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Families of Sets with Intersecting Clusters

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In Memory of Professor Chao Ko

Abstract. A family of k-subsets $A_1, A_2, \ldots, A_d$ on $[n] = \{1, 2, \ldots, n\}$ is called a (d, c)-cluster if the union $A_1 \cup A_2 \cup \cdots \cup A_d$ contains at most ck elements with $c < d$. Let $\mathcal{F}$ be a family of k-subsets of an n-element set. We show that for $k \geq 2$ and $n \geq k + 2$, if every $(k, 2)$-cluster of $\mathcal{F}$ is intersecting, then $\mathcal{F}$ contains no $(k-1)$-dimensional simplices. This leads to an affirmative answer to Mubayi's conjecture for $d = k$ based on Chvátal's simplex theorem. We also show that for any d satisfying $3 \leq d \leq k$ and $n \geq \frac{dk}{d-1}$, if every $(d, \frac{d+1}{2})$-cluster is intersecting, then $|\mathcal{F}| \leq \binom{n-1}{k-1}$ with equality only when $\mathcal{F}$ is a complete star. This result is an extension of both Frankl's theorem and Mubayi's theorem.

Keywords: clusters of subsets, Chvátal's simplex theorem, d-simplex, Erdős–Ko–Rado theorem
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1 Introduction

This paper is concerned with the study of families of subsets with intersecting clusters. The first result is a proof of an extreme case of a conjecture recently proposed by

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Mubayi [5] on intersecting families with the aid of Chvátal's simplex theorem. The second result is an extension of both Frankl's theorem and Mubayi's theorem. It should be noted that we have used these two theorems themselves as starting points in proving this extension.

Let us review some notation and terminology. The set $\{1, 2, \ldots, n\}$ is usually denoted by $[n]$, and the family of all k-subsets of a finite set X is denoted by X^k or $\binom{X}{k}$. A family $\mathcal{F}$ of sets is said to be intersecting if every two sets in $\mathcal{F}$ have a nonempty intersection. A family $\mathcal{F}$ of sets in X^k is called a complete star if $\mathcal{F}$ consists of all k-subsets containing x for some $x \in X$.

The classical Erdős–Ko–Rado (EKR) theorem [2] is stated as follows.

Theorem 1.1 (the EKR theorem)**.** *Let $n \geq 2k$, and let $\mathcal{F} \subseteq \binom{[n]}{k}$ be an intersecting family; then $|\mathcal{F}| \leq \binom{n-1}{k-1}$. Furthermore, for $n > 2k$, the equality holds only when $\mathcal{F}$ is a complete star.*

The following generalization of the EKR theorem is due to Frankl [3].

Theorem 1.2 (Frankl)**.** *Let $k \geq 2$, $d \geq 2$, and $n \geq dk/(d-1)$. Suppose that $\mathcal{F} \subseteq [n]^k$ such that every d sets of $\mathcal{F}$ have a nonempty intersection. Then $|\mathcal{F}| \leq \binom{n-1}{k-1}$ with equality only when $\mathcal{F}$ is a complete star.*

The following conjecture due to Erdős on triangle-free families implies Frankl's theorem for $d \geq 3$. Recall that a d-dimensional simplex, or a d-simplex for short, is defined to be a family of $d+1$ sets $A_1, A_2, \ldots, A_{d+1}$ such that every d of them have a nonempty intersection, but $A_1 \cap A_2 \cap \cdots \cap A_{d+1} = \emptyset$. A 2-dimensional simplex is called a triangle. This conjecture has been proved by Mubayi and Verstraëte [7].

Conjecture 1.3 (Erdős)**.** *For $n \geq \frac{3k}{2}$, if $\mathcal{F} \subseteq [n]^k$ contains no triangle, then $|\mathcal{F}| \leq \binom{n-1}{k-1}$ with equality only when $\mathcal{F}$ is a complete star.*

However, as a generalization of Erdős' conjecture, Chvátal [1] proposed the following conjecture, which remains open in the general case.

Conjecture 1.4 (Chvátal's simplex conjecture)**.** *Let $k \geq d+1 \geq 3$, $n \geq k(d+1)/d$, and $\mathcal{F} \subseteq [n]^k$. If $\mathcal{F}$ contains no d-dimensional simplices, then $|\mathcal{F}| \leq \binom{n-1}{k-1}$ with equality only when $\mathcal{F}$ is a complete star.*

Chvátal [1] has shown that Conjecture 1.4 holds for $d = k-1$, which we call Chvátal's simplex theorem.

Theorem 1.5 (Chvátal's simplex theorem)**.** *For $n \geq k+2 \geq 5$, if $\mathcal{F} \subseteq [n]^k$ contains no $(k-1)$-dimensional simplices, then $|\mathcal{F}| \leq \binom{n-1}{k-1}$ with equality only when $\mathcal{F}$ is a complete star.*

Frankl and Füredi [4] have shown that Chvátal's conjecture holds for sufficiently large n.

Theorem 1.6 (Frankl and Füredi)**.** *For $k \geq d+2 \geq 4$, there exists n_0 such that, for $n > n_0$, if $\mathcal{F} \subseteq [n]^k$ contains no d-dimensional simplices, then $|\mathcal{F}| \leq \binom{n-1}{k-1}$ with equality only when $\mathcal{F}$ is a complete star.*

As will be seen, a recent conjecture proposed by Mubayi [5] is related to Chvátal's simplex theorem. Here we introduce the terminology of clusters of subsets. A family of k-subsets $A_1, A_2, \ldots, A_d$ of $[n]$ is called a (d,c)-cluster if $|A_1 \cup A_2 \cup \cdots \cup A_d| \leq ck$, where $c < d$ is a constant that may depend on d. A cluster $\{A_1, A_2, \ldots, A_d\}$ is said to be intersecting if $A_1 \cap A_2 \cap \cdots \cap A_d \neq \emptyset$.

Conjecture 1.7 (Mubayi's conjecture)**.** *Let $k \geq d \geq 3$ and $n \geq dk/(d-1)$. Suppose that $\mathcal{F} \subseteq [n]^k$ such that every $(d,2)$-cluster of $\mathcal{F}$ is intersecting; i.e., for any $A_1, A_2, \ldots, A_d \in \mathcal{F}$, $|A_1 \cup A_2 \cup \cdots \cup A_d| \leq 2k$ implies $A_1 \cap A_2 \cap \cdots \cap A_d \neq \emptyset$. Then $|\mathcal{F}| \leq \binom{n-1}{k-1}$ with equality only when $\mathcal{F}$ is a complete star.*

Mubayi [5] has shown that this conjecture holds for $d = 3$ (Theorem 1.8). He has also proved that his conjecture holds for $d = 4$ when n is sufficiently large [6].

Theorem 1.8 (Mubayi)**.** *Let $k \geq 3$ and $n \geq \frac{3k}{2}$. Suppose that $\mathcal{F} \subseteq [n]^k$ is a family such that every $(3,2)$-cluster $A_1, A_2, A_3 \in \mathcal{F}$ is intersecting; then $|\mathcal{F}| \leq \binom{n-1}{k-1}$ with equality only when $\mathcal{F}$ is a complete star.*

In this paper, we study the case $d = k$ of Mubayi's conjecture in connection with Chvátal's simplex theorem. We show that in this case the conditions for Mubayi's conjecture imply the nonexistence of any $(k-1)$-dimensional simplex. Therefore, Chvátal's simplex theorem leads to Mubayi's conjecture for $d = k$. As the main result of this paper, we present a theorem on families of subsets with intersecting clusters which can be viewed as an extension of both Frankl's theorem (Theorem 1.2) and Mubayi's theorem (Theorem 1.8).

2 Families of subsets with intersecting clusters

In this section, we first consider a special case of Mubayi's conjecture for $k = d$. We show that this case can be deduced from Chvátal's simplex theorem (Theorem 1.5). Then we study families of k-subsets with intersecting $(d, \frac{d+1}{2})$-clusters and obtain a theorem as an extension of both Frankl's theorem (Theorem 1.2) and Mubayi's theorem (Theorem 1.8). Our proof is based on the EKR theorem and Frankl's theorem. We will also use a similar strategy as in the proof of Mubayi's theorem [5].

Theorem 2.1. *Let $k \geq 3$ and $n \geq k+2$. Suppose that $\mathcal{F} \subseteq [n]^k$ is a family of subsets of $[n]$ such that every $(k,2)$-cluster is intersecting. Then $\mathcal{F}$ contains no $(k-1)$-dimensional simplices.*

Proof. Suppose to the contrary that $A_1, A_2, \ldots, A_k \in \mathcal{F}$ form a $(k-1)$-dimensional simplex; namely, every $k-1$ of them have a nonempty intersection but

$$A_1 \cap A_2 \cap \cdots \cap A_k = \emptyset. \tag{1}$$

It follows that any two distinct families $\{A_{i_1}, A_{i_2}, \ldots, A_{i_{k-1}}\}$ and $\{A_{j_1}, A_{j_2}, \ldots, A_{j_{k-1}}\}$ cannot have a common element, because the union of these two families equals $\{A_1, A_2, \ldots, A_k\}$. Without loss of generality, let

$$i \in A_1 \cap \cdots \cap A_{i-1} \cap A_{i+1} \cap \cdots \cap A_k.$$

That is, i belongs to every subset A_j other than A_i. It follows that $\{1, \ldots, i-1, i+1, \ldots, k\} \subset A_i$. Since A_i is a k-subset, A_i must contain an element in $\{k+1, \ldots, n\}$. So we have

$$|A_1 \cup A_2 \cup \cdots \cup A_k| \leq 2k.$$

This means that $\{A_1, A_2, \ldots, A_k\}$ is a $(k,2)$-cluster that is not intersecting, contradicting the assumption of the theorem. So we conclude that $\mathcal{F}$ does not contain any $(k-1)$-dimensional simplex. This completes the proof. □

The following theorem is the main result of this paper.

Theorem 2.2. *Let $k \geq d \geq 3$ and $n \geq \frac{dk}{d-1}$. Suppose that $\mathcal{F} \subseteq [n]^k$ is a family of subsets of $[n]$ such that every $(d, \frac{d+1}{2})$-cluster is intersecting (i.e., for any $A_1, A_2, \ldots, A_d \in \mathcal{F}$, $|A_1 \cup A_2 \cup \cdots \cup A_d| \leq \frac{d+1}{2}k$ implies that $\cap_{i=1}^d A_i \neq \emptyset$). Then $|\mathcal{F}| \leq \binom{n-1}{k-1}$ with equality only when $\mathcal{F}$ is a complete star.*

The next lemma gives an upper bound on the number of edges in a graph with intersecting clusters, and it will be used in the proof of Theorem 2.2.

Lemma 2.3. *Let $n > d \geq 3$. Suppose that $\mathcal{F} \subseteq [n]^2$ is a family of 2-subsets of $[n]$ such that every $(d, \frac{d+1}{2})$-cluster is intersecting. Then $|\mathcal{F}| \leq n-1$ with equality only when $\mathcal{F}$ is a complete star.*

Proof. Since $\mathcal{F}$ is a family of 2-subsets, we may consider it as a graph G with vertex set $[n]$. The conditions in the lemma imply that any d edges $A_1, A_2, \ldots, A_d$ of G either intersect at a common vertex or cover at least $d+2$ vertices (for $d=3$, G does not contain any triangle because every $(3,2)$-cluster is intersecting).

We proceed by induction on n. For $n = d+1$, since any d edges cover at most $n = d+1$ vertices, any d edges of G must intersect at a common vertex and thus form a star. This implies that

$$|\mathcal{F}| = |E(G)| \leq d = n-1$$

with equality only when $\mathcal{F}$ (or G) is a complete star.

Assume that $n \geq d+2$ and that the lemma holds for $n-1$. We first claim that G must contain a vertex of degree one. Otherwise, every vertex of G has degree at least two, which implies that for every connected component C of G we have

$$|V(C)| \leq |E(C)|. \tag{2}$$

Let $C_1, C_2, \ldots, C_m$ be the connected components of G ordered by the condition

$$|E(C_1)| \geq |E(C_2)| \geq \cdots \geq |E(C_m)|.$$

We aim to find d edges that form a nonintersecting $(d, \frac{d+1}{2})$-cluster to reach a contradiction. Let us consider two cases.

Case 1. $|C_1| \geq d$. Since C_1 is not a star, it contains a path P with three edges. Since $d \geq 3$, we can add $d-3$ edges to P to obtain a connected subgraph H of C_1. Let $A_1, A_2, \ldots, A_d$ be d edges of H. Then we have

$$|A_1 \cup A_2 \cup \cdots \cup A_d| = |V(H)| \leq |E(H)| + 1 = d+1.$$

Since H is not a star, we have $A_1 \cap A_2 \cap \cdots \cap A_d = \emptyset$.

Case 2. $|C_1| < d$. Let $r \geq 1$ be the integer such that

$$b = \sum_{i=1}^{r} |E(C_i)| < d \quad \text{and} \quad \sum_{i=1}^{r+1} |E(C_i)| \geq d.$$

It is clear that C_{r+1} has at least $d-b$ edges. We now take any connected subgraph H of C_{r+1} with $d-b$ edges. Since H is connected, we have

$$|E(H)| \geq |V(H)| - 1. \tag{3}$$

Let $A_1, A_2, \ldots, A_d$ be d edges in $C_1, C_2, \ldots, C_r, H$. From (2) and (3) it follows that

$$\begin{aligned} &|A_1 \cup A_2 \cup \cdots \cup A_d| \\ &\quad = |V(C_1)| + |V(C_2)| + \cdots + |V(C_r)| + |V(H)| \\ &\quad \leq |E(C_1)| + |E(C_2)| + \cdots + |E(C_r)| + |E(H)| + 1 \\ &\quad = d+1. \end{aligned}$$

Noting that $C_1, C_2, \ldots, C_r$ and H are disjoint, we have $A_1 \cap A_2 \cap \cdots \cap A_d = \emptyset$.

In summary, we have reached the conclusion that G has a vertex with degree one. Let v be a vertex of degree one in G, and let G' be the induced subgraph obtained from G by deleting the vertex v. Clearly, G' is a graph with $n-1$ vertices in which every d edges $A_1, A_2, \ldots, A_d$ either intersect at a common vertex or cover at least $d+2$

vertices. By the inductive hypothesis, we have $|E(G')| \le n-2$ with equality only if G' is a complete star. Hence

$$|\mathcal{F}| = |E(G)| = |E(C)| + 1 \le n - 1$$

with equality only if $\mathcal{F}$ (or G) is a complete star. $\square$

The following lemma is an extension of Lemma 3 of Mubayi [5]. While the proof of Mubayi relies on the EKR theorem, our proof is based on the above Lemma 2.3 and Frankl's theorem (Theorem 1.2). We will also use a similar approach as in the proof of Mubayi's theorem [5].

Lemma 2.4. *Let $k \ge d \ge 2$, $t \ge 2$, and $2 \le l \le k$. Let $S_1, S_2, \dots, S_t$ be pairwise disjoint k-subsets and $X = S_1 \cup S_2 \cup \cdots \cup S_t$. Suppose that $\mathcal{F}$ is a family of l-subsets of X satisfying the conditions (1) $S_i \in \mathcal{F}$ for all i if $l = k$; (2) for every $A_1, A_2, \dots, A_d \in \mathcal{F}$ and $1 \le i \le t$, $A_1 \cap A_2 \cap \cdots \cap A_d \cap S_i = \emptyset$ implies $|A_1 \cup A_2 \cup \cdots \cup A_d - S_i| > \frac{dl}{2}$. Then we have $|\mathcal{F}| < \binom{tk-1}{l-1}$.*

Proof. For $d = 2$, the above lemma reduces to Lemma 3 in [5]. So we may assume that $d \ge 3$. Let $n = |X| = tk$. We consider the following two cases.

Case 1. $l = 2$. We claim that any $(d, \frac{d+1}{2})$-cluster of $\mathcal{F}$ is intersecting; namely, for any $A_1, A_2, \dots, A_d \in \mathcal{F}$, we have either $A_1 \cap A_2 \cap \cdots \cap A_d \ne \emptyset$ or $|A_1 \cup A_2 \cup \cdots \cup A_d| \ge d+2$. To this end, we assume that $A_1 \cap A_2 \cap \cdots \cap A_d = \emptyset$. This gives $A_1 \cap A_2 \cap \cdots \cap A_d \cap S_i = \emptyset$ for any S_i. Since $X = \cup S_i$ is the ground set of $\mathcal{F}$, there exists S_m such that $A_1 \cap S_m \ne \emptyset$. As $A_1 \cap A_2 \cap \cdots \cap A_d \cap S_m = \emptyset$ and $l = 2$, in view of condition (2) we get

$$|A_1 \cup A_2 \cup \cdots \cup A_d - S_m| > d.$$

Furthermore, the condition $A_1 \cap S_m \ne \emptyset$ yields

$$|A_1 \cup A_2 \cup \cdots \cup A_d| > d + 1.$$

So the claim holds.

Since $d \ge 3$, by Lemma 2.3, we find that $|\mathcal{F}| \le n - 1$, where $n = tk$. So it remains to show that it is impossible for $|\mathcal{F}|$ to reach the upper bound $n - 1$. Assume that $|\mathcal{F}| = n - 1$. Again, by Lemma 2.3, $\mathcal{F}$ must be a complete star; namely, $\mathcal{F}$ consists of all 2-subsets of X for some x in X. Without loss of generality, we may assume that $x \in S_1$. Let A_1 be a 2-subset from $\mathcal{F}$ such that $A_1 \subseteq S_1$. Since $d - 1 \le k$, we may choose $(d-1)$ 2-subsets $A_2, A_3, \dots, A_d$ such that $A_i \in \mathcal{F}$ and $A_i - x \subseteq S_2$ for $2 \le i \le d$. This implies that

$$A_1 \cap A_2 \cap \cdots \cap A_d \cap S_2 = \emptyset$$

and

$$|(A_1 \cup A_2 \cup \cdots \cup A_d) - S_2| = 2 < d,$$

contradicting condition (2). Thus we have $|\mathcal{F}| < n-1 = tk-1$. So the lemma is proved for $l = 2$.

Case 2. $l \geq 3$. So we have $k \geq l \geq 3$. We use induction on t.

We first consider the case $t = 2$, namely, $X = S_1 \cup S_2$. We will show that $A_1 \cap A_2 \cap \cdots \cap A_d \neq \emptyset$ for any $A_1, A_2, \ldots, A_d \in \mathcal{F}$. If this were not true, there would exist subsets $A_1, A_2, \ldots, A_d \in \mathcal{F}$ for which

$$A_1 \cap A_2 \cap \cdots \cap A_d = \emptyset. \tag{4}$$

Let $A = A_1 \cup A_2 \cup \cdots \cup A_d$. It is clear that A contains at most dl elements. Since S_1 and S_2 are disjoint, so are $A \cap S_1$ and $A \cap S_2$. Therefore, either $A \cap S_1$ or $A \cap S_2$ contains at most half of the elements in A. We may assume without loss of generality that

$$|A \cap S_1| \leq \frac{dl}{2}.$$

Note that (4) implies $A_1 \cap A_2 \cap \cdots \cap A_d \cap S_2 = \emptyset$. Since $X = S_1 \cup S_2$, we get

$$|A - S_2| = |A \cap S_1| \leq \frac{dl}{2},$$

contradicting condition (2). Thus we deduce that $A_1 \cap A_2 \cap \cdots \cap A_d \neq \emptyset$ for any $A_1, A_2, \ldots, A_d \in \mathcal{F}$. By Frankl's theorem (Theorem 1.2), we obtain

$$|\mathcal{F}| \leq \binom{2k-1}{l-1}. \tag{5}$$

Next we prove that equality in (5) can never be reached. Let us assume that

$$|\mathcal{F}| = \binom{2k-1}{l-1}. \tag{6}$$

Since $d \geq 3$, by Frankl's theorem, $\mathcal{F}$ is a complete star; that is, $\mathcal{F}$ consists of all l-subsets of $[2k]$ containing an element x for some x in $[2k]$. Without loss of generality, we may assume that $x \in S_1$. Thus $\mathcal{F}$ contains every subset A_i which is either of the form $B \cup \{x\}$ for $B \in [S_1 - x]^{l-1}$ or of the form $C \cup \{x\}$ for $C \in [S_2]^{l-1}$. Since $d \leq k$ and $3 \leq l \leq k$, we have

$$d - 1 \leq k \leq \binom{k}{l-1}.$$

Now we may choose $A_1 \in \mathcal{F}$ with $A_1 \subseteq S_1$ and $d-1$ sets $A_2, A_3, \ldots, A_d \in \mathcal{F}$ with $A_i - x \subseteq S_2$ for each $i \geq 2$. Since $A_1 \cap S_2 = \emptyset$, $A_1 \cap A_2 \cdots \cap A_d \cap S_2 = \emptyset$. Moreover, since $A_i - x \subseteq S_2$ for $i = 2, 3, \ldots, d$, we have

$$|(A_1 \cup A_2 \cup \cdots \cup A_d) - S_2| = |A_1| = l < \frac{dl}{2},$$

contradicting condition (2). It follows that $|\mathcal{F}| < \binom{2k-1}{l-1}$, and hence the lemma is valid for $t = 2$.

Next suppose that $t \geq 3$ and that the result holds for $t-1$. We first show that there exists at most one set S_m such that

$$|\mathcal{F} \cap [S_m]^l| \geq \frac{d}{2}.$$

Suppose, to the contrary, that there exist two sets, say S_1 and S_2, such that

$$|\mathcal{F} \cap [S_i]^l| \geq \frac{d}{2}$$

for $i = 1, 2$. Then we have

$$|\mathcal{F} \cap [S_1]^l| + |\mathcal{F} \cap [S_2]^l| \geq d.$$

Since $|\mathcal{F} \cap [S_1]^l| \geq \frac{d}{2} \geq 1$ and $|\mathcal{F} \cap [S_2]^l| \geq \frac{d}{2} \geq 1$, we are able to choose d sets $A_1, A_2, \ldots, A_d$ from $(\mathcal{F} \cap [S_1]^l) \cup (\mathcal{F} \cap [S_2]^l)$ such that $A_1 \subseteq S_1$ and $A_2 \subseteq S_2$. Since $|(A_1 \cup A_2 \cup \cdots \cup A_d)| \leq dl$ and $S_1 \cap S_2 = \emptyset$, we have either

$$|(A_1 \cup A_2 \cup \cdots \cup A_d) \cap S_1| \leq \frac{dl}{2} \tag{7}$$

or

$$|(A_1 \cup A_2 \cup \cdots \cup A_d) \cap S_2| \leq \frac{dl}{2}. \tag{8}$$

Without loss of generality, assuming that (7) is valid, we see that

$$|(A_1 \cup A_2 \cup \cdots \cup A_d) - S_2| = |(A_1 \cup A_2 \cup \cdots \cup A_d) \cap S_1| \leq \frac{dl}{2}.$$

However, the choice of $A_1, A_2, \ldots, A_d$ ensures that $A_1 \cap A_2 \cap \cdots \cap A_d \cap S_2 = \emptyset$, contradicting condition (2). This leads to the conclusion that there exists at most one set S_m such that

$$|\mathcal{F} \cap [S_m]^l| \geq \frac{d}{2}.$$

Without loss of generality, let us assume that $m = t$. Thus we have

$$|\mathcal{F} \cap [S_i]^l| \leq \frac{d-1}{2}$$

for $i = 1, \ldots, t-1$. Set

$$\mathcal{H}_i = \{F \in \mathcal{F} : |F \cap S_i| = l-1\}$$

and

$$\deg_{\mathcal{H}_i}(B) = |\{F \in \mathcal{H}_i : B \subset F\}|$$

for each $1 \leq i \leq t$.

We claim that there exists at least one set S_i ($i \in \{1, \dots, t\}$) such that

$$|\mathcal{H}_i| \leq \binom{k}{l-1} \quad \text{and} \quad |\mathcal{F} \cap [S_i]^l| \leq \frac{d-1}{2}.$$

Suppose that the above claim is not true. Then

$$|\mathcal{H}_i| \geq \binom{k}{l-1} + 1 \tag{9}$$

for $i = 1, \dots, t-1$. Moreover, if $|\mathcal{F} \cap [S_t]^l| \leq \frac{d-1}{2}$, then

$$|\mathcal{H}_t| \geq \binom{k}{l-1} + 1.$$

By (9), there exists an $(l-1)$-subset B of S_1 such that

$$\deg_{\mathcal{H}_1}(B) \geq 2. \tag{10}$$

Assume that $A_1, A_2 \in \mathcal{H}_1$ are chosen subject to the conditions $B \subset A_1$ and $B \subset A_2$. Since

$$|\mathcal{H}_2| \geq \binom{k}{l-1} + 1 > d-2,$$

we can choose $A_3, \dots, A_d$ from $\mathcal{H}_2$. Since $A_1 \cap A_2 = B \subseteq S_1$,

$$A_1 \cap \dots \cap A_d \cap S_2 = \emptyset$$

and

$$|A_1 \cup \dots \cup A_d - S_2| \leq (l+1) + (d-2) = l + d - 1 \leq \frac{dl}{2}$$

for $d \geq 4$ and $l \geq 3$. So we have reached a contradiction to condition (2) when $d \geq 4$.

Consider the case $d = 3$. Let $\{x_i\} = A_i - B$ for $i = 1, 2$. Since $A_1, A_2 \in \mathcal{H}_1$, we have $x_i \notin S_1$. Let $x_1 \in S_{i_0}$ for some $i_0 \geq 2$. Choose A_3 to be either in $\mathcal{H}_{i_0}$ or $\mathcal{F} \cap [S_{i_0}]^l$. Since $A_1 \cap A_2 = B \in S_1$ and $S_1 \cap S_2 = \emptyset$, we have

$$A_1 \cap A_2 \cap A_3 \cap S_{i_0} = \emptyset$$

and

$$|A_1 \cup A_2 \cup A_3 - S_{i_0}| \leq (l-1) + 1 + 1 = l + 1 \leq \frac{dl}{2}$$

for $l \geq 3$ and $d = 3$, contradicting condition (2) again. Thus the claim is verified.

Without loss of generality, we assume that

$$|\{F \in \mathcal{F} : |F \cap S_1| = l-1\}| = |\mathcal{H}_1| \leq \binom{k}{l-1} \quad \text{and} \quad |F \cap [S_1]^l| \leq \frac{d-1}{2}.$$

For any $F \in \mathcal{F}$, we may express F as $F_1 \cup F_2$, where $F_1 = F \cap S_1$ and $F_2 = F - F_1$. For a fixed F_1 of size $l-r$ ($1 \leq r \leq l$), let $\mathcal{F}_r$ be the family of all r-sets $F_2 \subset S_2 \cup S_3 \cup \cdots \cup S_t$ such that $F_1 \cup F_2 \in \mathcal{F}$.

We claim that $\mathcal{F}_r$ satisfies the conditions of the lemma. Otherwise, we may assume that there exist $A_1, A_2, \ldots, A_d \in \mathcal{F}_r$ and $i \in \{2, \ldots, t\}$ such that $A_1 \cap A_2 \cap \cdots \cap A_d \cap S_i = \emptyset$ and

$$|(A_1 \cup A_2 \cup \cdots \cup A_d) - S_i| \leq \frac{d}{2} r.$$

Now, let $A'_j = A_j \cup F_1$ for $1 \leq j \leq d$. Clearly, $A'_1, A'_2, \ldots, A'_d \in \mathcal{F}$ and $A'_1 \cap A'_2 \cap \cdots \cap A'_d \cap S_i = \emptyset$. Recalling that $l \geq r$, we find

$$\begin{aligned} |(A'_1 \cup A'_2 \cup \cdots \cup A'_d) - S_i| &= |F_1| + |(A_1 \cup A_2 \cup \cdots \cup A_d) - S_i| \\ &\leq l - r + \frac{dr}{2} = l + \frac{d-2}{2} r \leq l + \frac{d-2}{2} l = \frac{dl}{2}, \end{aligned}$$

contradicting condition (2). Thus we have shown that $\mathcal{F}_r$ satisfies the conditions of the lemma. For $r \geq 2$, by the inductive hypothesis, we see that

$$|\mathcal{F}_r| < \binom{(t-1)k-1}{r-1}.$$

Since $l \geq 3$ and $d \leq k$, it is easy to check that

$$\sum_{r=2}^{l} \binom{k}{l-r} - d \geq 0.$$

Hence $|\mathcal{F}|$ can be bounded as follows:

$$\begin{aligned} |\mathcal{F}| &\leq \sum_{r=2}^{l} \binom{k}{l-r} |\mathcal{F}_r| + |\{F \in \mathcal{F} : |F \cap S_1| = l-1\}| + |\mathcal{F} \cap [S_1]^l| \\ &\leq \sum_{r=1}^{l} \binom{k}{l-r} \binom{(t-1)k-1}{r-1} - \sum_{r=1}^{l} \binom{k}{l-r} + \binom{k}{l-1} + \frac{d-1}{2} \\ &< \binom{tk-1}{l-1} - \sum_{r=2}^{l} \binom{k}{l-r} + d \leq \binom{tk-1}{l-1}. \end{aligned}$$

This completes the proof. $\square$

We are now ready to prove Theorem 2.2.

Proof of Theorem 2.2. For $d = 3$, the result follows from Theorem 1.8. So we assume $d \geq 4$. Let $S_1, S_2, \ldots, S_t$ be a maximum subfamily of pairwise disjoint k-subsets from $\mathcal{F}$. We proceed by induction on t. If $t = 1$, then $\mathcal{F}$ is intersecting, and the result follows from Theorem 1.1 when $n \geq 2k$. When $\frac{dk}{d-1} \leq n < 2k$, for any $A_1, \ldots, A_d \in \mathcal{F}$, $|A_1 \cup \cdots \cup A_d| \leq n < 2k$, it follows that their intersection is nonempty from the condition of the theorem. Hence the theorem reduces to Theorem 1.2 in this case. Now we may assume that $t \geq 2$ and that the theorem holds for $t - 1$. Note that $t = 1$ is the only case when $\mathcal{F}$ can be a complete star. It will be shown that $|\mathcal{F}| < \binom{n-1}{k-1}$.

If $n = tk$, we set $l = k$. The condition on $\mathcal{F}$ in Theorem 2.2 implies the conditions on $\mathcal{F}$ in Lemma 2.4 with d replaced by $d - 1$. In fact, suppose that there exist $A_1, A_2, \ldots, A_{d-1} \in \mathcal{F}$ for which $A_1 \cap A_2 \cap \cdots \cap A_{d-1} \cap S_i = \emptyset$. Since every $(d, \frac{d+1}{2})$-cluster of $\mathcal{F}$ is intersecting, we see that

$$|A_1 \cup A_2 \cup \cdots \cup A_{d-1} \cup S_i| > \frac{d-1}{2}k,$$

and hence

$$|A_1 \cup A_2 \cup \cdots \cup A_{d-1} - S_i| > \frac{d+1}{2}k - k = \frac{d-1}{2}k.$$

Thus the theorem follows from Lemma 2.4 in this case.

We now assume $n > tk$ and let

$$Y = [n] - \bigcup_{i=1}^{t} S_i. \tag{11}$$

Given the choice of $S_1, S_2, \ldots, S_t$, Y does not contain any subset $A \in \mathcal{F}$. Set

$$\mathcal{F}' = \{F \in \mathcal{F} \colon |F \cap Y| = k - 1\}.$$

We claim that if $|Y| = n - tk \geq k$, then

$$|\mathcal{F}'| \leq \binom{n-tk}{k-1}. \tag{12}$$

If the claim is not true, then we have

$$|\mathcal{F}'| \geq \binom{n-tk}{k-1} + 1 \geq k + 1 > d.$$

Therefore, there exists a $(k-2)$-subset $B \subset Y$ such that

$$\deg_{\mathcal{F}'}(B) \geq |Y| - k + 3 = (n - tk) - k + 3. \tag{13}$$

Otherwise, we would have

$$|\mathcal{F}'| \le \frac{((n-tk)-k+2)\binom{n-tk}{k-2}}{k-1} = \binom{n-tk}{k-1}.$$

Since the number of $(k-1)$-subsets of Y containing B is equal to $|Y|-k+2$, there exists a $(k-1)$-subset C in Y containing B such that $\deg_{\mathcal{F}'}(C) \ge 2$. Let $A_1, A_2 \in \mathcal{F}'$ be such that $A_1 \cap A_2 = C \subset Y$. It is easy to see that

$$A_1 \cap A_2 \cap S_i = \emptyset$$

for each $1 \le i \le t$. Let $A_3, A_4, \dots, A_{d-1}$ be additional subsets in $\mathcal{F}'$ such that $B \subseteq A_i$ for each i if $|Y|-k+3 \ge d-1$. We deduce that

$$A_1 \cap \cdots \cap A_{d-1} \cap S_i = \emptyset$$

for each $1 \le i \le t$. Moreover,

$$|A_1 \cup \cdots \cup A_{d-1}| \le k-2+2(d-2)+1 = k+2d-5 \quad \text{if} \quad |Y|-k+3 \ge d-1$$

and

$$|A_1 \cup \cdots \cup A_{d-1}| \le |Y|+d-1 \le k+2d-6 \quad \text{if} \quad |Y|-k+3 < d-1.$$

Let S_h be such that $S_h \cap A_1 \ne \emptyset$. Since $k \ge d \ge 4$, we see that

$$|(A_1 \cup \cdots \cup A_{d-1}) \cup S_h| \le k+2d-5+(k-1) = 2k+2d-6 \le \frac{d+1}{2}k,$$

contradicting the assumption of the theorem. So the claim is justified.

Note that for any member F in $\mathcal{F}$, we can write it as $F = F_1 \cup F_2$, where $F_1 = F \cap Y$ and $F_2 = F - F_1$. We now consider all possible ways to construct F in the above form. Let F_1 be a given subset of Y size $k-l$ $(1 \le l \le k)$. By the definition of Y in (11), F_2 is a subset $\cup_{i=1}^t S_i$. Let $\mathcal{F}_l$ be the family of all l-sets $F_2 \subset \cup_{i=1}^t S_i$ such that $F_1 \cup F_2 \in \mathcal{F}$. It remains to prove that $\mathcal{F}_l$ satisfies the conditions in Lemma 2.4 with d replaced by $d-1$. For $l = k$, the assumption of the theorem implies that for every $A_1, A_2, \dots, A_{d-1} \in \mathcal{F}_k$, if $A_1 \cap A_2 \cap \cdots \cap A_{d-1} \cap S_i = \emptyset$, then

$$|A_1 \cup A_2 \cup \cdots \cup A_{d-1} \cup S_i| > \frac{d+1}{2}k,$$

which yields that

$$|A_1 \cup A_2 \cup \cdots \cup A_{d-1} - S_i| > \frac{d-1}{2}k.$$

Therefore, the assertion holds when $l = k$. For $l < k$, if the assertion is not valid, then there exist $A_1, A_2, \ldots, A_{d-1} \in \mathcal{F}_l$ such that $A_1 \cap A_2 \cdots \cap A_{d-1} \cap S_i = \emptyset$ and

$$|A_1 \cup A_2 \cup \cdots \cup A_{d-1} - S_i| \leq \frac{d-1}{2}l.$$

Setting $A'_i = A_i \cup F_1$ for $i \leq d-1$, we deduce that $A'_i \in \mathcal{F}$, $A'_1 \cap A'_2 \cdots \cap A'_{d-1} \cap S_i = \emptyset$, and

$$\begin{aligned}|(A'_1 \cup A'_2 \cup \cdots \cup A'_{d-1}) \cup S_i| &= |F_1| + |(A_1 \cup A_2 \cup \cdots \cup A_{d-1}) - S_i| + |S_i| \\ &\leq k - l + \frac{d-1}{2}l + k = 2k + \frac{d-3}{2}l \leq 2k + \frac{d-3}{2}k = \frac{d+1}{2}k,\end{aligned}$$

contradicting the assumption of the theorem. Up to now, we have shown that $\mathcal{F}_l$ satisfies the conditions in Lemma 2.4. For $l \geq 2$, by Lemma 2.4 we find that

$$|\mathcal{F}_l| < \binom{tk-1}{l-1}.$$

Evidently, for $|Y| = n - tk \leq k-2$, we have

$$|\{F \in \mathcal{F} \colon |F \cap Y| = k-1\}| = 0.$$

For the case $|Y| = k-1$, we have

$$|\{F \in \mathcal{F} \colon |F \cap Y| = k-1\}| < d-1 \leq k-1.$$

Otherwise we can choose $d-1$ sets $A_1, \ldots, A_{d-1} \in \mathcal{F}$ together with S_1 in violation of the assumption of the theorem. When $|Y| \geq k$, it follows from (12) that

$$|\{F \in \mathcal{F} \colon |F \cap Y| = k-1\}| \leq \binom{n-tk}{k-1},$$

which implies

$$|\{F \in \mathcal{F} \colon |F \cap Y| = k-1\}| < \sum_{l=1}^{k} \binom{n-tk}{k-l}.$$

Finally,

$$\begin{aligned}|\mathcal{F}| &\leq \sum_{l=2}^{k} \binom{|Y|}{k-l} |\mathcal{F}_l| + |\{F \in \mathcal{F} : |F \cap Y| = k-1\}| \\ &\leq \sum_{l=2}^{k} \binom{|Y|}{k-l} \left[\binom{tk-1}{l-1} - 1\right] + |\{F \in \mathcal{F} : |F \cap Y| = k-1\}|\end{aligned}$$

$$
\begin{aligned}
&= \sum_{l=1}^{k} \binom{|Y|}{k-l} \left[\binom{tk-1}{l-1} - 1 \right] + |\{F \in \mathcal{F} : |F \cap Y| = k-1\}| \\
&= \sum_{l=1}^{k} \binom{n-tk}{k-l} \binom{tk-1}{l-1} - \sum_{l=1}^{k} \binom{n-tk}{k-l} + |\{F \in \mathcal{F} : |F \cap Y| = k-1\}| \\
&< \binom{n-1}{k-1},
\end{aligned}
$$

as required. This completes the proof.

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References

[1] C. Chvátal, *An extremal set-intersection theorem*, J. London Math. Soc., 12 (1974/1975), pp. 355–359.

[2] P. Erdős, C. Ko, and R. Rado, *Intersection theorems for systems of finite sets*, Quart. J. Math. Oxford (2), 12 (1961), pp. 313–320.

[3] P. Frankl, *On Sperner families satisfying an additional condition*, J. Combin. Theory Ser. A, 20 (1976), pp. 1–11.

[4] P. Frankl and Z. Füredi, *Exact solution of some Turan-type problems*, J. Combin. Theory Ser. A, 45 (1987), pp. 226–262.

[5] D. Mubayi, *Erdős-Ko-Rado for three sets*, J. Combin. Theory Ser. A, 113 (2006), pp. 547–550.

[6] D. Mubayi, *An intersection theorem for four sets*, Adv. Math., 215 (2007), pp. 601–615.

[7] D. Mubayi and J. Verstraëte, *Proof of a conjecture of Erdős on triangles in set systems*, Combinatorica, 25 (2005), pp. 599–614.