [14]. Foata and Han [12] found a combinatorial interpretation of the equidistribution of the fmaj index and the finv index for signed permutations, which implies the generating function formula for the case $k = 2$, that is,

$$\sum_{\pi \in B_n} q^{\mathrm{fmaj}(\pi)} = [2]_q [4]_q \cdots [2n]_q. \quad (1.4)$$

The second result of this paper is a combinatorial treatment of a relation on the q-derangement numbers $D_n^k(q)$ with respect to S_n^k. This relation implies the formula for $d_n^k(q)$, as given by Faliharimalala and Zeng [11]. For $n \geq 1$, let

$$\mathcal{D}_n := \{\sigma \in S_n \colon \sigma(i) \neq i \text{ for all } i \in [n]\}$$

be the set of derangements on S_n. Gessel defined the q-derangement numbers by

$$d_n(q) := \sum_{\sigma \in \mathscr{D}_n} q^{\mathrm{maj}(\sigma)}$$

and proved that

$$d_n(q) = [n]_q! \sum_{k=0}^{n} \frac{(-1)^k q^{\binom{k}{2}}}{[k]_q!}, \tag{1.5}$$

where $[n]_q! = [1]_q[2]_q \cdots [n]_q$. Wachs [16] found a combinatorial proof of the above formula. Chow [8] generalized the argument of Wachs to the type B case. Chow defined

$$\mathscr{D}_n^B := \{\sigma \in B_n \colon \sigma(i) \neq i \text{ for all } i \in [n]\}$$

as the set of derangements in B_n and

$$d_n^B(q) := \sum_{\sigma \in \mathscr{D}_n^B} q^{\mathrm{fmaj}(\sigma)}.$$

It has been shown that

$$d_n^B(q) = [2]_q[4]_q \cdots [2n]_q \sum_{k=0}^{n} \frac{(-1)^k q^{2\binom{k}{2}}}{[2]_q[4]_q \cdots [2k]_q}. \tag{1.6}$$

The notion of derangements of type B can be generalized to S_n^k, as given by Faliharimalala and Zeng [11]. Define

$$\mathscr{D}_n^k := \{\sigma \in S_n^k : \sigma(i) \neq i_0 \text{ for all } i \in [n]\}$$

and

$$d_n^k(q) := \sum_{\sigma \in \mathscr{D}_n^k} q^{\mathrm{fmaj}_k(\sigma)}.$$

Faliharimalala and Zeng have shown that

$$d_n^k(q) = [k]_q[2k]_q \cdots [nk]_q \sum_{j=0}^{n} \frac{(-1)^j q^{k\binom{j}{2}}}{[k]_q[2k]_q \cdots [jk]_q}. \tag{1.7}$$

The argument of Chow for $d_n^B(q)$ can be extended to $d_n^k(q)$. Our proof is based on the structure of labeled partitions with colored permutations, which is an extension of the combinatorial approach of Chen and Xu [7] for ordinary permutations. We will present the proof for the case $k = 3$, which is valid for the general case.

The third result is concerned the following formula of Gessel and Simon [17] on signed q-counting of permutations with respect to the major index:

$$\sum_{\pi \in S_n} \mathrm{sign}(\pi) q^{\mathrm{maj}(\pi)} = [1]_q [2]_{-q} [3]_q [4]_{-q} \cdots [n]_{(-1)^{n-1}q}.$$

Note that a combinatorial proof of the above formula has been given by Wachs [17] based on permutations. We will present an involution on labeled partitions which serves as a combinatorial proof of the above formula. Moreover, our involution can be extended to signed permutations. This gives a combinatorial proof of the following formula of Adin-Gessel-Roichman [3] for signed q-counting of signed permutations with respect to the fmaj index:

$$\sum_{\pi \in B_n} \mathrm{sign}(\pi) q^{\mathrm{fmaj}(\pi)} = [2]_{-q} [4]_q \cdots [2n]_{(-1)^n q}.$$

2 Labeled Partitions and the fmaj_k Index

In this section, we introduce the notion of labeled partitions with colored permutations. Using this structure, we give a combinatorial proof of the following formula for the generating function of the fmaj_k indices of colored permutations in S_n^k, given by Adin and Roichman [2], see also Haglund, Loehr and Remmel [14], Faliharimalala and Zeng [11].

Theorem 2.1. *For $n \geq 1$, we have*

$$\sum_{\pi \in S_n^k} q^{\mathrm{fmaj}_k(\pi)} = [k]_q [2k]_q \cdots [nk]_q.$$

Recall that given a colored permutation $\pi \in S_n^k$, $N_j(\pi)$ is the number of elements $\pi(i) \in \pi$ with subscript j, where $j = 1, 2, \ldots, k-1$. The fmaj_k index which is originally defined algebraically by Adin and Roichman has the following equivalent form

$$\mathrm{fmaj}_k(\pi) = k\mathrm{maj}(\pi) + N_1(\pi) + 2N_2(\pi) + \cdots + (k-1)N_{k-1}(\pi).$$

Clearly, Theorem 2.1 is a generalization of the formulas for permutations and signed permutations. We shall give a combinatorial proof of Theorem 2.1 by using labeled partitions with colored permutations.

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ be an integer partition with at most n parts where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$, see Andrews [4]. We write $|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_n$. A labeled partition associated with S_n^3 is defined as a pair (λ, π), where λ is a partition with at most n parts and $\pi = \pi(1)\pi(2)\cdots\pi(n)$ is a colored permutation in S_n^3. We can also employ the two-row notation to represent a labeled partition

$$\begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \pi(1) & \pi(2) & \cdots & \pi(n) \end{pmatrix}.$$

A labeled partition (λ, π) is said to be standard if $\pi(i) > \pi(i+1)$ implies $\lambda_i > \lambda_{i+1}$. Equivalently, a labeled partition (λ, π) is standard if $\lambda_i = \lambda_{i+1}$ implies $\pi(i) < \pi(i+1)$. Given a colored element w_i, we use $c(w_i)$ to denote the color or subscript i, and use $d(w_i)$ to denote the element w after removing the color i.

Let P_n^3 denote the set of partitions with at most n parts such that each part is divisible by 3. For any $\pi \in S_n^3$, we denote by Q_π the set of standard labeled partitions such that $\lambda_i - \mathrm{c}(\pi(i))$ is divisible by 3.

Lemma 2.2. *For any $\pi \in S_n^3$, there is a bijection $g_\pi \colon \lambda \to (\mu, \pi)$ from P_n^3 to Q_π such that $|\lambda| + \mathrm{fmaj}_3(\pi) = |\mu|$.*

Proof. Define μ to be

$$\mu = (\lambda_1 + 3a_1 + \mathrm{c}(\pi(1)), \lambda_2 + 3a_2 + \mathrm{c}(\pi(2)), \ldots, \lambda_n + 3a_n + \mathrm{c}(\pi(n))),$$

where a_i is the number of descents in $\pi(i)\pi(i+1)\cdots\pi(n)$. From this definition, we see that μ is a partition and $\mu_i - \mathrm{c}(\pi(i))$ is divisible by 3. It suffices to show that (μ, π) is standard. We have the following two cases.

Case 1: $\lambda_i > \lambda_{i+1}$. We have

$$\lambda_i + 3a_i + \mathrm{c}(\pi(i)) = \mu_i > \mu_{i+1} = \lambda_{i+1} + 3a_{i+1} + \mathrm{c}(\pi(i+1)),$$

since $\lambda_i - \lambda_{i+1} \geq 3, a_i \geq a_{i+1}$ and $|\mathrm{c}(\pi(i)) - \mathrm{c}(\pi(i+1))| < 3$.

Case 2: $\lambda_i = \lambda_{i+1}$. We further consider the following two subcases:

(i) $\pi(i) > \pi(i+1)$. It is easy to verify that

$$\lambda_i + 3a_i + \mathrm{c}(\pi(i)) = \mu_i > \mu_{i+1} = \lambda_{i+1} + 3a_{i+1} + \mathrm{c}(\pi(i+1)).$$

(ii) $\pi(i) < \pi(i+1)$. If $\pi(i)$ and $\pi(i+1)$ have the same subscript, then we have

$$\lambda_i + 3a_i + \mathrm{c}(\pi(i)) = \mu_i = \mu_{i+1} = \lambda_{i+1} + 3a_{i+1} + \mathrm{c}(\pi(i+1)).$$

Otherwise, we find that the subscript of $\pi(i)$ is greater than that of $\pi(i+1)$. This implies that

$$\lambda_i + 3a_i + \mathrm{c}(\pi(i)) = \mu_i > \mu_{i+1} = \lambda_{i+1} + 3a_{i+1} + \mathrm{c}(\pi(i+1)).$$

Hence the labeled partition (μ, π) is standard. Conversely, given a labeled partition $(\mu, \pi) \in Q_\pi$, we can recover the partition $\lambda \in P_n^3$ by reversing the steps of the above procedure. ∎

As a consequence of the above bijection, we obtain the following identity.

Theorem 2.3. *For $n \geq 1$, we have*

$$\sum_{\pi \in S_n^3} q^{\mathrm{fmaj}_3(\pi)} = [3]_q [6]_q \cdots [3n]_q. \tag{2.8}$$

Proof. We consider the following equivalent form of (2.8):

$$\frac{1}{(q^3; q^3)_n} \sum_{\pi \in S_n^3} q^{\mathrm{fmaj}_3(\pi)} = \frac{1}{(1-q)^n},$$

where

$$(q^3; q^3)_n = (1-q^3)(1-q^6) \cdots (1-q^{3n}).$$

Let W_n be the set of sequences of n nonnegative integers. Note that $\frac{1}{(q^3;q^3)_n}$ and $\frac{1}{(1-q)^n}$ are the generating functions for numbers of elements in P_n^3 and W_n, respectively. We wish to construct a bijection $\phi\colon (\lambda\,, \pi) \to s$ from (P_n^3, S_n^3) to W_n such that

$$|\lambda| + \mathrm{fmaj}_3(\pi) = |s|,$$

where $|s|$ denotes the sum of entries of s. The bijection ϕ can be described as follows:

Step 1. Use the bijection in Lemma 2.2 to derive a standard labeled partition (μ, π) from (λ, π).

Step 2. Based on the two row representation of the labeled partition (μ, π), we permute the columns to make the second row become the identity permutation by ignoring the subscripts of the elements in π. Let s denote the first row of the resulted array.

It is not difficult to see that the above procedure is reversible. The inverse of ϕ consists of four steps.

Step 1. For a sequence $s = (s(1), s(2), \ldots, s(n)) \in W_n$, we construct a two row array

$$\begin{pmatrix} s(1) & s(2) & \cdots & s(n) \\ 1 & 2 & \cdots & n \end{pmatrix}.$$

Step 2. For each element $i \in [n]$, we may construct a colored permutation $1_{c_1} 2_{c_2} \cdots n_{c_n}$, where $c_i = s(i) \pmod 3$. Clearly, we have $s^*(i) = s(i) - c_i$ is divisible by 3. So we are led to the following array

$$\begin{pmatrix} s^*(1) & s^*(2) & \cdots & s^*(n) \\ 1_{c_1} & 2_{c_2} & \cdots & n_{c_n} \end{pmatrix}.$$

Step 3. Permute the columns of the above array to make the first row $s^*(j_1) s^*(j_2) \cdots s^*(j_n)$ in decreasing order. Moreover, we rearrange the elements in the second row in increasing order if they correspond to the same elements in the first row. Let us denote the resulted labeled partition by

$$\begin{pmatrix} s^*(j_1) & s^*(j_2) & \cdots & s^*(j_n) \\ \delta(1)_{e_1} & \delta(2)_{e_2} & \cdots & \delta(n)_{e_n} \end{pmatrix}.$$

Step 4. Recover the initial labeled partition (λ, π) from the array produced in Step 3 by the following rule:

$$(\lambda^*, \pi) = \begin{pmatrix} s^*(j_1) - 3a_1 & s^*(j_2) - 3a_2 & \cdots & s^*(j_n) - 3a_n \\ \delta(1)_{e_1} & \delta(2)_{e_2} & \cdots & \delta(n)_{e_n} \end{pmatrix},$$

where a_k is the number of descents in the colored permutation $\delta(k)_{e_k} \cdots \delta(n)_{e_n}$.

It is routine to check that the above procedure is feasible. Moreover, one can verify that $\phi \cdot \phi^{-1} = \text{id}$ and $\phi^{-1} \cdot \phi = \text{id}$, where id is the identity map. This completes the proof. ∎

For example, let $n = 7$, $\lambda = (18, 18, 18, 9, 9, 6, 3)$ and $\pi = 3_2\, 4_2\, 6_0\, 5_1\, 7_2\, 2_1\, 1_2$. We obtain $s = (5, 10, 29, 29, 16, 27, 14)$ by the following two steps:

$$\begin{pmatrix} 18 & 18 & 18 & 9 & 9 & 6 & 3 \\ 3_2 & 4_2 & 6_0 & 5_1 & 7_2 & 2_1 & 1_2 \end{pmatrix} \xrightarrow{\text{Step 1}} \begin{pmatrix} 29 & 29 & 27 & 16 & 14 & 10 & 5 \\ 3_2 & 4_2 & 6_0 & 5_1 & 7_2 & 2_1 & 1_2 \end{pmatrix}$$

$$\xrightarrow{\text{Step 2}} (5, 10, 29, 29, 16, 27, 14).$$

The reverse process from s to (λ, π) is illustrated as follows:

$$(5, 10, 29, 29, 16, 27, 14)$$

$$\xrightarrow{\text{Step 1}} \begin{pmatrix} 5 & 10 & 29 & 29 & 16 & 27 & 14 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{pmatrix} \xrightarrow{\text{Step 2}} \begin{pmatrix} 3 & 9 & 27 & 27 & 15 & 27 & 12 \\ 1_2 & 2_1 & 3_2 & 4_2 & 5_1 & 6_0 & 7_2 \end{pmatrix}$$

$$\xrightarrow{\text{Step 3}} \begin{pmatrix} 27 & 27 & 27 & 15 & 12 & 9 & 3 \\ 3_2 & 4_2 & 6_0 & 5_1 & 7_2 & 2_1 & 1_2 \end{pmatrix} \xrightarrow{\text{Step 4}} \begin{pmatrix} 18 & 18 & 18 & 9 & 9 & 6 & 3 \\ 3_2 & 4_2 & 6_0 & 5_1 & 7_2 & 2_1 & 1_2 \end{pmatrix}.$$

3 Labeled Partitions and q-Derangements Numbers

In this section, we give a combinatorial treatment of a relation on the q-derangement numbers for S_n^k. This relation leads to the formula of Faliharimalala and Zeng for $d_n^k(q)$. We will give the proof for the case $k = 3$. It is easily seen that the argument applies to the general case.

Following Wachs [16] and Chow [8], we define the reduction of a colored permutation σ on a set of positive integers $A = \{a_1 < a_2 < \cdots < a_k\}$ by substituting the element a_i with i while keeping the colors of the elements. A position i is called a fixed point of a colored permutation $\pi(1)\pi(2)\cdots\pi(n)$ if $\pi(i) = i_0$. The derangement part of a colored permutation

$\sigma \in S_n^3$, denoted by $dp(\sigma)$, is the reduction of the sequence obtained from σ by removing the fixed elements. For example, $dp(8_0\, 1_2\, 5_1\, 4_0\, 3_1\, 6_0\, 7_1\, 2_2) = 6_0\, 1_2\, 4_1\, 3_1\, 5_1\, 2_2$.

We have the following extension of the relation of Wachs [16] to colored permutations.

Theorem 3.1. *Let $\alpha \in \mathscr{D}_k^3$. For $0 \le k \le n$, we have*

$$\sum_{dp(\sigma)=\alpha,\, \sigma\in S_n^3} q^{\mathrm{fmaj}_3(\sigma)} = q^{\mathrm{fmaj}_3(\alpha)} \begin{bmatrix} n \\ k \end{bmatrix}_{q^3}. \tag{3.9}$$

It should be noted that the above theorem can be proved by the method of Wachs [16] which has been extended by Chow [8] to signed permutations. We will give a combinatorial proof based on labeled partitions with colored permutations.

For any $\pi = \pi(1)\pi(2)\cdots\pi(k) \in S_k^3$, we can insert a fixed point j with $1 \le j \le k+1$ into π to obtain a permutation $\bar{\pi}$ in S_{k+1}^3. Precisely, $\bar{\pi}$ is given by

$$\bar{\pi} = \pi'(1)\pi'(2)\cdots\pi'(j-1)j_0\pi'(j)\cdots\pi'(k),$$

where

$$\pi'(i) = \begin{cases} (c(\pi(i)))d(\pi(i)), & \text{if } d(\pi(i)) < j, \\ (c(\pi(i)))(d(\pi(i))+1), & \text{otherwise.} \end{cases}$$

In other words, $\bar{\pi}$ is the unique permutation with i being a fixed point such that the reduction of the sequence obtained from $\bar{\pi}$ by deleting the element at position i equals π. For example, let $\pi = 4_2\, 1_0\, 2_0\, 6_1\, 5_1\, 3_2$ and $j = 3$. We have $\bar{\pi} = 5_2\, 1_0\, 3_0\, 2_0\, 7_1\, 6_1\, 4_2$.

Proof of Theorem 3.1. First of all, we reformulate the relation (3.9) in the equivalent form

$$\frac{1}{(q^3;q^3)_n} \sum_{dp(\sigma)=\alpha,\sigma\in S_n^3} q^{\mathrm{fmaj}_3(\sigma)} = \frac{1}{(q^3;q^3)_k(q^3;q^3)_{n-k}} q^{\mathrm{fmaj}_3(\alpha)}. \tag{3.10}$$

We proceed to make use of labeled partitions to give a combinatorial proof of (3.10). Let R_α be the set of colored permutations $\sigma \in S_n^3$ such that $dp(\sigma) = \alpha$. We aim to give a bijection $\theta\colon (\lambda,\sigma) \to (\beta,\gamma)$ from (P_n^3, R_α) to (P_k^3, P_{n-k}^3) such that

$$|\lambda| + \mathrm{fmaj}_3(\sigma) = |\beta| + |\gamma| + \mathrm{fmaj}_3(\alpha). \tag{3.11}$$

This bijection consists of the following three steps.

Step 1. Apply the bijection g_σ given in Lemma 2.2 to get a standard labeled partition (λ^*, σ) from λ.

Step 2. Let $\sigma(i_1), \sigma(i_2), \ldots, \sigma(i_{n-k})$ be the fixed points, and $\sigma(j_1), \sigma(j_2), \ldots, \sigma(j_k)$ the non-fixed points of σ. We decompose λ^* into two parts, namely, $\lambda^*(i_1), \lambda^*(i_2), \ldots, \lambda^*(i_{n-k})$ and $\lambda^*(j_1), \lambda^*(j_2), \ldots, \lambda^*(j_k)$. Let $\gamma = (\lambda^*(i_1), \lambda^*(i_2), \ldots, \lambda^*(i_{n-k}))$ and $\beta^* = (\lambda^*(j_1), \lambda^*(j_2), \ldots, \lambda^*(j_k))$.

Step 3. Apply g_α^{-1} to (β^*, α) and denote the resulted partition by β.

To prove that the above procedure is feasible, it is necessary to show that β^* generated in Step 2 satisfies the condition that (β^*, α) belongs to Q_α so that one can apply g_α^{-1}. Observe that for any $1 \le q \le k$, $\sigma(j_q)$ and $\alpha(q)$ have the same subscript since $\alpha(q)$ is obtained by the reduction operation. It follows that

$$\beta^*(q) - \mathrm{c}(\alpha(q)) = \lambda^*(j_q) - \mathrm{c}(\alpha(q))$$

is divisible by 3 for any $1 \le q \le k$. To verify that (β^*, α) is standard, it suffices to show if $\sigma(p) > \sigma(q)$ with $\sigma(p+1), \ldots, \sigma(q-1)$ being at the positions of fixed points, then we have $\lambda_p^* > \lambda_q^*$. When $q = p+1$, we see that $\lambda_p^* > \lambda_q^*$ since (λ^*, σ) is standard. When $q > p+1$, it is easy to see that either $\sigma(p) > \sigma(p+1)$ or $\sigma(q-1) > \sigma(q)$. Therefore, we have either $\lambda_p^* > \lambda_{p+1}^*$ or $\lambda_{q-1}^* > \lambda_q^*$. Since λ^* is a partition, we have $\lambda_p^* > \lambda_q^*$. Hence the bijection is well defined.

It remains to show that the above procedure is reversible. We proceed to construct the inverse map η from (P_k^3, P_{n-k}^3) to (P_n^3, R_α), which consists of three steps.

Step 1. Apply g_α to β and denote the resulted partition by $(\tilde{\beta}, \alpha)$.

Step 2. Let $(\tilde{\lambda}^0, \sigma^0) = (\tilde{\beta}, \alpha)$. We insert γ_i into $(\tilde{\lambda}^{i-1}, \sigma^{i-1})$ to get $(\tilde{\lambda}^i, \sigma^i)$. Find the first position r in $\tilde{\lambda}^{i-1}$ such that the insertion of γ_i to this position will generate a partition. We denote this partition by $\tilde{\lambda}^i$. Obviously, we have $\tilde{\lambda}^i_{r-1} > \tilde{\lambda}^i_r = \gamma_i$. Suppose that $\tilde{\lambda}^i_r = \cdots = \tilde{\lambda}^i_t > \tilde{\lambda}^i_{t+1}$ for some $t \ge r$. If $r = t$, we set $s = r$. Otherwise, we look for a position s, from left to right, subject to the condition

$$\sigma^{i-1}(s-1) < s_0 \le \sigma^{i-1}(s),$$

here we treat $\sigma^{i-1}(r-1)$ as $-\infty$ and $\sigma^{i-1}(t+1)$ as ∞. In this way, we obtain σ^i from σ^{i-1} by inserting s_0 as a fixed point. In fact, this procedure guarantees that the subsequence $\sigma^i(r), \sigma^i(r+1), \ldots, \sigma^i(t)$ is increasing. That is, $(\tilde{\lambda}^i, \sigma^i)$ is a standard labeled partition. On the other hand, since $\gamma \in P_{n-k}^3$ and each fixed point has subscript 0, we find that γ_i is divisible by 3 for each $1 \le i \le n-k$ and thus $(\tilde{\lambda}^i, \sigma^i) \in Q_{\sigma^i}$.

Step 3. Apply $g_{\sigma^{n-k}}^{-1}$ to $(\tilde{\lambda}^{n-k}, \sigma^{n-k})$ and denote the resulted partition by λ^{n-k}.

We claim that λ^{n-k} and σ^{n-k} are equal to λ and σ, respectively. This implies that η is the inverse of θ. From Lemma 2.2, it is easily seen that $\beta^* = \tilde{\beta}$. Since $\tilde{\lambda}^{n-k}$ is the partition obtained from $\tilde{\beta}$ by inserting $\gamma_1, \ldots, \gamma_{n-k}$, we have $\lambda^* = \tilde{\lambda}^{n-k}$.

It remains to show that $\sigma^{n-k} = \sigma$. It suffices to verify σ^{n-k} and σ have the same fixed points. By removing the common fixed points, let us use f, or f_0, to be more precise, since the color of f is 0, to denote the first fixed point of σ, which is different from the first fixed point of f' of σ^{n-k}. It is clear that

$$\sigma(f-1) < f_0 \le \sigma(f+1) - 1.$$

By the choice of f', we infer that $f' < f$. On the other hand, $\lambda^*(f) = \lambda^*(f')$. Since (λ^*, σ)

and $(\lambda^*, \sigma^{n-k})$ are both standard labeled partitions, we have

$$\sigma(f') < \sigma(f'+1) < \cdots < \sigma(f),$$

and

$$\sigma^{n-k}(f') < \sigma^{n-k}(f'+1) < \cdots < \sigma^{n-k}(f).$$

Using the fact that $\lambda^*(f) = \lambda^*(f')$, $\sigma^{n-k}(f)$ and $\sigma^{n-k}(f')$ have the same subscript, we deduce that $\sigma^{n-k}(f)$ has the subscript 0 as $\sigma^{n-k}(f')$.

Recall that by assumption $\sigma(f) = f$ and $\sigma^{n-k}(f') = f'$. Since

$$\sigma(f') < \sigma(f'+1) < \cdots < \sigma(f)$$

and $\sigma(f) = f$, it follows that $\sigma(f') \leq f'$. Recalling that f is the first fixed point of σ, we obtain

$$\alpha(f') = \sigma(f') < f'.$$

From the construction of σ^{n-k}, we get

$$\sigma^{n-k}(f') \leq \alpha(f') < f',$$

which contradicts the assumption that $\sigma^{n-k}(f') = f'$. This implies that $\sigma = \sigma^{n-k}$. Again by Lemma 2.2, we conclude that $\lambda = \lambda^{n-k}$. Hence η is the inverse map of θ. This completes the proof. ∎

For example, let $n = 8, \lambda = (18, 12, 12, 12, 9, 9, 6, 3)$ and $\sigma = 5_2\, 1_0\, 2_0\, 4_0\, 8_1\, 6_0\, 7_1\, 3_2$. We have

$$g_\sigma(\lambda) = \begin{pmatrix} 29 & 21 & 21 & 21 & 16 & 10 & 5 \\ 5_2 & 1_0 & 2_0 & 4_0 & 7_1 & 6_1 & 3_2 \end{pmatrix}.$$

The fixed points of σ are 4_0 and 6_0, and $\alpha = dp(\sigma) = 4_2\, 1_0\, 2_0\, 6_1\, 5_1\, 3_2$. Decomposing $(29, 21, 21, 21, 16, 15, 10, 5)$, we get $((29, 21, 21, 16, 10, 5), (21, 15))$. Applying g_α^{-1} to $\beta^* = (29, 21, 21, 16, 10, 5)$ gives $\beta = (18, 12, 12, 9, 6, 3)$ and $\gamma = (21, 15)$.

Conversely, given $\alpha = 4_2\, 1_0\, 2_0\, 6_1\, 5_1\, 3_2$ and $(\beta, \gamma) = ((18, 12, 12, 9, 6, 3), (21, 15))$, we have $\tilde{\beta} = (29, 21, 21, 16, 10, 5)$. The insertion process is illustrated as follows,

$$\begin{pmatrix} 29 & 21 & 21 & 16 & 10 & 5 \\ 4_2 & 1_0 & 2_0 & 6_1 & 5_1 & 3_2 \end{pmatrix} \xrightarrow{\gamma_1=21} \begin{pmatrix} 29 & 21 & 21 & 21 & 16 & 10 & 5 \\ 5_2 & 1_0 & 2_0 & 4_0 & 7_1 & 6_1 & 3_2 \end{pmatrix}$$

$$\xrightarrow{\gamma_2=15} \begin{pmatrix} 29 & 21 & 21 & 21 & 16 & 15 & 10 & 5 \\ 5_2 & 1_0 & 2_0 & 4_0 & 8_1 & 6_0 & 7_1 & 3_2 \end{pmatrix}.$$

So we get $\tilde{\lambda}^{n-k} = (29, 21, 21, 21, 16, 15, 10, 5)$, $\sigma^{n-k} = 5_2\, 1_0\, 2_0\, 4_0\, 8_1\, 6_0\, 7_1\, 3_2$. Finally, we obtain $\lambda^{n-k} = g_{\sigma^{n-k}}^{-1} = (18, 12, 12, 12, 9, 9, 6, 3)$.

4 Involutions on Labeled Partitions

In this section, we give an involution on labeled partitions which leads to a combinatorial interpretation of a formula of Gessel and Simon on signed q-counting of the major indices. This involution can be easily extended to signed permutations. This gives a combinatorial proof of a formula of Adin, Gessel and Roichman on signed q-counting of fmaj indices.

Recall that the sign of a signed permutation is defined in terms of the generators of B_n as a Coxeter group. Consider the generating set $\{s_0, s_1, s_2, \ldots, s_{n-1}\}$ of B_n, where

$$s_0 := [-1, 2, 3, \ldots, n], \quad \text{and} \quad s_i := [1, 2, \ldots, i-1, i+1, i, i+2, \ldots, n]$$

for $1 \leq i \leq n-1$. The sign of a signed permutation π is defined by

$$\operatorname{sign}(\pi) := (-1)^{l(\pi)},$$

where $l(\pi)$ is the standard length of π with respect to the generators of B_n.

The following theorem is due to Gessel and Simon [17].

Theorem 4.1.

$$\sum_{\pi \in S_n} \operatorname{sign}(\pi) q^{\operatorname{maj}(\pi)} = [1]_q [2]_{-q} [3]_q [4]_{-q} \cdots [n]_{(-1)^{n-1}q}. \tag{4.12}$$

A combinatorial proof of the above formula has been given by Wachs [17]. Here we shall give an involution on labeled partitions and shall show that this involution can be easily extended to the following type B formula due to Adin, Gessel and Roichman [3].

Theorem 4.2.

$$\sum_{\pi \in B_n} \operatorname{sign}(\pi) q^{\operatorname{fmaj}(\pi)} = [2]_{-q} [4]_q \cdots [2n]_{(-1)^n q}. \tag{4.13}$$

To describe our involution on labeled partitions as a proof of (4.12), we may reformulate it into the following equivalent form:

$$\frac{1}{(q;q)_n} \sum_{\pi \in S_n} \operatorname{sign}(\pi) q^{\operatorname{maj}(\pi)} = \frac{1}{(1-q)(1+q)(1-q)(1+q)\cdots(1-(-1)^{n-1}q)}. \tag{4.14}$$

Proof of Theorem 4.1. We consider the two cases according to the parity of n.

Case 1. n is even, i.e., $n = 2k$. In this case (4.14) takes the form

$$\frac{1}{(q;q)_{2k}} \sum_{\pi \in S_{2k}} \operatorname{sign}(\pi) q^{\operatorname{maj}(\pi)} = \frac{1}{(1-q^2)^k}. \tag{4.15}$$

Notice that the right hand side of (4.15) is the generating function of sequences $(a_1, a_2, \ldots, a_{2k})$ satisfying $a_{2i-1} = a_{2i}$ for $i = 1, 2, \ldots, k$. Meanwhile, the left hand side of (4.15) is the generating function of labeled partitions on S_n with at most $2k$ parts under the assumption that a labeled partition (λ, π) carries the sign of the permutation π. To be more specific, such labeled partitions are called signed labeled partitions. We wish to construct an involution on the set H of signed labeled partitions (λ, π) such that the generating function of the fixed points of the involution equals the right hand side of (4.15). This involution consists of three steps.

Step 1. Let (λ, π) be a labeled partition such that $\pi \in S_{2k}$ and $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{2k})$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{2k} \geq 0$. If $|\pi^{-1}(1) - \pi^{-1}(2)| \neq 1$, we define

$$\phi^1(\pi)(i) = \begin{cases} \pi(i), & i \neq \pi^{-1}(1) \text{ and } \pi^{-1}(2), \\ 2, & i = \pi^{-1}(1), \\ 1, & i = \pi^{-1}(2). \end{cases}$$

Obviously, (λ, π) and $(\lambda, \phi^1(\pi))$ have opposite signs and $\mathrm{maj}(\pi) = \mathrm{maj}(\phi^1(\pi))$. Consequently, we have

$$\mathrm{maj}(\pi) + |\lambda| = \mathrm{maj}(\phi^1(\pi)) + |\lambda|,$$

and so these two elements cancel each other.

In the case that $|\pi^{-1}(1) - \pi^{-1}(2)| = 1$, we have $\mathrm{maj}(\pi) \neq \mathrm{maj}(\phi^1(\pi))$. So we consider that the set H^1 of signed labeled partitions (λ, π) such that $|\pi^{-1}(1) - \pi^{-1}(2)| = 1$. Repeating the above procedure, some elements in H^1 will be cancelled. At this stage, we consider the positions of the elements 3 and 4. Similarly, if $|\pi^{-1}(3) - \pi^{-1}(4)| \neq 1$, we define

$$\phi^2(\pi)(i) = \begin{cases} \pi(i), & i \neq \pi^{-1}(3) \text{ and } \pi^{-1}(4), \\ 4, & i = \pi^{-1}(3), \\ 3, & i = \pi^{-1}(4). \end{cases}$$

Therefore, (λ, π) and $(\lambda, \phi^2(\pi))$ have the opposite signs and

$$\mathrm{maj}(\pi) + |\lambda| = \mathrm{maj}(\phi^2(\pi)) + |\lambda|.$$

In other words, these two elements cancel out in the set H^1.

Similarly, we use H^2 to denote the subset of H^1 such that $|\pi^{-1}(3) - \pi^{-1}(4)| = 1$. Repeating the same procedure, we may consider the elements $\{5, 6\}, \{7, 8\}, \ldots, \{2k-1, 2k\}$ and obtain a sequence of subsets $H^k \subseteq H^{k-1} \subseteq \cdots \subseteq H^1$. Let ϕ^i $(1 \leq i \leq k)$ denote the functions defined in the above procedure. It is not difficult to see that for a labeled partition (λ, π) in H^k, we have

$$|\pi^{-1}(1) - \pi^{-1}(2)| = 1, |\pi^{-1}(3) - \pi^{-1}(4)| = 1, \ldots, |\pi^{-1}(2k-1) - \pi^{-1}(2k)| = 1.$$

Namely, any odd number $2i - 1$ is next to $2i$ in π for all $i = 1, \ldots, k$.

Step 2. For any labeled partition

$$(\lambda, \pi) = \begin{pmatrix} \lambda_1 & \cdots & \lambda_{\pi^{-1}(2)} & \lambda_{\pi^{-1}(1)} & \cdots & \lambda_{2k} \\ \pi(1) & \cdots & 2 & 1 & \cdots & \pi(2k) \end{pmatrix},$$

we define $(f^1(\lambda), g^1(\pi))$ as follows

$$(f^1(\lambda), g^1(\pi)) = \begin{pmatrix} \lambda_1+1 & \cdots & \lambda_{\pi^{-1}(2)}+1 & \lambda_{\pi^{-1}(1)} & \cdots & \lambda_{2k} \\ \pi(1) & \cdots & 1 & 2 & \cdots & \pi(2k) \end{pmatrix},$$

where $f^1(\lambda)$ is the partition obtained from λ by adding 1 to the first $\pi^{-1}(2)$ parts of λ and $g^1(\pi)$ is the permutation obtained from π by exchanging the positions of 1 and 2.

Note that (λ, π) and $(f^1(\lambda), g^1(\pi))$ have opposite signs. Moreover,

$$\mathrm{maj}(\pi) + |\lambda| = \mathrm{maj}(g^1(\pi)) + |f^1(\lambda)|.$$

Therefore, (λ, π) and $(f^1(\lambda), g^1(\pi))$ cancel out in H^k. Observe that the resulted labeled partition $(f^1(\lambda), g^1(\pi))$ has the additional property that $f^1(\lambda)_{\pi^{-1}(1)}$ is greater than $f^1(\lambda)_{\pi^{-1}(2)}$. By inspection, we see that after cancellation, the remaining elements in H^k are of the following form

$$(\lambda, \pi) = \begin{pmatrix} \lambda_1 & \cdots & \lambda_{\pi^{-1}(1)} & \lambda_{\pi^{-1}(2)} & \cdots & \lambda_{2k} \\ \pi(1) & \cdots & 1 & 2 & \cdots & \pi(2k) \end{pmatrix}$$

where $\lambda_{\pi^{-1}(1)} = \lambda_{\pi^{-1}(2)}$. Let H_1^k denote the set of remaining elements in H^k that of the above form.

We iterate the above process for H_1^k with respect the relative positions of 3 and 4. It is easily seen that for any labeled partition (λ, π) in H_1^k, 1 appears before 2 in π and $\lambda_{\pi^{-1}(1)} = \lambda_{\pi^{-1}(2)}$. Now, for any element $(\lambda, \pi) \in H_1^k$, if

$$(\lambda, \pi) = \begin{pmatrix} \lambda_1 & \cdots & \lambda_{\pi^{-1}(4)} & \lambda_{\pi^{-1}(3)} & \cdots & \lambda_{2k} \\ \pi(1) & \cdots & 4 & 3 & \cdots & \pi(2k) \end{pmatrix},$$

then we can find another labeled partition $(f^2(\lambda), g^2(\pi)) \in H_1^k$

$$(f^2(\lambda), g^2(\pi)) = \begin{pmatrix} \lambda_1+1 & \cdots & \lambda_{\pi^{-1}(4)}+1 & \lambda_{\pi^{-1}(3)} & \cdots & \lambda_{2k} \\ \pi(1) & \cdots & 3 & 4 & \cdots & \pi(2k) \end{pmatrix}.$$

Again, (λ, π) and $(f^2(\lambda), g^2(\pi))$ cancel each other in H_1^k. Notice that $f^2(\lambda)_{\pi^{-1}(3)}$ is greater than $f^2(\lambda)_{\pi^{-1}(4)}$. So the remaining labeled partitions after the above cancelation are of the following form

$$(\lambda, \pi) = \begin{pmatrix} \lambda_1 & \cdots & \lambda_{\pi^{-1}(3)} & \lambda_{\pi^{-1}(4)} & \cdots & \lambda_{2k} \\ \pi(1) & \cdots & 3 & 4 & \cdots & \pi(2k) \end{pmatrix},$$

where $\lambda_{\pi^{-1}(3)} = \lambda_{\pi^{-1}(4)}$. We now denote the set of the remaining labeled partitions by H_2^k and continue the above process. In the end, we get $H_k^k \subseteq H_{k-1}^k \subseteq \cdots \subseteq H_1^k$. Moreover, in the above process we have defined the functions f^i and g^i for $i = 1, 2, \ldots, k$.

Evidently, for any labeled partition (λ, π) in H_k^k and for any $i \in \{1, \ldots, k\}$, $2i-1$ appears immediately before $2i$ and $\lambda_{\pi^{-1}(2i-1)} = \lambda_{\pi^{-1}(2i)}$. It is also clear that all the labeled partitions in H_k^k have positive signs.

Step 3. Permute the columns of the labeled partitions (λ, π) in H_k^k so that the elements in π are rearranged in increasing order. Taking the first row of the resulted two row array, we will get a sequence $(a_1, a_2, \ldots, a_{2k-1}, a_{2k})$ such that $a_{2i-1} = a_{2i}$ $(i = 1, \ldots, k)$ whose generating function is the right hand side of (4.15).

It is easy to see that the relation (4.15) can be justified by the above algorithm. Hence Theorem 4.1 holds when n is even.

Case 2. n is odd, i.e., $n = 2k+1$. We need to show that

$$\frac{1}{(q;q)_{2k+1}} \sum_{\pi \in S_{2k+1}} \mathrm{sign}(\pi) q^{\mathrm{maj}(\pi)} = \frac{1}{(1-q^2)^k (1-q)}. \tag{4.16}$$

The proof is similar to the reasoning when n is even. We may employ the same operations in Step 1 and Step 2 by ignoring the element $2k+1$ while making the pairs $\{1, 2, \}, \{3, 4\}, \ldots, \{2k-1, 2k\}$. The only difference lies in Step 3. When we take the first row of the resulted two row array, we encounter a sequence $(a_1, a_2, \ldots, a_{2k-1}, a_{2k}, a_{2k+1})$ such that $a_{2i-1} = a_{2i}$ $(i = 1, \ldots, k)$. Moreover, a_{2k+1} can be any positive integer. This completes the proof of (4.16).

So far we have constructed a sign reversing involution

$$(\theta, \chi) \colon (\lambda, \pi) \to (\theta(\lambda), \chi(\pi)).$$

To be more specific, the map (θ, χ) is given by

$$(\theta(\lambda), \chi(\pi)) = \begin{cases} (\lambda, \phi^1(\pi)), & \text{if} (\lambda, \pi) \in H \setminus H^1, \\ (\lambda, \phi^2(\pi)), & \text{if} (\lambda, \pi) \in H^1 \setminus H^2, \\ \cdots & \\ (\lambda, \phi^k(\pi)), & \text{if} (\lambda, \pi) \in H^{k-1} \setminus H^k, \\ (f^1(\lambda), g^1(\pi)), & \text{if} (\lambda, \pi) \in H^k \setminus H_1^k, \\ (f^2(\lambda), g^2(\pi)), & \text{if} (\lambda, \pi) \in H_1^k \setminus H_2^k, \\ \cdots & \\ (f^k(\lambda), g^k(\pi)), & \text{if} (\lambda, \pi) \in H_{k-1}^k \setminus H_k^k, \\ (\lambda, \pi), & \text{if} (\lambda, \pi) \in H_k^k, \end{cases}$$

where $\phi^i(\pi)$, $f^i(\lambda)$ and $g^i(\pi)$ are defined in the above algorithm. It is easy to verify that the map is sign reversing, that is, if (λ, π) is not a fixed point of the map (θ, χ), then we have $\mathrm{sign}(\theta(\lambda), \chi(\pi)) = -\mathrm{sign}(\lambda, \pi)$ and

$$|\theta(\lambda)| + \mathrm{maj}(\chi(\pi)) = |\lambda| + \mathrm{maj}(\pi).$$

The fixed points of the map (θ, χ) correspond to the right hand side of (4.12). This completes the proof. ∎

We now turn to the proof of Theorem 4.2, and we need a characterization of the length function of signed permutations [6, Propostion 3.1 and Corollary 3.2].

Lemma 4.3. *Let $\sigma \in B_n$, we have*

$$l(\sigma) = inv(\sigma) + \sum_{\{1\leq i\leq n|\sigma(i)<0\}} |\sigma(i)|,$$

where $inv(\sigma)$ is defined with respect to the order

$$\bar{n} < \cdots < \bar{1} < 1 < \cdots < n.$$

Observe that in the definition of the fmaj index on B_n we have imposed the order

$$\bar{1} < \cdots < \bar{n} < 1 < \cdots < n$$

or in the notation of colored permutations,

$$1_1 < \cdots < n_1 < 1_0 < \cdots < n_0.$$

The above lemma is useful for the construction of a sign reversing involution for the formula (4.13) for B_n. Given a signed permutation $\sigma \in B_n$, we may construct a signed permutation σ' as follows. If 1 and 2 have different signs or 1 and 2 have the same sign but are not adjacent in σ, then we exchange 1 and 2 without changing the signs. By Lemma 4.3, we see that the σ' and σ have opposite signs and $\mathrm{fmaj}(\sigma) = \mathrm{fmaj}(\sigma')$.

For example, let $\sigma = 4_0\, 2_1\, 5_1\, 1_0\, 3_1$. We have $\sigma' = 4_0\, 1_1\, 5_1\, 2_0\, 3_1$. Clearly, σ and σ' have opposite signs.

Using the above rule, we can easily extend the above involution for permutations to signed permutations. The details are omitted.

Acknowledgments. We would like to thank the referees for helpful suggestions. This work was supported by the 973 Project, the PCSIRT Project of the Ministry of Education, the Ministry of Science and Technology, and the National Science Foundation of China.

References

[1] R.M. Adin, F. Brenti and Y. Roichman, Descent numbers and major indices for the hyperoctahedral group, Adv. Appl. Math. 27 (2001) 210–224.

[2] R.M. Adin and Y. Roichman, The flag major index and group actions on polynomial rings, European J. Combin. 22 (2001) 431–446.

[3] R.M. Adin, I.M. Gessel and Y. Roichman, Signed Mahonians, J. Combin. Theory, Ser. A 109 (2005) 25–43.

[4] G. E. Andrews, The Theory of Partitions, Addison-Wesley Publishing Co., 1976.

[5] E. Bagno, A. Butman, and D. Garber, Statistics on the multi-colored permutation groups, Electron. J. Combin. 14 (2007) #R24.

[6] F. Brenti, q-Eulerian polynomials arising from Coxeter groups, European J. Combin. 15 (1994) 417–441.

[7] W.Y.C. Chen and D.H. Xu, Labeled Partitions and the q-Derangement Numbers, SIAM J. Discrete Math. 22 (2008) 1099–1104.

[8] C.-O. Chow, On derangement polynomials of type B, Sém. Lothar. Combin. 55 (2006), Article B55b.

[9] C.-O. Chow and W.C. Shiu, Asymptotic normality of the distribution of q-derangement numbers for the wreath product $C_r \wr \mathfrak{S}_n$, preprint, 2008.

[10] H.L.M. Faliharimalala and J. Zeng, Derangements and Euler's difference table for $C_l \wr S_n$, Electron. J. Combin. 15 (2008) #R65.

[11] H.L.M. Faliharimalala and J. Zeng, Fix-Euler-Mahonian statistics on wreath product, arXiv:Math.CO/0810.2731.

[12] D. Foata, G.-N. Han, Signed words and permutations, I; A fundamental transformation, Proc. Amer. Math. Soc., 135 (2007) 31–40.

[13] A.M. Garsia and I.M. Gessel, Permutation statistics and partitions, Adv. Math. 31 (1979) 288–305.

[14] J. Haglund, N. Loehr and J.B. Remmel, Statistics on wreath products, perfect matchings, and signed words, European J. Combin. 26 (2005) 835–868.

[15] P. A. MacMahon, Combinatory Analysis, Vol 1. Cambridge Univ. Press, London and New York (repreinted by Chelsea, New York, 1960).

[16] M. Wachs, On q-derangement numbers, Proc. Amer. Math. Soc. 106 (1989), 273–278.

[17] M. Wachs, An involution for signed Eulerian numbers, Discrete Math. 99 (1992) 59–62.

[18] J.D. Wagner, The permutation enumeration of wreath products $C_k \wr S_n$ of cyclic and symmetric groups, Adv. Appl. Math. 30 (2003) 343–368.