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The Ratio Monotonicity of the Boros-Moll Polynomials

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Abstract. In their study of a quartic integral, Boros and Moll discovered a special class of Jacobi polynomials, which we call the Boros-Moll polynomials. Kauers and Paule proved the conjecture of Moll that these polynomials are log-concave. In this paper, we show that the Boros-Moll polynomials possess the ratio monotone property which implies the log-concavity and the spiral property. We conclude with a conjecture which is stronger than Moll's conjecture on the ∞-log-concavity.

1 Introduction

In this paper, we aim to show that the Boros-Moll polynomials satisfy the ratio monotone property which implies the log-concavity and the spiral property. Boros and Moll [3, 4, 5, 6, 7, 10] explored a special class of Jacobi polynomials in their study of a quartic integral. They have shown that for any $a > -1$ and any nonnegative integer m,

$$\int_0^\infty \frac{1}{(x^4+2ax^2+1)^{m+1}}dx = \frac{\pi}{2^{m+3/2}(a+1)^{m+1/2}}P_m(a), \tag{1.1}$$

where

$$P_m(a) = \sum_{j,k}\binom{2m+1}{2j}\binom{m-j}{k}\binom{2k+2j}{k+j}\frac{(a+1)^j(a-1)^k}{2^{3(k+j)}}. \tag{1.2}$$

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Using Ramanujan's Master Theorem, Boros and Moll [6, 10] derived the following formula

$$P_m(a) = 2^{-2m} \sum_k 2^k \binom{2m-2k}{m-k} \binom{m+k}{k} (a+1)^k, \tag{1.3}$$

which indicates that the coefficients of a^i in $P_m(a)$ is positive for $0 \le i \le m$. Let $d_i(m)$ be defined by

$$P_m(a) = \sum_{i=0}^{m} d_i(m) a^i. \tag{1.4}$$

The polynomials $P_m(a)$ will be called the Boros-Moll polynomials, and the sequence $\{d_i(m)\}_{0 \le i \le m}$ of the coefficients will be called a Boros-Moll sequence. From (1.4), it follows that

$$d_i(m) = 2^{-2m} \sum_{k=i}^{m} 2^k \binom{2m-2k}{m-k} \binom{m+k}{k} \binom{k}{i}. \tag{1.5}$$

The readers can find in [2] many proofs of this formula. Recall that $P_m(a)$ can be expressed as a hypergeometric function

$$P_m(a) = 2^{-2m} \binom{2m}{m} {}_2F_1(-m, m+1; \frac{1}{2} - m; \frac{a+1}{2}),$$

from which one sees that $P_m(a)$ can be viewed as the Jacobi polynomial $P_m^{(\alpha,\beta)}(a)$ with $\alpha = m + \frac{1}{2}$ and $\beta = -(m + \frac{1}{2})$, where $P_m^{(\alpha,\beta)}(a)$ is given by

$$P_m^{(\alpha,\beta)}(a) = \sum_{k=0}^{m} (-1)^{m-k} \binom{m+\beta}{m-k} \binom{m+k+\alpha+\beta}{k} \left(\frac{1+a}{2}\right)^k.$$

Boros and Moll [4] proved that the sequence $\{d_i(m)\}_{0 \le i \le m}$ is unimodal and the maximum element appears in the middle, namely,

$$d_0(m) < d_1(m) < \cdots < d_{[\frac{m}{2}]}(m) > d_{[\frac{m}{2}]+1}(m) > \cdots > d_m(m).$$

They also established the unimodality by taking a different approach [5]. Moll [10] conjectured that the sequence $\{d_i(m)\}_{0 \le i \le m}$ is log-concave. Kauers and Paule [9] proved this conjecture based on four recurrence relations found using a computer algebra approach. Two of these four recurrences have been independently derived by Moll [11] using the WZ-method. Moreover, as will be seen, the two recurrences derived by Moll easily imply the other two given by Kauers and Paule. These recursions will be discussed in Section 2.

Recall that a sequence $\{a_i\}_{0\le i\le m}$ of positive numbers is said to be log-concave if

$$\frac{a_0}{a_1} \le \frac{a_1}{a_2} \le \cdots \le \frac{a_{m-1}}{a_m}.$$

A polynomial is said to be log-concave if the sequence of its coefficients is log-concave. It is easy to see that if a sequence is log-concave then it is unimodal. A sequence $\{a_i\}_{0\le i\le m}$ of positive numbers is said to be spiral if

$$a_m \le a_0 \le a_{m-1} \le a_1 \le \cdots \le a_{[\frac{m}{2}]}.$$

Similarly, a polynomial is said to be spiral if its sequence of coefficients is spiral. It is easily seen that a log-concave sequence is not necessarily spiral, and vice versa. For example, $(2, 10, 3, 1)$ is spiral but not log-concave, whereas $(3, 5, 4, 2, 1)$ is lgo-concave but not spiral. Chen and Xia [8] discovered that the q-derangement numbers are both spiral and log-concave, and introduced the *ratio monotone* property defined below, which implies both log-concavity and the spiral property. The purpose of this paper is to show that the Boros-Moll polynomials possess the ratio monotone property.

A sequence $\{a_i\}_{0\le i\le m}$ of positive numbers is said to be ratio monotone if

$$\frac{a_0}{a_{m-1}} \le \frac{a_1}{a_{m-2}} \le \cdots \le \frac{a_{i-1}}{a_{m-i}} \le \frac{a_i}{a_{m-1-i}} \le \cdots \le \frac{a_{[\frac{m}{2}]-1}}{a_{m-[\frac{m}{2}]}} \le 1 \tag{1.6}$$

and

$$\frac{a_m}{a_0} \le \frac{a_{m-1}}{a_1} \le \cdots \le \frac{a_{m-i}}{a_i} \le \frac{a_{m-1-i}}{a_{i+1}} \le \cdots \le \frac{a_{m-[\frac{m-1}{2}]}}{a_{[\frac{m-1}{2}]}} \le 1. \tag{1.7}$$

If every inequality relation in (1.6) and (1.7) becomes strict, we say that the sequence is strictly ratio monotone. It is easy to see that the ratio monotonicity implies log-concavity. In deeded, from (1.6) and (1.7), we deduce that

$$\frac{a_i}{a_{i-1}} \ge \frac{a_{m-1-i}}{a_{m-i}} \quad \text{and} \quad \frac{a_{i+1}}{a_i} \le \frac{a_{m-1-i}}{a_{m-i}}.$$

This gives

$$\frac{a_i}{a_{i-1}} \ge \frac{a_{i+1}}{a_i}.$$

The main result of this paper is stated as follows.

Theorem 1.1. *Let $m \ge 2$ be an integer. Then the Boros-Moll sequence $\{d_i(m)\}_{0\le i\le m}$ satisfies the strictly ratio monotone property. To be precise, we have*

$$\frac{d_m(m)}{d_0(m)} < \frac{d_{m-1}(m)}{d_1(m)} < \cdots < \frac{d_{m-i}(m)}{d_i(m)} < \frac{d_{m-i-1}(m)}{d_{i+1}(m)} < \cdots < \frac{d_{m-[\frac{m-1}{2}]}(m)}{d_{[\frac{m-1}{2}]}(m)} < 1 \tag{1.8}$$

and

$$\text{(1.9)}\quad \frac{d_0(m)}{d_{m-1}(m)} < \frac{d_1(m)}{d_{m-2}(m)} < \cdots < \frac{d_{i-1}(m)}{d_{m-i}(m)} < \frac{d_i(m)}{d_{m-i-1}(m)} < \cdots < \frac{d_{[\frac{m}{2}]-1}(m)}{d_{m-[\frac{m}{2}]}(m)} < 1.$$

As a corollary of Theorem 1.1, we obtain the spiral property of the Boros-Moll sequences. It is not clear whether there is a simpler way to verify this property directly.

Corollary 1.2. *Let $m \geq 2$ be an integer. Then the Boros-Moll sequence $\{d_i(m)\}_{0\leq i\leq m}$ is spiral.*

The following example illustrates our main result. For $m = 8$, we have

$$\begin{aligned} P_8(a) =& \frac{4023459}{32768} + \frac{3283533}{4096}a + \frac{9804465}{4096}a^2 + \frac{8625375}{2048}a^3 + \frac{9695565}{2048}a^4 \\ &+ \frac{1772199}{512}a^5 + \frac{819819}{512}a^6 + \frac{109395}{256}a^7 + \frac{6435}{128}a^8. \end{aligned}$$

The strictly ratio monotone property is illustrated as follows:

$$\frac{\frac{6435}{128}}{\frac{4023459}{32768}} < \frac{\frac{109395}{256}}{\frac{3283533}{4096}} < \frac{\frac{819819}{512}}{\frac{9804465}{4096}} < \frac{\frac{1772199}{512}}{\frac{8625375}{2048}} < 1,$$

$$\frac{\frac{4023459}{32768}}{\frac{109395}{256}} < \frac{\frac{3283533}{4096}}{\frac{819819}{512}} < \frac{\frac{9804465}{4096}}{\frac{1772199}{512}} < \frac{\frac{8625375}{2048}}{\frac{9695565}{2048}} < 1.$$

The spiral property of $P_8(x)$ is reflected by following order of the coefficients:

$$\begin{aligned} \frac{6435}{128} < \frac{4023459}{32768} < \frac{109395}{256} < \frac{3283533}{4096} < \frac{819819}{512} \\ < \frac{9804465}{4096} < \frac{1772199}{512} < \frac{8625375}{2048} < \frac{9695565}{2048}. \end{aligned}$$

Based on the Moll conjecture on the ∞-log-concavity of the sequences $\{d_i(m)\}_{0\leq i\leq m}$, we conclude this paper with a stronger conjecture that these polynomials are infinitely ratio monotone. Numerical evidence seems to be supportive of this conjecture.

2 Recurrence Relations

We first give a brief review of Kauers and Paule's approach to proving the log-concavity of the Boros-Moll sequence [9]. Our work employs the four recurrences

$$\text{(2.1)}\qquad d_i(m+1) = \frac{m+i}{m+1}d_{i-1}(m) + \frac{(4m+2i+3)}{2(m+1)}d_i(m), \qquad 0 \leq i \leq m+1,$$

$$(2.2)\qquad d_i(m+1) = \frac{(4m-2i+3)(m+i+1)}{2(m+1)(m+1-i)} d_i(m) - \frac{i(i+1)}{(m+1)(m+1-i)} d_{i+1}(m), \qquad 0 \le i \le m,$$

$$(2.3)\qquad d_i(m+2) = \frac{-4i^2+8m^2+24m+19}{2(m+2-i)(m+2)} d_i(m+1) - \frac{(m+i+1)(4m+3)(4m+5)}{4(m+2-i)(m+1)(m+2)} d_i(m), \qquad 0 \le i \le m+1,$$

and for $0 \le i \le m+1$,

$$(2.4)\quad (m+2-i)(m+i-1)d_{i-2}(m) - (i-1)(2m+1)d_{i-1}(m) + i(i-1)d_i(m) = 0.$$

These recurrences are derived by Kauers and Paule [9] with the RISC package MultiSum [12]. In fact, the recurrences (2.3) and (2.4) are also derived independently by Moll [11], and the other two relations (2.1) and (2.2) can be easily deduced from (2.3) and (2.4). Based on the four recurrence relations, Kauers and Paule [9] used a computer algebra system to derive the next theorem, from which the log-concavity of the Boros-Moll sequence is derived.

Theorem 2.1. *For $0 < i < m$, we have*

$$(2.5)\qquad d_i(m+1) \ge \frac{4m^2+7m+i+3}{2(m+1-i)(m+1)} d_i(m).$$

The inequality (2.5) is also of vital importance for our proof of the ratio monotonicity of the Boros-Moll sequences. We note that the above inequality (2.5) is very tight. In other words, the ratio

$$\frac{(4m^2+7m+i+3)d_i(m)}{2(m+1-i)(m+1)d_i(m+1)},$$

seems to be very close to 1. For example, for $m = 100$, the smallest ratio is 0.998348.

In order to establish the strict ratio monotonicity, we need a slightly sharper version of (2.5). For example, we will show that the inequality in (2.5) is strict for $1 \le i \le m-1$.

Theorem 2.2. *Let $m \ge 2$. We have*

$$(2.6)\qquad d_i(m+1) > \frac{4m^2+7m+i+3}{2(m+1-i)(m+1)} d_i(m), \qquad 1 \le i \le m-1,$$

and

$$(2.7)\qquad d_0(m+1) = \frac{4m+3}{2(m+1)} d_0(m),$$

$$d_m(m+1) = \frac{(2m+3)(2m+1)}{2(m+1)} d_m(m) = \frac{(2m+3)(2m+1)}{2(m+1)} 2^{-m} \binom{2m}{m}. \tag{2.8}$$

To make this paper self-contained, we will present a detailed proof of the above improvement of Theorem 2.1. Before doing so, we remark that (2.3) and (2.4) can be also derived from (2.1) and (2.2). Equating the right hand sides of (2.1) and (2.2) and replacing i by $i-1$, we get (2.4). Substituting i with $i+1$ and m with $m+1$ in (2.1) and (2.2), respectively, we obtain two expressions for $d_{i+1}(m+1)$. This yields

(2.9)
$$\begin{aligned} d_i(m+2) =& \frac{(4m-2i+7)(m+i+2)}{2(m+2)(m+2-i)} d_i(m+1) \\ &- \frac{i(i+1)}{(m+2)(m+2-i)} \left(\frac{m+i+1}{m+1} d_i(m) + \frac{(4m+2i+5)}{2(m+1)} d_{i+1}(m) \right) \\ =& \frac{(4m-2i+7)(m+i+2)}{2(m+2)(m+2-i)} d_i(m+1) - \frac{i(i+1)(m+i+1)}{(m+2)(m+2-i)(m+1)} d_i(m) \\ &- \frac{i(i+1)(4m+2i+5)}{(m+2)(m+2-i)(2m+2)} d_{i+1}(m). \end{aligned}$$

On the other hand, from (2.2), we have

$$d_{i+1}(m) = -\frac{(m+1)(m+1-i)}{i(i+1)} d_i(m+1) + \frac{(m+i+1)(4m-2i+3)}{2i(i+1)} d_i(m). \tag{2.10}$$

Substituting (2.10) into (2.9), we obtain (2.3).

We now present a proof of Theorem 2.2.

Proof. Clearly, (2.7) follows from (2.1) by setting $i=0$, and (2.8) can be obtained from (2.2) by setting $i=m$.

We proceed to prove (2.6) by induction on m. It is easy to verify that (2.6) holds for $m=2$. We assume that (2.6) holds for $n \geq 2$, namely,

$$d_i(n+1) > \frac{4n^2+7n+i+3}{2(n+1-i)(n+1)} d_i(n), \qquad 1 \leq i \leq n-1. \tag{2.11}$$

We aim to show that (2.6) holds for $n+1$, that is,

$$d_i(n+2) > \frac{4(n+1)^2+7(n+1)+i+3}{2(n+2)(n+2-i)} d_i(n+1), \quad 1 \leq i \leq n. \tag{2.12}$$

Observe that for $1 \leq i \leq n-1$,

$$2(n+i+1)(4n+3)(4n+5)(n+1-i)(n+1) - 2(4n^2+7n+i+3)$$

$$\times (n+1)(n+1-i)(4n+4i+5) = -4i(1+2i)(n+1)(n+1-i) < 0.$$

Hence we have for $1 \le i \le n-1$,

$$\frac{4n^2+7n+i+3}{2(n+1-i)(n+1)} > \frac{(n+i+1)(4n+3)(4n+5)}{2(n+1)(n+1-i)(4n+4i+5)}. \tag{2.13}$$

From the inequalities (2.13) and (2.11), we find that for $1 \le i \le n-1$,

$$d_i(n+1) > \frac{(n+i+1)(4n+3)(4n+5)}{2(n+1)(n+1-i)(4n+4i+5)} d_i(n). \tag{2.14}$$

It is easy to check that

$$\frac{\dfrac{(n+i+1)(4n+3)(4n+5)}{4(n+2-i)(n+1)(n+2)}}{\dfrac{-4i^2+8n^2+24n+19}{2(n+2-i)(n+2)} - \dfrac{4(n+1)^2+7(n+1)+i+3}{2(n+2-i)(n+2)}} = \frac{(n+i+1)(4n+3)(4n+5)}{2(n+1)(n+1-i)(4n+4i+5)}.$$

Hence the inequality (2.14) can be rewritten as

$$d_i(n+1) > \frac{\dfrac{(n+i+1)(4n+3)(4n+5)}{4(n+2-i)(n+1)(n+2)}}{\dfrac{-4i^2+8n^2+24n+19}{2(n+2-i)(n+2)} - \dfrac{4(n+1)^2+7(n+1)+i+3}{2(n+2-i)(n+2)}} d_i(n).$$

It follows that

$$\frac{-4i^2+8n^2+24n+19}{2(n+2-i)(n+2)} d_i(n+1) - \frac{(n+i+1)(4n+3)(4n+5)}{4(n+2-i)(n+1)(n+2)} d_i(n) \tag{2.15}$$
$$> \frac{4(n+1)^2+7(n+1)+i+3}{2(n+2-i)(n+2)} d_i(n+1).$$

From the recurrence relation (2.3), the left hand side of (2.15) equals $d_i(n+2)$. Thus we have verified the inequality (2.12) for $1 \le i \le n-1$. It is still necessary to show that (2.12) is true for $i = n$, that is,

$$d_n(n+2) > \frac{4(n+1)^2+8n+10}{4(n+2)} d_n(n+1). \tag{2.16}$$

Using the formula (1.5), we get

$$d_n(n+1) = 2^{-n-2}(2n+3)\binom{2n+2}{n+1},$$

$$d_n(n+2) = \frac{(n+1)(4n^2+18n+21)}{2^{n+4}(2n+3)}\binom{2n+4}{n+2}.$$

It is easily checked that for $n \geq 1$,

$$\frac{d_n(n+2)}{d_n(n+1)} = \frac{(n+1)(4n^2+18n+21)}{2(n+2)(2n+3)} > \frac{4(n+1)^2+8n+10}{4(n+2)}.$$

Hence the proof is complete by induction. □

3 Preliminary Inequalities

To prove the ratio monotone property of the Boros-Moll polynomials, we will establish the some inequalities based on the recurrence relations derived by Kauers and Paule [9] and Moll [11].

Lemma 3.1. *Let $m \geq 2$ be an integer. Then we have*

$$\frac{m-j}{j+1} > \frac{d_{j+1}(m)}{d_j(m)}, \qquad 1 \leq j \leq m-1. \tag{3.1}$$

Proof. From (2.2) and Theorem 2.2, we find that for $1 \leq j \leq m-1$,

$$\begin{aligned}(4m-2j+3)(m+j+1)d_j(m) - 2j(j+1)d_{j+1}(m) &= 2(m+1-j)(m+1)d_j(m+1)\\ &> (4m^2+7m+j+3)d_j(m),\end{aligned}$$

which implies (3.1). □

The following lemma gives an upper bound on the ratio $d_i(m+1)/d_i(m)$, which is crucial for the proof of the main result of this paper (Theorem 1.1).

Lemma 3.2. *Let $m \geq 2$ be a positive integer. We have for $0 \leq i \leq m$,*

$$d_i(m+1) \leq B(m,i)d_i(m), \tag{3.2}$$

where $B(m,i)$ is defined by

$$B(m,i) = \frac{A(m,i)}{2(i+2)(4m+2i+5)(m+1)(m-i+1)} \tag{3.3}$$

with

$$\begin{aligned}A(m,i) = {} & 30+96m^2+94m+37i+72m^2i+8m^2i^2-i^3\\ & +99mi+5i^2+13mi^2+16m^3i+32m^3.\end{aligned} \tag{3.4}$$

Proof. We proceed by induction on m. It is easily seen that the lemma holds for $m = 2$. We assume that the lemma is true for $n \geq 2$, i.e.,

$$d_i(n+1) \leq B(n,i)d_i(n), \qquad 0 \leq i \leq n, \tag{3.5}$$

where $B(n,i)$ is defined by (3.3). It will be shown that the lemma holds for $n+1$, that is,

$$d_i(n+2) \leq B(n+1,i)d_i(n+1), \quad 0 \leq i \leq n+1. \tag{3.6}$$

For $0 \leq i \leq n$, let

$$\begin{aligned} F(n,i) =& (4n+2i+9)(i+2)(4n+5)(4n+3)(n+i+1), \\ G(n,i) =& -2(-90-23i-202n+51i^3+60i^2-144n^2-32n^3 \\ & -80n^2i-8n^2i^2-97ni+13ni^2-16n^3i+16ni^3+8i^4)(n+1). \end{aligned}$$

We claim that

$$\frac{F(n,i)}{G(n,i)} \geq B(n,i), \qquad 0 \leq i \leq n. \tag{3.7}$$

Keeping in mind that $A(n,i)$ is defined by (3.4), it is easy to check that

$$\begin{aligned} & 2(i+2)(4n+2i+5)(n+1)(n-i+1)F(n,i) - A(n,i)G(n,i) \\ &= (128n^4i^4 - 32n^3i^5 - 80n^2i^6 - 16ni^7) + (618n^3i^4 - 222ni^6 - 16i^7 - 284n^2i^5) \\ &\quad + (844ni^3 - 170i^4) + (1502n^2i^3 - 338i^5) + (984n^2i^4 - 142i^6) \\ &\quad + (844n^3i^3 - 590ni^5) + 256n^5i^2 + 720i + 10i^3 + 788i^2 + 3984n^2i \\ &\quad + 2656ni + 3568ni^2 + 3136n^3i + 4600n^3i^2 + 256n^5i \\ &\quad + 1344n^4i + 324ni^4 + 176n^4i^3 + 5908n^2i^2 + 1728n^4i^2. \end{aligned}$$

We are now in a position to see that the above expression is always nonnegative since the expression in every parenthesis is nonnegative for $0 \leq i \leq n$. For example,

$$128n^4i^4 - 32n^3i^5 - 80n^2i^6 - 16ni^7 \geq 128n^4i^4 - 32n^4i^4 - 80n^4i^4 - 16n^4i^4 = 0.$$

Thus we have

$$2(i+2)(4n+2i+5)(n+1)(n-i+1)F(n,i) - A(n,i)G(n,i) \geq 0. \tag{3.8}$$

It is easy to see that $G(n,i)$ is positive for $0 \le i \le n$, and hence (3.7) can be deduced from (3.8). From the inductive hypothesis (3.5) and (3.8), it follows that for $0 \le i \le n$,

$$\frac{F(n,i)}{G(n,i)} d_i(n) \ge B(n,i) d_i(n) \ge d_i(n+1). \tag{3.9}$$

It is a routine to verify that

$$\frac{(n+1+i)(4n+3)(4n+5)}{4(n+1)(n+2)(n+2-i)\left(\frac{-4i^2+8n^2+24n+19}{2(n+2-i)(n+2)} - B(n+1,i)\right)} = \frac{F(n,i)}{G(n,i)}.$$

From the above identity and (3.9), it follows that for $0 \le i \le n$,

$$\begin{aligned} &\frac{(n+1+i)(4n+3)(4n+5)d_i(n)}{4(n+1)(n+2)(n+2-i)\left(\frac{-4i^2+8n^2+24n+19}{2(n+2-i)(n+2)} - B(n+1,i)\right)} \\ &\qquad = \frac{F(n,i)}{G(n,i)} d_i(n) \ge d_i(n+1). \end{aligned} \tag{3.10}$$

Since

$$\frac{-4i^2+8n^2+24n+19}{2(n+2-i)(n+2)} - B(n+1,i)$$

is positive for $0 \le i \le n$, (3.10) can be rewritten as

$$\begin{aligned} &\frac{-4i^2+8n^2+24n+19}{2(n+2-i)(n+2)} d_i(n+1) \\ &\qquad - \frac{(n+1+i)(4n+3)(4n+5)}{4(n+1)(n+2)(n+2-i)} d_i(n) \le B(n+1,i) d_i(n+1). \end{aligned} \tag{3.11}$$

From the recurrence relation (2.3), we see that

$$\frac{-4i^2+8n^2+24n+19}{2(n+2-i)(n+2)} d_i(n+1) - \frac{(n+1+i)(4n+3)(4n+5)}{4(n+1)(n+2)(n+2-i)} d_i(n) = d_i(n+2). \tag{3.12}$$

In view of (3.11) and (3.12), we find that the inequality (3.6) holds for $0 \le i \le n$. It remains to verify that (3.6) holds for $i = n+1$, that is,

$$d_{n+1}(n+2) \le B(n+1,n+1) d_{n+1}(n+1). \tag{3.13}$$

By the definition (3.3) of $B(n,i)$, we have

$$B(n+1,n+1) = \frac{501 + 212n^3 + 692n^2 + 975n + 24n^4}{2(n+3)(6n+11)(n+2)}.$$

From the formula (1.5) for $d_i(m)$, we get

$$d_{n+1}(n+1) = 2^{-n-1}\binom{2n+2}{n+1}$$

and

$$d_{n+1}(n+2) = 2^{-n-2}\binom{2n+3}{n+1} + 2^{-n-2}(n+2)\binom{2n+4}{n+2}.$$

Therefore, for $n \geq 0$, we have

$$\frac{d_{n+1}(n+2)}{d_{n+1}(n+1)} = \frac{(2n+3)(2n+5)}{2(n+2)} \leq \frac{501 + 212n^3 + 692n^2 + 975n + 24n^4}{2(n+3)(6n+11)(n+2)}.$$

This completes the proof of the lemma. □

Lemma 3.3. *Let $B(m,j)$ be defined by (3.3) and $m \geq 2$ be an integer. Then we have for $1 \leq j \leq m$,*

$$d_{j-1}(m) \leq \frac{2(m+1)B(m,j) - (4m+2j+3)}{2(m+j)} d_j(m). \tag{3.14}$$

Proof. From the recurrence relation (2.1) and Lemma 3.2, we find that for $0 \leq j \leq m$,

$$\begin{aligned} 2(m+1)d_j(m+1) &= 2(m+j)d_{j-1}(m) + (4m+2j+3)d_j(m) \\ &\leq 2(m+1)B(m,j)d_j(m), \end{aligned} \tag{3.15}$$

where $B(m,j)$ is defined by (3.3). Then (3.15) implies (3.14). □

Lemma 3.4. *Let m be a positive integer. For $0 \leq i \leq \frac{m}{2}$, we have*

$$\frac{2(2m-i)}{2(m+1)B(m,m-i) - (6m-2i+3)} > \frac{2(m+1)B(m,i) - (4m+2i+3)}{2(m+i)}, \tag{3.16}$$

where $B(m,i)$ is defined by (3.3).

Proof. For $0 \leq i \leq m$, let

$$N(m,i) = 2(2m-i)(m-i+2)(6m-2i+5)(i+1), \tag{3.17}$$

$$\begin{aligned} M(m,i) = 4(3m-i)(2m-i)(m-i)^2 + (80m^3 - 155m^2 i) \\ + (80m^2 - 108mi) + (20m - 20i) + (94mi^2 - 19i^3) + 28i^2, \end{aligned} \tag{3.18}$$

$$C(m,i) = i(24m^2 + 52m + 8m^2 i + 37mi + 4i^3 + 12mi^2 + 20 + 19i^2 + 28i), \tag{3.19}$$

$$(3.20) \qquad D(m,i) = 2(i+2)(4m+2i+5)(m-i+1)(i+m).$$

Note that $N(m,i), M(m,i), C(m,i)$ and $D(m,i)$ are all nonnegative for $0 \le i \le \frac{m}{2}$, since the sum in every parenthesis in (3.17), (3.18), (3.19) and (3.20) is nonnegative for $0 \le i \le \frac{m}{2}$. It is easy to check that

$$\begin{aligned}
&N(m,i)D(m,i) - C(m,i)M(m,i)\\
&\quad = (312m^5i^2 + 36m^2i^5 + 276m^3i^4 - 612m^4i^3 - 12mi^6) + (2040m^4i^2 - 2533m^3i^3)\\
&\qquad + (129mi^5 - 43i^6) + (384m^6 - 752m^5i) + (3568m^4 - 3328m^3i)\\
&\qquad + (1952m^5 - 2792m^4i) + (4280m^3i^2 - 2976m^2i^3) + (2800m^3 - 1240m^2i)\\
&\qquad + (3868m^2i^2 - 1080mi^3) + 1240mi^2 + 1488mi^4 + 540i^4 + 800m^2 + 1159m^2i^4.
\end{aligned}$$

Observe that the expression in every parenthesis in the above sum is nonnegative for $0 \le i \le \frac{m}{2}$. Moreover, one sees the term $800m^2$ is certainly positive. It follows that

$$(3.21) \qquad N(m,i)D(m,i) - C(m,i)M(m,i) > 0, \qquad 0 \le i \le \frac{m}{2}.$$

Recall that $B(n,i)$ is defined by (3.3). It is easy to check that

$$\frac{2(m+1)B(m,i) - (4m+2i+3)}{2(m+i)} = \frac{C(m,i)}{D(m,i)},$$

$$\frac{2(2m-i)}{2(m+1)B(m,m-i) - (6m-2i+3)} = \frac{N(m,i)}{M(m,i)}.$$

Thus the inequality (3.21) is equivalent to (3.16). This completes the proof of the lemma. □

4 Proof of the Main Theorem

Using the preliminary inequalities presented in the previous section, we are ready to give a proof of Theorem 1.1.

Proof. It is clear that Theorem 1.1 holds for $m = 2, 3, 4$. We now assume that $m \ge 5$. First we consider (1.8). In order to verify

$$(4.1) \qquad \frac{d_m(m)}{d_0(m)} < \frac{d_{m-1}(m)}{d_1(m)},$$

we invoke the formula (1.5) to get

$$(4.2) \qquad \frac{d_1(m)}{d_0(m)} = \frac{2^{-2m}\sum_{k=1}^{m} 2^k\binom{2m-2k}{m-k}\binom{m+k}{m}k}{2^{-2m}\sum_{k=0}^{m} 2^k\binom{2m-2k}{m-k}\binom{m+k}{m}} < \frac{\sum_{k=1}^{m} 2^k\binom{2m-2k}{m-k}\binom{m+k}{m}m}{\sum_{k=1}^{m} 2^k\binom{2m-2k}{m-k}\binom{m+k}{m}} = m,$$

and

$$\frac{d_{m-1}(m)}{d_m(m)} = \frac{2^{-m}\binom{2m-1}{m} + 2^{-m}\binom{2m}{m}m}{2^{-m}\binom{2m}{m}} > m. \tag{4.3}$$

Combining (4.2) and (4.3), we obtain

$$\frac{d_1(m)}{d_0(m)} < \frac{d_{m-1}(m)}{d_m(m)},$$

which yields (4.1).

The next step is to show that

$$\frac{d_{m-i}(m)}{d_i(m)} < \frac{d_{m-i-1}(m)}{d_{i+1}(m)}, \qquad 1 \le i \le \left[\frac{m-1}{2}\right] - 1. \tag{4.4}$$

By the assumption $m \ge 5$, we have $\left[\frac{m-1}{2}\right] - 1 \ge 1$. Substituting j with i in (3.1), we have for $1 \le i \le \left[\frac{m-1}{2}\right] - 1$,

$$\frac{d_{i+1}(m)}{d_i(m)} < \frac{m-i}{i+1}. \tag{4.5}$$

On the other hand, since $1 \le i \le \left[\frac{m-1}{2}\right] - 1$, we have $m - \left[\frac{m-1}{2}\right] \le m - i - 1 \le m - 2$. Hence we may substitute j with $m - i - 1$ in (3.1) to deduce that

$$\frac{d_{m-i-1}(m)}{d_{m-i}(m)} > \frac{m-i}{i+1}. \tag{4.6}$$

From (4.5) and (4.6), it follows that for $1 \le i \le \left[\frac{m-1}{2}\right] - 1$,

$$\frac{d_{i+1}(m)}{d_i(m)} < \frac{m-i}{i+1} < \frac{d_{m-i-1}(m)}{d_{m-i}(m)}.$$

Hence we have verified (4.4).

It remains to show that the last ratio in (1.8) is smaller than 1. Since $\left[\frac{m-1}{2}\right] < m - \left[\frac{m-1}{2}\right]$, it is easily seen that for $m - \left[\frac{m-1}{2}\right] \le k \le m$, we have

$$\binom{k}{\left[\frac{m-1}{2}\right]} \ge \binom{k}{m - \left[\frac{m-1}{2}\right]}.$$

Based on the formula (1.5) and the above relation, we obtain that

$$d_{\left[\frac{m-1}{2}\right]}(m) = 2^{-2m} \sum_{k=\left[\frac{m-1}{2}\right]}^{m} 2^k \binom{2m-2k}{m-k}\binom{m+k}{k}\binom{k}{\left[\frac{m-1}{2}\right]}$$

$$> 2^{-2m} \sum_{k=m-\left[\frac{m-1}{2}\right]}^{m} 2^k \binom{2m-2k}{m-k}\binom{m+k}{k}\binom{k}{\left[\frac{m-1}{2}\right]}$$

$$\geq 2^{-2m} \sum_{k=m-\left[\frac{m-1}{2}\right]}^{m} 2^k \binom{2m-2k}{m-k}\binom{m+k}{k}\binom{k}{m-\left[\frac{m-1}{2}\right]}$$

$$= d_{m-\left[\frac{m-1}{2}\right]}(m),$$

leading to the relation

$$\frac{d_{m-\left[\frac{m-1}{2}\right]}(m)}{d_{\left[\frac{m-1}{2}\right]}(m)} < 1.$$

This completes the proof of (1.8).

We now turn our attention to the proof (1.9), which will rely on the bound $B(n, i)$ and Lemmas 3.3 and 3.4. First, rewrite (3.14) as

$$\frac{d_{i-1}(m)}{d_i(m)} \leq \frac{2(m+1)B(m,i) - (4m+2i+3)}{2(m+i)}, \quad 1 \leq i \leq m. \tag{4.7}$$

For $1 \leq i \leq \left[\frac{m}{2}\right]$, we have $m - \left[\frac{m}{2}\right] \leq m - i \leq m - 1$. It follows that

$$\begin{aligned} &2(m+1)B(m,j) - (4m+2j+3) \\ &= \frac{j(24m^2 + 8m^2 j + 52m + 37mj + 19j^2 + 28j + 20 + 12mj^2 + 4j^3)}{(j+2)(4m+2j+5)(m-j+1)}, \end{aligned} \tag{4.8}$$

which is positive for $1 \leq j \leq m$. Substituting j with $m - i$ in (4.8), we obtain that

$$2(m+1)B(m, m-i) - (6m - 2i + 3) > 0, \qquad 1 \leq i \leq \left[\frac{m}{2}\right].$$

Hence we can substitute j with $m - i$ in (3.14) to deduce that for $1 \leq i \leq \left[\frac{m}{2}\right]$,

$$\frac{d_{m-i}(m)}{d_{m-i-1}(m)} \geq \frac{2(2m-i)}{2(m+1)B(m,m-i) - (6m-2i+3)}. \tag{4.9}$$

Combining (4.7), (4.9) and Lemma 3.4, we obtain that for $1 \leq i \leq \left[\frac{m}{2}\right]$,

$$\frac{d_{i-1}(m)}{d_i(m)} < \frac{d_{m-i}(m)}{d_{m-i-1}(m)},$$

which can be restated as

$$\frac{d_{i-1}(m)}{d_{m-i}(m)} < \frac{d_i(m)}{d_{m-i-1}(m)}, \qquad 1 \leq i \leq \left[\frac{m}{2}\right]. \tag{4.10}$$

At this point, it is necessary to show that

$$\frac{d_{[\frac{m}{2}]-1}(m)}{d_{m-[\frac{m}{2}]}(m)} < 1. \tag{4.11}$$

For $i = \left[\frac{m}{2}\right]$, (4.10) becomes

$$\frac{d_{[\frac{m}{2}]-1}(m)}{d_{m-[\frac{m}{2}]}(m)} < \frac{d_{[\frac{m}{2}]}(m)}{d_{m-[\frac{m}{2}]-1}(m)}. \tag{4.12}$$

When m is even, we have $\left[\frac{m}{2}\right] = m - \left[\frac{m}{2}\right]$. From (4.12) it follows that

$$\frac{d_{[\frac{m}{2}]-1}(m)}{d_{m-[\frac{m}{2}]}(m)} < \frac{d_{m-[\frac{m}{2}]}(m)}{d_{[\frac{m}{2}]-1}(m)},$$

which implies (4.11). When m is odd, we have $\left[\frac{m}{2}\right] = m - \left[\frac{m}{2}\right] - 1$. Then (4.11) immediately follows from (4.12). This completes the proof of Theorem 1.1. $\square$

5 A Conjecture

Moll made a conjecture on a property of the Boros-Moll sequences which is stronger than the log-concavity. Given a sequence $A = \{a_i\}_{0\le i\le n}$, define the operator $\mathcal{L}$ by $\mathcal{L}(A) = S = \{b_i\}_{0\le i\le n}$, where

$$b_i = a_i^2 - a_{i-1}a_{i+1}, \qquad 0 \le i \le n,$$

with the convention that $a_{-1} = a_{n+1} = 0$. We say that $\{a_i\}_{0\le i\le n}$ is k-log-concave if $\mathcal{L}^j(\{a_i\}_{0\le i\le n})$ is log-concave for every $0 \le j \le k-1$, and that $\{a_i\}_{0\le i\le n}$ is ∞-log-concave if $\mathcal{L}^k(\{a_i\}_{0\le i\le n})$ is log-concave for every $k \ge 0$. Similarly, we say that $\{a_i\}_{0\le i\le n}$ is j-ratio-monotone (resp. j-strictly-ratio-monotone) if $\mathcal{L}^k(\{a_i\}_{0\le i\le n})$ is ratio monotone (resp. strictly ratio monotone) for every $0 \le k \le j-1$, and that $\{a_i\}_{0\le i\le n}$ is ∞-ratio-monotone (resp. ∞-strictly-ratio-monotone) if $\mathcal{L}^k(\{a_i\}_{0\le i\le n})$ is ratio monotone (resp. strictly ratio monotone) for every $k \ge 0$.

Moll [10] has conjectured that the Boros-Moll sequence $\{d_i(m)\}_{0\le i\le m}$ is ∞-log-concave. We propose a stronger conjecture.

Conjecture 5.1. *Suppose that $m \ge 2$ is a positive integer, then the Boros-Moll sequence $\{d_i(m)\}_{0\le i\le m}$ is ∞-strictly-ratio-monotone.*

We have verified that the Boros-Moll sequence $\{d_i(m)\}_{0\le i\le m}$ is 2-strictly-ratio-monotone for $2 \le m \le 100$. For example, $\mathcal{L}(\{d_i(8)\}_{0\le i\le 8})$ is given by

$$b_0 = \frac{16188222324681}{1073741824}, \qquad b_1 = \frac{46804848752277}{134217728}, \qquad b_2 = \frac{39484127036475}{16777216},$$

$$b_3 = \frac{53734360083525}{8388608}, \qquad b_4 = \frac{32860456870725}{4194304}, \qquad b_5 = \frac{4614148779669}{1048576},$$
$$b_6 = \frac{284363773551}{262144}, \qquad b_7 = \frac{836466345}{8192}, \qquad b_8 = \frac{41409225}{16384}.$$

It is easy to verify that

$$\frac{b_8}{b_0} < \frac{b_7}{b_1} < \frac{b_6}{b_2} < \frac{b_5}{b_3} < 1, \qquad \frac{b_0}{b_7} < \frac{b_1}{b_6} < \frac{b_2}{b_5} < \frac{b_3}{b_4} < 1.$$

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