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NON-TERMINATING BASIC HYPERGEOMETRIC SERIES AND THE q-ZEILBERGER ALGORITHM

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Abstract We present a systematic method for proving non-terminating basic hypergeometric identities. Assume that k is the summation index. By setting a parameter x to xq^n, we may find a recurrence relation of the summation by using the q-Zeilberger algorithm. This method applies to almost all non-terminating basic hypergeometric summation formulae in the work of Gasper and Rahman. Furthermore, by comparing the recursions and the limit values, we may verify many classical transformation formulae, including the Sears–Carlitz transformation, transformations of the very well-poised ${}_8\phi_7$ series, the Rogers–Fine identity and the limiting case of Watson's formula that implies the Rogers–Ramanujan identities.

Keywords: basic hypergeometric series; q-Zeilberger algorithm; Bailey's very well-poised ${}_6\psi_6$ summation formula; Sears–Carlitz transformation; Rogers–Fine identity

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1. Introduction

The aim of this paper is to develop a systematic method for proving non-terminating basic hypergeometric series summation and transformation formulae. The idea of finding recurrence relations and proving basic hypergeometric series identities by iteration has been used very often (see [**9–11**, **23**]). However, a systematic method does not seem to exist within the scope of computer algebra for proving non-terminating hypergeometric summation and transformation formulae. One obstacle lies in the infinities of the summation ranges. In this paper, we find that the q-Zeilberger algorithm can be used as a mechanism for proving many basic hypergeometric summation and transformation formulae. To prove transformation formulae by using our approach, we show that, subject to certain conditions, a series is uniquely determined by a recurrence relation and a limit value (Theorem 3.1).

Wilf and Zeilberger developed an algorithmic proof theory for identities on hypergeometric series and basic hypergeometric series [**42**, **48**, **49**]. For the purpose of this paper, we are concerned with the q-Zeilberger algorithm. Koornwinder [**37**], Paule and

Riese [**41**] and Böing and Koepf [**18**] studied further algorithmic proofs of basic hypergeometric identities. Most of the theory and implementations are restricted to the case of terminating identities. For non-terminating identities, Gessel [**26**] and Koornwinder [**38**] provided computer proofs of Gauss's summation formula and Saalschütz's summation formula by means of a combination of Zeilberger's algorithm and asymptotic estimates. Vidunas [**46**] (see also [**36, 39**]) presented a method to evaluate ${}_2F_1(\genfrac{}{}{0pt}{}{a,b}{c}|-1)$ series for the case when $c-a+b$ is an integer and developed the MAPLE program infhsum.mpl for the extension of Zeilberger's algorithm to non-terminating series.

Our method can be described as follows. Let

$$f(a,\dots,c)=\sum_{k=0}^{\infty} t_k(a,\dots,c)$$

be a basic hypergeometric series with parameters $a,\dots,c$, which are chosen for the purpose of establishing a recurrence relation of the form

$$p_0(a,\dots,c)f(a,\dots,c)+p_1(a,\dots,c)f(aq,\dots,cq)+\cdots+p_d(a,\dots,c)f(aq^d,\dots,cq^d)=0, \tag{1.1}$$

where d is a positive integer and $p_0,\dots,p_d$ are polynomials. To this end, we try to find polynomials $p_0,\dots,p_d$ and a sequence $(g_0,g_1,\dots)$ such that

$$p_0(a,\dots,c)t_k(a,\dots,c)+p_1(a,\dots,c)t_k(aq,\dots,cq)+\cdots \\ +p_d(a,\dots,c)t_k(aq^d,\dots,cq^d)=g_{k+1}-g_k. \tag{1.2}$$

Assume that $g_0=\lim_{k\to\infty} g_k=0$. Then (1.1) follows immediately by summing over k in (1.2).

The main idea of this paper is to use the q-Zeilberger algorithm [**18, 35, 37, 41, 48**] to find p_i and g_k. The bridge to the q-Zeilberger algorithm is the introduction of a new variable n by setting the parameters $a,\dots,c$ to $aq^n,\dots,cq^n$. Then the summand $t_k(aq^n,\dots,cq^n)$ becomes a bivariate q-hypergeometric term. Applying the q-Zeilberger algorithm, we can always obtain an equation of the form (1.2).

When the recurrence relation (1.1) is of first order (i.e. $d=1$) or involves only two terms, $f(a,\dots,c)$ equals its limit value $\lim_{N\to\infty} f(aq^N,\dots,cq^N)$ multiplied by an infinite product. Using this approach, we can derive almost all non-terminating summation formulae listed in the appendix of [**25**], including bilateral series formulae.

When the recurrence relation involves at least three terms, we show that $f(a,\dots,c)$ is determined uniquely by the recurrence relation and its limit value under suitable convergence conditions (Theorem 3.1). Therefore, to prove an identity, it suffices to verify that both sides satisfy the same recurrence relation and that the identity holds for the limiting case. Using this method, we can prove many classical transformation formulae.

Let us introduce some basic notation. The sets of integers and of non-negative integers are denoted by $\mathbb{Z}$ and $\mathbb{N}$, respectively. Throughout the paper, q is a fixed non-zero complex number with $|q|<1$.

The q-shifted factorial is defined for any complex parameter a by

$$(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k) \quad \text{and} \quad (a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty} \quad \forall n \in \mathbb{Z}.$$

For notational brevity, we write

$$(a_1, \ldots, a_m; q)_n = (a_1;q)_n \cdots (a_m;q)_n,$$

where n is an integer or infinity. Furthermore, the basic hypergeometric series are defined by

$${}_r\phi_s \left[\begin{matrix} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix}; q, z \right] = \sum_{k=0}^{\infty} \frac{(a_1, \ldots, a_r; q)_k}{(b_1, \ldots, b_s; q)_k} \frac{z^k}{(q;q)_k} \left((-1)^k q^{\binom{k}{2}} \right)^{s-r+1},$$

and the bilateral basic hypergeometric series are defined by

$${}_r\psi_s \left[\begin{matrix} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix}; q, z \right] = \sum_{k=-\infty}^{\infty} \frac{(a_1, \ldots, a_r; q)_k}{(b_1, \ldots, b_s; q)_k} z^k \left((-1)^k q^{\binom{k}{2}} \right)^{s-r}.$$

2. Summation formulae

In this section, we present a method of proving non-terminating basic hypergeometric identities by using the q-Zeilberger algorithm. Given a term of the form

$$\frac{(a_1', \ldots, a_u'; q)_\infty}{(b_1', \ldots, b_v'; q)_\infty} \frac{(a_1, \ldots, a_r; q)_k}{(b_1, \ldots, b_s; q)_k} q^{d\binom{k}{2}} z^k,$$

by setting some parameters $a, \ldots, c$ to $aq^n, \ldots, cq^n$, we get a bivariate q-hypergeometric term $t_k(aq^n, \ldots, cq^n)$ in n and k. By the q-Zeilberger algorithm, we obtain a bivariate q-hypergeometric term $g_{n,k}$ and polynomials $p_i(q^n, a, \ldots, c)$ which are independent of k such that

$$\begin{aligned} p_0(q^n, a, \ldots, c)t_k(aq^n, \ldots, cq^n) + p_1(q^n, a, \ldots, c)t_k(aq^{n+1}, \ldots, cq^{n+1}) \\ + \cdots + p_d(q^n, a, \ldots, c)t_k(aq^{n+d}, \ldots, cq^{n+d}) = g_{n,k+1} - g_{n,k}. \end{aligned} \tag{2.1}$$

Suppose that $g_{0,0} = \lim_{k\to\infty} g_{0,k} = 0$. By setting $n = 0$ in (2.1) and summing over k, we derive a recurrence relation of the form (1.1) for $f(a, \ldots, c) = \sum_{k=0}^{\infty} t_k(a, \ldots, c)$.

When the recursion (1.1) involves only two terms, say $f(a, \ldots, c)$ and $f(aq^d, \ldots, cq^d)$, we have

$$f(a, \ldots, c) = \lim_{N\to\infty} f(aq^{dN}, \ldots, cq^{dN}) \lim_{N\to\infty} \prod_{i=0}^{N-1} \left(-\frac{p_d(aq^{di}, \ldots, cq^{di})}{p_0(aq^{di}, \ldots, cq^{di})} \right).$$

Therefore, the evaluation of $f(a, \ldots, c)$ becomes the evaluation of its limit value

$$\lim_{N\to\infty} f(aq^{dN}, \ldots, cq^{dN}),$$

which is much simpler and is usually an infinite product.

2.1. Unilateral summations

We now present an example to show how to obtain an infinite product expression from an infinite summation.

Example 2.1. The q-binomial theorem:

$$f(a,z) = {}_1\phi_0 \left[\begin{matrix} a \\ - \end{matrix} ; q, z \right] = \frac{(az;q)_\infty}{(z;q)_\infty}, \quad |z| < 1.$$

The q-binomial theorem was derived by Cauchy [**20**], Jacobi [**32**] and Heine [**29**]. Heine's proof consists of using series manipulations to derive the recurrence relation

$$(1-z)f(a,z) = (1-az)f(a,qz). \tag{2.2}$$

Gasper [**24**] provided another proof using a recurrence relation with respect to the parameter a:

$$f(a,z) = (1-az)f(aq,z).$$

Our computer-generated proof is similar to Heine's proof. The recurrence relation generated by the q-Zeilberger algorithm turns out to be (2.2). Let $u_k(z)$ be the summand

$$u_k(z) = \frac{(a;q)_k}{(q;q)_k} z^k$$

and let $u_{n,k} = u_k(zq^n)$. By the q-Zeilberger algorithm, we obtain

$$(-azq^n + 1)u_{n+1,k} + (zq^n - 1)u_{n,k} = g_{n,k+1} - g_{n,k}, \tag{2.3}$$

where $g_{n,k} = (1-q^k)u_{n,k}$. Denote the left-hand side of (2.3) by $t_{n,k}$. Then

$$\sum_{k=0}^{\infty} t_{n,k} = -g_{n,0} + \lim_{k\to\infty} g_{n,k} = 0 \quad \forall n \geqslant 0,$$

implying that

$$f(a,zq^n) = \frac{1-azq^n}{1-zq^n} f(a,zq^{n+1}) \quad \forall n \geqslant 0.$$

Hence,

$$\begin{aligned} f(a,z) &= \frac{1-az}{1-z} f(a,zq) \\ &= \frac{1-az}{1-z}\frac{1-azq}{1-zq} f(a,zq^2) \\ &\vdots \\ &= \lim_{N\to\infty} \frac{(az;q)_N}{(z;q)_N} f(a,zq^N) \\ &= \frac{(az;q)_\infty}{(z;q)_\infty}, \end{aligned}$$

where $\lim_{N\to\infty} f(a,zq^N) = 1$ holds by Tannery's theorem [**45**, p. 292].

Theorem 2.2 (Tannery's theorem). *Suppose that $s(n) = \sum_{k\geqslant 0} f_k(n)$ is a convergent series for each n. If there exists a convergent series $\sum_{k\geqslant 0} M_k$ such that $|f_k(n)| \leqslant M_k$, then*

$$\lim_{n\to\infty} s(n) = \sum_{k=0}^{\infty} \lim_{n\to\infty} f_k(n).$$

The following summation formulae (most of which come from the appendix of [**25**]) can be verified by the above method. We list in Table 1 only the recursions and the limit values $\lim_{N\to\infty} f(aq^{dN}, \ldots, cq^{dN})$. The recursions are computed by using the MAPLE package hsum6.mpl developed by Koepf [**35**]. The limit values are obtained by straightforward estimations and Tannery's theorem. Notice that most of the summands are of the form $a_n x^n$ with a_n being bounded. Thus, Tannery's theorem can be applied for $|x| < 1$.

2.1.1. *More examples*

A q-analogue of Watson's ${}_3F_2$ sum:

$$f(a,c) = {}_8\phi_7 \left[\begin{matrix} \mu, q\mu^{1/2}, -q\mu^{1/2}, a^2, b^2q, c, -c, -\dfrac{abq}{c} \\ \mu^{1/2}, -\mu^{1/2}, -\dfrac{bcq}{a}, -\dfrac{ac}{b}, -abq, abq, c^2 \end{matrix}; q, \frac{c}{ab}\right],$$

where $\mu = -abc$.

By the q-Kummer sum, we have

$$\lim_{N\to\infty} f(aq^N, cq^N) = {}_2\phi_1 \left[\begin{matrix} b^2q, -\dfrac{abq}{c} \\ -\dfrac{bcq}{a} \end{matrix}; q, \frac{c}{ab}\right] = \frac{(-q;q)_\infty (b^2q^2, c^2q/a^2; q^2)_\infty}{(c/ab, -bcq/a; q)_\infty}.$$

By computation, one derives that

$$f(a,c) = \frac{(1+abcq)(1+c/b)(1+abcq^2)(1-a^2q)(1-c/b)}{(1-c^2q)(1-abq)(1+abq)(1+acq/b)(1+ac/b)} f(aq, cq).$$

Thus, we have

$$f(a,c) = \frac{(-abcq, -c/b, c/b, -q; q)_\infty (a^2q, b^2q^2, c^2q/a^2; q^2)_\infty}{(abq, -abq, -ac/b, c/ab, -bcq/a; q)_\infty (c^2q; q^2)_\infty}.$$

A q-analogue of Whipple's ${}_3F_2$ sum:

$$f(c) = {}_8\phi_7 \left[\begin{matrix} -c, q(-c)^{1/2}, -q(-c)^{1/2}, a, \dfrac{q}{a}, c, -d, -\dfrac{q}{d} \\ (-c)^{1/2}, -(-c)^{1/2}, -\dfrac{cq}{a}, -ac-q, \dfrac{cq}{d}, cd \end{matrix}; q, c\right].$$

Table 1. Recursions and the limit values $\lim_{N\to\infty} f(aq^{dN},\ldots,cq^{dN})$

name	summation	recurrence relation	limit value
q-exponential	$\sum_{k=0}^{\infty} \frac{z^k}{(q;q)_k}$	$f(z) = \frac{1}{1-z} f(zq)$	1
q-exponential	$\sum_{k=0}^{\infty} \frac{q^{\binom{k}{2}} z^k}{(q;q)_k}$	$f(z) = (1+z) f(zq)$	1
Lebesgue [**7**, p. 21]	$\sum_{k=0}^{\infty} \frac{(x;q)_k q^{\binom{k+1}{2}}}{(q;q)_k}$	$f(x) = (1-xq) f(xq^2)$	$(-q;q)_\infty$
Lebesgue generalization [**5**]	$\sum_{k=0}^{\infty} \frac{(a,b;q)_k q^{\binom{k+1}{2}}}{(q;q)_k (abq;q^2)_k}$	$f(a,b) = \frac{(1-aq)(1-bq)}{(1-abq)(1-abq^3)} f(aq^2, bq^2)$	$(-q;q)_\infty$
${}_1\phi_1$	${}_1\phi_1 \left[\begin{matrix} a \\ c \end{matrix}; q, \frac{c}{a}\right]$	$f(c) = \frac{1-c/a}{1-c} f(cq)$	1
q-Gauss	${}_2\phi_1 \left[\begin{matrix} a, b \\ c \end{matrix}; q, \frac{c}{ab}\right]$	$f(c) = \frac{(1-c/a)(1-c/b)}{(1-c)(1-c/ab)} f(cq)$	1

Table 1. (Cont.)

name	summation	recurrence relation	limit value
q-Kummer (Bailey–Daum) sum	${}_2\phi_1\left[\begin{matrix} a, b \\ \frac{aq}{b} \end{matrix}; q, -\frac{q}{b}\right]$	$f(a) = \frac{(1-aq^2/b^2)(1-aq)}{(1-aq^2/b)(1-aq/b)} f(aq^2)$	$\frac{(-q;q)_\infty}{(-q/b;q)_\infty}$
a q-analogue of Bailey's ${}_2F_1(-1)$	${}_2\phi_2\left[\begin{matrix} a, \frac{q}{a} \\ -q, b \end{matrix}; q, -b\right]$	$f(b) = \frac{(1-ab)(1-bq/a)}{(1-bq)(1-b)} f(bq^2)$	1
a q-analogue of Gauss's ${}_2F_1(-1)$	${}_2\phi_2\left[\begin{matrix} a^2, b^2 \\ abq^{1/2}, -abq^{1/2} \end{matrix}; q, -q\right]$	$f(a,b) = \frac{(1-a^2q)(1-b^2q)}{(1-a^2b^2q)(1-a^2b^2q^3)} f(aq, bq)$	$(-q;q)_\infty$
q-Dixon sum	${}_4\phi_3\left[\begin{matrix} a^2, -qa, b, c \\ -a, = \frac{a^2q}{b}, \frac{a^2q}{c} \end{matrix}; q, \frac{qa}{bc}\right]$	$f(a) = \frac{(1-a^2q/bc)(1-a^2q^2/bc)(1+aq)}{(1-a^2q/b)(1-a^2q^2/b)(1-aq/bc)} \times \frac{(1-aq/b)(1-aq/c)(1-a^2q)}{(1-a^2q/c)(1-a^2q^2/c)} f(aq)$	1

By computation we have the recurrence relation

$$f(c) = \frac{(1+cq^2)(1+c)(1-cq^2/ad)(1-acq/d)}{(1+acq)(1-cdq)(1+cq^2/a)(1-cq^2/d)} \times \frac{(1-acd)(1-cdq/a)(1+cq)^2}{(1+ac)(1-cd)(1+cq/a)(1-cq/d)} f(cq^2).$$

Since $f(0) = 1$, we obtain

$$f(c) = \frac{(-c, -cq; q)_\infty (acd, acq/d, cdq/a, cq^2/ad; q^2)_\infty}{(cd, cq/d, -ac, -cq/a; q)_\infty}.$$

The sum of a very well-poised ${}_6\phi_5$ series:

$$f(a) = {}_6\phi_5 \left[\begin{matrix} a, qa^{1/2}, -qa^{1/2}, b, c, d \\ a^{1/2}, -a^{1/2}, \frac{aq}{b}, \frac{aq}{c}, \frac{aq}{d} \end{matrix} ; q, \frac{aq}{bcd} \right].$$

By the q-Zeilberger algorithm, we find

$$f(a) = \frac{(1-aq/cd)(1-aq/bc)(1-aq/bd)(1-aq)}{(1-aq/bcd)(1-aq/b)(1-aq/c)(1-aq/d)} f(aq).$$

Since $f(0) = 1$, we obtain

$$f(a) = \frac{(aq, aq/bc, aq/bd, aq/cd; q)_\infty}{(aq/b, aq/c, aq/d, aq/bcd; q)_\infty}.$$

2.2. Two-Term summation formulae

Many classical two-term non-terminating summation formulae can be dealt with by using the same method as single summation formulae. It turns out that, for many two-term summation formulae, the two summands share the same recurrence relation. Moreover, the boundary values $\lim_{k\to\infty} g_{0,k}$ for the two summands cancel out. So we still obtain homogeneous recurrence relations which lead to infinite products. We give three examples from the appendix of [**25**] and present a detailed proof for the first example.

Example 2.3. A non-terminating form of the q-Vandermonde sum:

$$f(a,b,c) = {}_2\phi_1 \left[\begin{matrix} a, b \\ c \end{matrix} ; q, q \right] + \frac{(q/c, a, b; q)_\infty}{(c/q, aq/c, bq/c; q)_\infty} {}_2\phi_1 \left[\begin{matrix} \frac{aq}{c}, \frac{bq}{c} \\ \frac{q^2}{c} \end{matrix} ; q, q \right].$$

Since $\lim_{N\to\infty} f(aq^N, bq^N, cq^N)$ does not exist, we consider

$$g(a,b,c) = \frac{f(a,b,c)}{(q/c;q)_\infty}.$$

Let

$$u^{(1)}_{n,k} = \frac{1}{(q/cq^n;q)_\infty}\frac{(aq^n,bq^n;q)_k}{(cq^n,q;q)_k}q^k,$$

$$u^{(2)}_{n,k} = \frac{(aq^n,bq^n;q)_\infty}{(cq^n/q,aq/c,bq/c;q)_\infty}\frac{(aq/c,bq/c;q)_k}{(q^2/cq^n,q;q)_k}q^k.$$

We have

$$(abq^{n+1}-c)u^{(i)}_{n+1,k} + cu^{(i)}_{n,k} = g^{(i)}_{n,k+1} - g^{(i)}_{n,k}, \quad i=1,2,$$

where

$$g^{(1)}_{n,k} = \frac{c(1-abq^{2n+k})(1-q^k)}{q^k(1-aq^n)(1-bq^n)}u^{(1)}_{n,k},$$

$$g^{(2)}_{n,k} = \frac{c(cq^n-q^{k+1})(1-q^k)}{q^{k+1}(1-aq^n)(1-bq^n)}u^{(2)}_{n,k}.$$

Noting that $g^{(1)}_{n,0} = g^{(2)}_{n,0} = 0$ and

$$\lim_{k\to\infty} g^{(1)}_{n,k} = -\lim_{k\to\infty} g^{(2)}_{n,k} = -\frac{(aq^{n+1},bq^{n+1};q)_\infty}{q^n(1/cq^n,cq^{n+1},q;q)_\infty},$$

we get $g(a,b,c) = (1-abq/c)g(aq,bq,cq)$. Since

$$\lim_{N\to\infty} g(aq^N,bq^N,cq^N) = 0 + \frac{1}{(aq/c,bq/c;q)_\infty} = \frac{1}{(aq/c,bq/c;q)_\infty},$$

we get

$$f(a,b,c) = (q/c;q)_\infty g(a,b,c) = \frac{(q/c,abq/c;q)_\infty}{(aq/c,bq/c;q)_\infty}. \tag{2.4}$$

Example 2.4. A non-terminating form of the q-Saalschütz sum:

$$f(c) = {}_3\phi_2\left[\begin{matrix}a,b,c\\ e,f\end{matrix};q,q\right] + \frac{(q/e,a,b,c,qf/e;q)_\infty}{(e/q,aq/e,bq/e,cq/e,f;q)_\infty}{}_3\phi_2\left[\begin{matrix}aq/e,bq/e,cq/e\\ q^2/e,qf/e\end{matrix};q,q\right],$$

where $f = abcq/e$.

By computation, we have

$$f(c) = \frac{(1-bcq/e)(1-acq/e)}{(1-cq/e)(1-abcq/e)}f(cq),$$

and, by (2.4),

$$\lim_{N\to\infty} f(cq^N) = \frac{(q/e,abq/e;q)_\infty}{(aq/e,bq/e;q)_\infty}.$$

Thus, we get

$$f(c) = \frac{(bcq/e,acq/e,q/e,abq/e;q)_\infty}{(cq/e,abcq/e,aq/e,bq/e;q)_\infty}. \tag{2.5}$$

Example 2.5. Bailey's non-terminating extension of Jackson's ${}_8\phi_7$ sum:

$$f(a,b) = {}_8\phi_7\left[\begin{matrix} a, qa^{1/2}, -qa^{1/2}, b, c, d, e, f \\ a^{1/2}, -a^{1/2}, \frac{aq}{b}, \frac{aq}{c}, \frac{aq}{d}, \frac{aq}{e}, \frac{aq}{f} \end{matrix}; q, q\right]$$

$$- \frac{b}{a}\frac{(aq, c, d, e, f, bq/a, bq/c, bq/d, bq/e, bq/f; q)_\infty}{(aq/b, aq/c, aq/d, aq/e, aq/f, bc/a, bd/a, be/a, bf/a, b^2q/a; q)_\infty}$$

$$\times\, {}_8\phi_7\left[\begin{matrix} \frac{b^2}{a}, qba^{-1/2}, -qba^{-1/2}, b, \frac{bc}{a}, \frac{bd}{a}, \frac{be}{a}, \frac{bf}{a} \\ ba^{-1/2}, -ba^{-1/2}, \frac{bq}{a}, \frac{bq}{c}, \frac{bq}{d}, \frac{bq}{e}, \frac{bq}{f} \end{matrix}; q, q\right],$$

where $f = a^2q/bcde$.

By computation, we have

$$f(a,b) = \frac{(1-aq)(1-aq/cd)(1-aq/ce)(1-aq/de)}{(1-aq/cde)(1-aq/c)(1-aq/d)(1-aq/e)} f(aq, bq),$$

and, by (2.5),

$$\lim_{N\to\infty} f(aq^N, bq^N) = \frac{(b/a, aq/cf, aq/df, aq/ef; q)_\infty}{(aq/f, bc/a, bd/a, be/a; q)_\infty}.$$

Finally, we have

$$f(a,b) = \frac{(aq, aq/cd, aq/ce, aq/de, b/a, aq/cf, aq/df, aq/ef; q)_\infty}{(aq/cde, aq/c, aq/d, aq/e, aq/f, bc/a, bd/a, be/a; q)_\infty}.$$

2.3. Bilateral summations

Bilateral summations [**25**, Chapter 5] can also be dealt with by using the q-Zeilberger algorithm approach. We need the following special requirement for the recurrence relation (2.1):

$$\lim_{k\to-\infty} g_{n,k} = \lim_{k\to\infty} g_{n,k} = 0.$$

Here are some examples.

Example 2.6. Jacobi's triple product:

$$\sum_{k=-\infty}^{\infty} q^{\binom{k}{2}} z^k = \left(q, -z, -\frac{q}{z}; q\right)_\infty. \tag{2.6}$$

This well-known identity is due to Jacobi (see [**8**] and [**25**, p. 12]). Cauchy [**20**] gave a simple proof using the q-binomial theorem. For other proofs, see [**1**, **22**, **33**].

We give a q-Zeilberger style proof through a semi-finite form of the left-hand side of (2.6) [**21**]:

$$f(m) = \sum_{k=-\infty}^{\infty} \frac{q^{\binom{k}{2}} z^k}{(q^{m+1}; q)_k}, \quad m \geqslant 0.$$

Let $u_{m,k}$ be the summand. Applying the q-Zeilberger algorithm, we obtain

$$zu_{m+1,k} - (q^{m+1} + z)(1 - q^{m+1})u_{m,k} = g_{m,k+1} - g_{m,k},$$

where $g_{m,k} = (1 - q^{m+1})q^{m+1}u_{m,k}$. Since

$$\lim_{k\to\infty} g_{m,k} = \lim_{k\to-\infty} g_{m,k} = 0,$$

we have

$$f(m+1) = \left(1 + \frac{q^{m+1}}{z}\right)(1 - q^{m+1})f(m).$$

It follows that

$$\begin{aligned}\sum_{k=-\infty}^{\infty} q^{\binom{k}{2}} z^k &= \sum_{k=-\infty}^{\infty} \lim_{m\to\infty} u_{m,k} = \lim_{m\to\infty} f(m)\\ &= f(0)\left(q, -\frac{q}{z}; q\right)_\infty = \left(-z, q, -\frac{q}{z}; q\right)_\infty.\end{aligned}$$

Example 2.7. Ramanujan's ${}_1\psi_1$ sum:

$$f(b) = {}_1\psi_1\left[\begin{matrix}a\\ b\end{matrix}; q, z\right] = \frac{(q, b/a, az, q/az; q)_\infty}{(b, q/a, z, b/az; q)_\infty}, \quad |z|, \left|\frac{b}{az}\right| < 1.$$

This formula is due to Ramanujan. Andrews [**3**,**4**], Hahn [**27**], Jackson [**31**], Ismail [**30**], Andrews and Askey [**12**] and Berndt [**17**] have found different proofs.

The proof of Andrews and Askey [**12**] is based on the following recursion:

$$f(b) = \frac{1 - b/a}{(1 - b)(1 - b/az)} f(bq). \tag{2.7}$$

We now derive the recursion (2.7) by using the q-Zeilberger algorithm. Let

$$u_{n,k} = \frac{(a;q)_k}{(bq^n;q)_k} z^k.$$

Then

$$z(bq^n - a)u_{n+1,k} + (az - bq^n)(1 - bq^n)u_{n,k} = g_{n,k+1} - g_{n,k}, \tag{2.8}$$

where

$$g_{n,k} = (1 - bq^n)bq^n u_{n,k}.$$

Notice that (2.8) holds for any $k \in \mathbb{Z}$. Furthermore, when $|z| < 1$ and $|b/az| < 1$, we have

$$\lim_{k\to\pm\infty} g_{n,k} = (1 - bq^n)bq^n \lim_{k\to\pm\infty} u_{n,k} = 0.$$

Summing over $k \in \mathbb{Z}$ on both sides of (2.8), we immediately get (2.7), implying that

$$f(b) = \frac{(b/a;q)_\infty}{(b, b/az; q)_\infty} f(0).$$

By the q-binomial theorem,

$$f(q) = \sum_{k=0}^{\infty} \frac{(a;q)_k}{(q;q)_k} z^k = \frac{(az;q)_\infty}{(z;q)_\infty}.$$

Therefore,

$$f(b) = \frac{(b/a;q)_\infty}{(b,b/az;q)_\infty} \frac{(q,q/az;q)_\infty}{(q/a;q)_\infty} f(q) = \frac{(b/a,q,q/az,az;q)_\infty}{(b,b/az,q/a,z;q)_\infty}.$$

Example 2.8. A well-poised ${}_2\psi_2$ series:

$$f(b,c) = {}_2\psi_2 \left[\begin{matrix} b, c \\ \frac{aq}{b}, \frac{aq}{c} \end{matrix} ; q, -\frac{aq}{bc} \right], \quad \left| \frac{aq}{bc} \right| < 1.$$

By computation, we have

$$f(b,c) = \frac{(1-aq/bc)(1-aq^2/bc)}{(1+aq/bc)(1+aq^2/bc)} \frac{(1-aq^2/b^2)(1-aq^2/c^2)}{(1-q/b)(1-q/c)(1-aq/b)(1-aq/c)} f\left(\frac{b}{q}, \frac{c}{q}\right).$$

By Jacobi's triple product identity, we obtain

$$\lim_{N\to\infty} f\left(\frac{b}{q^N}, \frac{c}{q^N}\right) = \sum_{k=-\infty}^{\infty} q^{k^2}(-a)^k = \left(q^2, qa, \frac{q}{a}; q^2\right)_\infty.$$

Thus, we get

$$f(b,c) = \frac{(aq/bc;q)_\infty (aq^2/b^2, aq^2/c^2, q^2, qa, q/a; q^2)_\infty}{(-aq/bc, q/b, q/c, aq/b, aq/c; q)_\infty}.$$

Example 2.9. Bailey's sum of a well-poised ${}_3\psi_3$:

$$f(b,c,d) = {}_3\psi_3 \left[\begin{matrix} b, c, d \\ \frac{q}{b}, \frac{q}{c}, \frac{q}{d} \end{matrix} ; q, \frac{q}{bcd} \right].$$

We notice that applying the q-Zeilberger algorithm directly to

$$\frac{(b,c,d;q)_k}{(q/b,q/c,q/d;q)_k} \left(\frac{q}{bcd}\right)^k,$$

does not give a simple relation. Using Paule's idea [**40**] of symmetrizing a bilateral summation, we replace k by $-k$ to get a summation

$${}_3\psi_3 \left[\begin{matrix} b, c, d \\ \frac{q}{b}, \frac{q}{c}, \frac{q}{d} \end{matrix} ; q, \frac{q^2}{bcd} \right].$$

Now we apply the q-Zeilberger algorithm to the average of the above summands,

$$\frac{1+q^k}{2}\frac{(b,c,d;q)_k}{(q/b,q/c,q/d;q)_k}\left(\frac{q}{bcd}\right)^k,$$

and obtain that

$$f(b,c,d)=\frac{(1-q/bc)(1-q^2/bc)(1-q/bd)(1-q^2/bd)}{(1-q/b)(1-q/c)(1-q/d)} \times\frac{(1-q/cd)(1-q^2/cd)}{(1-q/bcd)(1-q^2/bcd)(1-q^3/bcd)}f\left(\frac{b}{q},\frac{c}{q},\frac{d}{q}\right).$$

By Jacobi's triple product identity, we have

$$\lim_{N\to\infty}f\left(\frac{b}{q^N},\frac{c}{q^N},\frac{d}{q^N}\right)=\sum_{k=-\infty}^{\infty}q^{3\binom{k}{2}}(-q)^k=(q^3,q,q^2;q^3)_\infty=(q;q)_\infty.$$

So we get

$$f(b,c,d)=\frac{(q,q/bc,q/bd,q/cd;q)_\infty}{(q/b,q/c,q/d,q/bcd;q)_\infty}.$$

Example 2.10. A basic bilateral analogue of Dixon's sum:

$$f(b,c,d)={}_4\psi_4\left[\begin{matrix}-qa,b,c,d\\ -a,\frac{a^2q}{b},\frac{a^2q}{c},\frac{a^2q}{d}\end{matrix};q,\frac{qa^3}{bcd}\right].$$

By computation, we get

$$\begin{aligned}f(b,c,d)=&\frac{(1-a^2q/bc)(1-a^2q^2/bc)(1-a^2q/bd)(1-a^2q^2/bd)}{(1-a^3q/bcd)(1-a^3q^2/bcd)(1-a^3q^3/bcd)}\\ &\times\frac{(1-a^2q/cd)(1-a^2q^2/cd)}{(1-q/b)(1-q/c)(1-q/d)}\\ &\times\frac{(1-aq/b)(1-aq/c)(1-aq/d)}{(1-a^2q/b)(1-a^2q/c)(1-a^2q/d)}f\left(\frac{b}{q},\frac{c}{q},\frac{d}{q}\right).\end{aligned}$$

Hence,

$$f(b,c,d)=\frac{(a^2q/bc,a^2q/bd,a^2q/cd,aq/b,aq/c,aq/d;q)_\infty}{(a^3q/bcd,q/b,q/c,q/d,a^2q/b,a^2q/c,a^2q/d;q)_\infty}S(a),\tag{2.9}$$

where

$$S(a)=\lim_{N\to\infty}f\left(\frac{b}{q^N},\frac{c}{q^N},\frac{d}{q^N}\right)=\sum_{k=-\infty}^{\infty}\frac{(-qa;q)_k}{(-a;q)_k}q^{3\binom{k}{2}}(-qa^3)^k.$$

In particular, replacing b, c, d in (2.9) by $-a$,c/q^N, d/q^N and taking the limit $N\to\infty$, we get

$$\lim_{N\to\infty}f\left(-a,\frac{c}{q^N},\frac{d}{q^N}\right)=\frac{(-q;q)_\infty}{(-q/a,-aq;q)_\infty}S(a).$$

By Jacobi's triple product identity, we have

$$\lim_{N\to\infty} f\left(-a, \frac{c}{q^N}, \frac{d}{q^N}\right) = \sum_{k=-\infty}^{\infty} q^{k^2}(-a^2)^k = \left(q^2, a^2q, \frac{q}{a^2}; q^2\right)_\infty,$$

which implies that

$$S(a) = \frac{(q, a^2q, q/a^2; q)_\infty}{(aq, q/a; q)_\infty}.$$

Therefore, we obtain

$$f(b,c,d) = \frac{(a^2q/bc, a^2q/bd, a^2q/cd, aq/b, aq/c, aq/d; q)_\infty}{(a^3q/bcd, q/b, q/c, q/d, a^2q/b, a^2q/c, a^2q/d; q)_\infty}\frac{(q, a^2q, q/a^2; q)_\infty}{(aq, q/a; q)_\infty}.$$

Example 2.11. Bailey's very well-poised ${}_6\psi_6$ series:

$$\begin{aligned} f(b,c,d,e) &= {}_6\psi_6\left[\begin{matrix} qa^{1/2}, -qa^{1/2}, b, c, d, e \\ a^{1/2}, -a^{1/2}, \frac{aq}{b}, \frac{aq}{c}, \frac{aq}{d}, \frac{aq}{e} \end{matrix}; q, \frac{qa^2}{bcde}\right] \\ &= \frac{(aq/bc, aq/bd, aq/be, aq/cd, aq/ce, aq/de, q, aq, q/a; q)_\infty}{(aq/b, aq/c, aq/d, aq/e, q/b, q/c, q/d, q/e, qa^2/bcde; q)_\infty}. \end{aligned}$$

This identity is due to Bailey [**16**]. Other proofs have been given by Slater and Lakin [**44**], Andrews [**6**], Schlosser [**43**] and Jouhet and Schlosser [**34**]. Askey and Ismail [**15**] gave a simple proof using the ${}_6\phi_5$ sum and an argument based on analytic continuation. Askey [**14**] also showed that it can be obtained from a simple difference equation and Ramanujan's ${}_1\psi_1$ sum.

Using our computational approach, we obtain

$$\begin{aligned} f(b,c,d,e) = &\frac{(1-aq/bc)(1-aq^2/bc)(1-aq/bd)(1-aq^2/bd)}{(1-aq/b)(1-aq/c)(1-aq/d)(1-aq/e)} \\ &\times \frac{(1-aq/be)(1-aq^2/be)(1-aq/cd)(1-aq^2/cd)}{(1-q/b)(1-q/c)(1-q/d)(1-q/e)} \\ &\times \frac{(1-aq/ce)(1-aq^2/ce)(1-aq/de)(1-aq^2/de)}{(1-a^2q/bcde)(1-a^2q^2/bcde)(1-a^2q^3/bcde)(1-a^2q^4/bcde)} \\ &\times f\left(\frac{b}{q}, \frac{c}{q}, \frac{d}{q}, \frac{e}{q}\right). \end{aligned}$$

By Jacobi's triple product identity, we have

$$\begin{aligned} \lim_{N\to\infty} f\left(\frac{b}{q^N}, \frac{c}{q^N}, \frac{d}{q^N}, \frac{e}{q^N}\right) &= \frac{1}{1-a}\sum_{k=-\infty}^{\infty}(1-aq^{2k})q^{4\binom{k}{2}}(qa^2)^k \\ &= \frac{1}{1-a}\sum_{k=-\infty}^{\infty} q^{\binom{k}{2}}(-a)^k \\ &= \left(q, aq, \frac{q}{a}; q\right)_\infty. \end{aligned}$$

Hence, we get

$$f(b,c,d,e)=\frac{(aq/bc,aq/bd,aq/be,aq/cd,aq/ce,aq/de,q,aq,q/a;q)_\infty}{(aq/b,aq/c,aq/d,aq/e,q/b,q/c,q/d,q/e,qa^2/bcde;q)_\infty}.$$

Example 2.12. The Askey beta integral [**13**]:

$$I(a,b,d,e)=\int_{-\infty}^{\infty}\frac{(at,bt;q)_\infty}{(-dt,et;q)_\infty}d_qt,$$

where

$$\int_{-\infty}^{\infty}f(t)d_qt=(1-q)\sum_{k=-\infty}^{\infty}f(q^k)q^k+(1-q)\sum_{k=-\infty}^{\infty}f(-q^k)q^k.$$

Applying the q-Zeilberger algorithm to the two infinite sums of $I(aq^n,bq^n,d,e)$ respectively, we obtain a homogenous recurrence for $I(a,b,d,e)$:

$$I(a,b,d,e)=\frac{(1-a/e)(1+a/d)(1-b/e)(1+b/d)}{(1+ab/deq)(1+ab/de)}I(aq,bq,d,e),$$

implying that

$$I(a,b,d,e)=\frac{(a/e,-a/d,b/e,-b/d;q)_\infty}{(-ab/deq;q)_\infty}I(0,0,d,e).$$

By the non-terminating form of the q-Vandermonde sum (2.4), we obtain

$$\begin{aligned}\frac{I(q,-q,d,e)}{1-q}&=\frac{(q,-q;q)_\infty}{(-d,e;q)_\infty}\left({}_2\phi_1\left[\begin{matrix}-d,e\\-q\end{matrix};q,q\right]+\frac{(-d,e;q)_\infty}{(d,-e;q)_\infty}{}_2\phi_1\left[\begin{matrix}d,-e\\-q\end{matrix};q,q\right]\right)\\&=\frac{(q,-q;q)_\infty}{(-d,e;q)_\infty}\frac{(-1,de;q)_\infty}{(d,-e;q)_\infty}.\end{aligned}$$

Therefore,

$$\begin{aligned}I(0,0,d,e)&=\frac{(q/de;q)_\infty}{(q^2/d^2,q^2/e^2;q^2)_\infty}I(q,-q,d,e)\\&=\frac{2(1-q)(q^2;q^2)_\infty^2(de,q/de;q)_\infty}{(q;q)_\infty(d^2,e^2,q^2/d^2,q^2/e^2;q^2)_\infty}.\end{aligned}$$

Finally, we get

$$I(a,b,c,d)=\frac{2(1-q)(q^2;q^2)_\infty^2(de,q/de,a/e,-a/d,b/e,-b/d;q)_\infty}{(q;q)_\infty(d^2,e^2,q^2/d^2,q^2/e^2;q^2)_\infty(-ab/deq;q)_\infty}.$$

3. Transformation formulae

In this section, we show that many classical transformation formulae of non-terminating basic hypergeometric series can be proved by using the q-Zeilberger algorithm. The basic idea is to find the same recurrence relations and limit values of two summations $f(a,\ldots,c)$ and $g(a,\ldots,c)$. Suppose we have obtained a recurrence relation of second order or higher order of the form (1.1) for both $f(a,\ldots,c)$ and $g(a,\ldots,c)$. Then the following theorem ensures that $f(a,\ldots,c)$ and $g(a,\ldots,c)$ must be equal as long as $\lim_{N\to\infty} f(aq^N,\ldots,cq^N)$ coincides with the limit $\lim_{N\to\infty} g(aq^N,\ldots,cq^N)$.

Theorem 3.1. *Let $f(z)$ be a continuous function defined on the disc $|z|\leqslant r$ and let $d\geqslant 2$ be an integer. Suppose that we have a recurrence relation*

$$f(z)=a_1(z)f(zq)+a_2(z)f(zq^2)+\cdots+a_d(z)f(zq^d). \tag{3.1}$$

For $i=1,\ldots,d$, we denote $a_i(0)$ by w_i. Suppose that there exists a real number $M>0$ such that

$$|a_i(z)-w_i|\leqslant M|z|,\quad 1\leqslant i\leqslant d,$$

and

$$\begin{aligned}|w_d|+|w_{d-1}+w_d|+\cdots+|w_2+\cdots+w_d|&<1,\\ w_1+w_2+\cdots+w_d&=1.\end{aligned}$$

Then $f(z)$ is uniquely determined by $f(0)$ and the functions $a_i(z)$.

Proof. By the recurrence relation (3.1), we have

$$f(z)=\sum_{i=1}^{d} A_n^{(i)} f(zq^{n+i}),$$

where $A_0^{(i)}=a_i(z)$ and

$$\left.\begin{aligned}A_{n+1}^{(i)}&=a_i(zq^{n+1})A_n^{(1)}+A_n^{(i+1)},\quad 1\leqslant i<d,\\ A_{n+1}^{(d)}&=a_d(zq^{n+1})A_n^{(1)}.\end{aligned}\right\} \tag{3.2}$$

Let

$$\lambda(x)=x^{d-1}-\sum_{i=2}^{d}\left|\sum_{j=i}^{d} w_j\right| x^{d-i}.$$

By the assumption, $\lambda(1)>0$. Hence, we may choose a real number p such that $|q|<p<1$ and $\lambda(p)>0$, namely,

$$\sum_{i=2}^{d} p^{d-i}\left|\sum_{j=i}^{d} w_j\right|<p^{d-1}.$$

Let

$$\begin{aligned}A&=\max\{|A_0^{(1)}|,\ldots,|A_d^{(1)}|\},\\ A'&=\max\left\{A\frac{dMr}{\lambda(p)},\frac{|A_1^{(1)}-A_0^{(1)}|}{p},\ldots,\frac{|A_d^{(1)}-A_{d-1}^{(1)}|}{p^d}\right\},\\ B&=\frac{dMr}{p^{d-2}}+\frac{A'p}{A}.\end{aligned}$$

We will use induction on n to show that

$$|A_n^{(1)}| \leqslant A(-B;p)_n, \tag{3.3}$$

$$|A_n^{(1)} - A_{n-1}^{(1)}| \leqslant A'p^n(-B;p)_n. \tag{3.4}$$

By definition, the inequalities (3.3) and (3.4) hold for $n = 1, \dots, d$. Suppose that $n \geqslant d$ and the inequalities hold for $1, 2, \dots, n$. From (3.2) we have that

$$\begin{aligned} A_{n+1}^{(1)} &= \sum_{i=1}^{d} a_i(zq^{n+2-i})A_{n+1-i}^{(1)} \\ &= \sum_{i=1}^{d}((a_i(zq^{n+2-i}) - w_i)A_{n+1-i}^{(1)}) + A_n^{(1)} + \sum_{i=2}^{d}\biggl((A_{n+1-i}^{(1)} - A_{n+2-i}^{(1)})\sum_{j=i}^{d} w_j\biggr). \end{aligned}$$

By the inductive hypotheses, it follows that

$$\begin{aligned} |A_{n+1}^{(1)}| &\leqslant \sum_{i=1}^{d} Mr|q^{n+2-i}|A(-B;p)_{n+1-i} + A(-B;p)_n \\ &\qquad + A'\sum_{i=2}^{d} p^{n+2-i}(-B;p)_{n+2-i}\biggl|\sum_{j=i}^{d} w_j\biggr| \\ &\leqslant A\biggl(1 + \frac{dMr}{p^{d-2}}p^n\biggr)(-B;p)_n + A'(-B;p)_n\sum_{i=2}^{d} p^{n+2-i}\biggl|\sum_{j=i}^{d} w_j\biggr| \\ &< A\biggl(1 + \biggl(\frac{dMr}{p^{d-2}} + \frac{A'p}{A}\biggr)p^n\biggr)(-B;p)_n \\ &= A(-B;p)_{n+1}. \end{aligned}$$

Similarly, by the inductive assumptions we have

$$\begin{aligned} &|A_{n+1}^{(1)} - A_n^{(1)}| \\ &\quad\leqslant \sum_{i=1}^{d} |a_i(zq^{n+2-i}) - w_i|\,|A_{n+1-i}^{(1)}| + \biggl|\sum_{i=2}^{d}\biggl((A_{n+1-i}^{(1)} - A_{n+2-i}^{(1)})\sum_{j=i}^{d} w_j\biggr)\biggr| \\ &\quad\leqslant A\frac{dMr}{p^{d-1}}p^{n+1}(-B;p)_n + A'(-B;p)_n\sum_{i=2}^{d} p^{n+2-i}\biggl|\sum_{j=i}^{d} w_j\biggr| \\ &\quad= A'\biggl(\frac{AdMr}{(A'p^{d-1}) + 1} - \frac{\lambda(p)}{p^{d-1}}\biggr)p^{n+1}(-B;p)_n \\ &\quad\leqslant A'p^{n+1}(-B;p)_{n+1}. \end{aligned}$$

Therefore, the inequalities (3.3) and (3.4) hold for $n+1$. Using (3.4) we reach the following inequality:

$$|A_n^{(1)} - A_{n-1}^{(1)}| \leqslant A'p^n(-B;p)_\infty.$$

So the limit $\lim_{n\to\infty} A_n^{(1)}$ exists. By (3.2), for any $1 \leqslant i \leqslant d$, the $\lim_{n\to\infty} A_n^{(i)}$ exists. Thus, we get

$$f(z) = f(0) \sum_{i=1}^{d} \lim_{n\to\infty} A_n^{(i)},$$

which completes the proof. □

Remarks 3.2.

(i) The condition that $f(z)$ is continuous in $|z| \leqslant r$ can be replaced by the assumption that $\lim_{N\to\infty} f(zq^N)$ exists.

(ii) The above theorem can be easily generalized to multi-variables.

We now give some examples. The first five are adopted from the appendix of [**25**].

Example 3.3. Heine's transformations of ${}_2\phi_1$ series:

$$
{}_2\phi_1 \left[\begin{matrix} a, b \\ c \end{matrix}; q, z\right] = \frac{(b, az; q)_\infty}{(c, z; q)_\infty} {}_2\phi_1 \left[\begin{matrix} \frac{c}{b}, z \\ az \end{matrix}; q, b\right] \tag{3.5}
$$

$$
= \frac{(c/b, bz; q)_\infty}{(c, z; q)_\infty} {}_2\phi_1 \left[\begin{matrix} \frac{abz}{c}, b \\ bz \end{matrix}; q, \frac{c}{b}\right] \tag{3.6}
$$

$$
= \frac{(abz/c; q)_\infty}{(z; q)_\infty} {}_2\phi_1 \left[\begin{matrix} \frac{c}{a}, \frac{c}{b} \\ c \end{matrix}; q, \frac{abz}{c}\right]. \tag{3.7}
$$

Let

$$f(z) = {}_2\phi_1 \left[\begin{matrix} a, b \\ c \end{matrix}; q, z\right].$$

We have

$$
f(z) = \frac{-c - q + (qa + qb)z}{q(z-1)} f(zq) + \frac{c - qabz}{q(z-1)} f(zq^2). \tag{3.8}
$$

By Theorem 3.1, for $|c/q| < 1$, $f(z)$ is uniquely determined by $f(0)$ and the recurrence relation (3.8). Let

$$g(z) = \frac{(b, az; q)_\infty}{(c, z; q)_\infty} {}_2\phi_1 \left[\begin{matrix} \frac{c}{b}, z \\ az \end{matrix}; q, b\right].$$

Then $g(z)$ satisfies the same recursion as (3.8). By the q-binomial theorem, we have

$$g(0) = \frac{(b; q)_\infty}{(c; q)_\infty} {}_1\phi_0 \left[\begin{matrix} \frac{c}{b} \\ - \end{matrix}; q, b\right] = 1 = f(0).$$

Therefore, (3.5) holds for $|c/q| < 1$. By analytic continuation, (3.5) holds for all $a, b, c, z \in \mathbb{C}$, provided that both sides are convergent. Similar arguments can justify (3.6) and (3.7).

Example 3.4. Jackson's transformations of ${}_2\phi_1$, ${}_2\phi_2$ and ${}_3\phi_2$ series:

$$ {}_2\phi_1\left[\begin{matrix}a,b\\c\end{matrix};q,z\right]=\frac{(az;q)_\infty}{(z;q)_\infty}{}_2\phi_2\left[\begin{matrix}a,\frac{c}{b}\\c,az\end{matrix};q,bz\right] \tag{3.9}$$

$$=\frac{(abz/c;q)_\infty}{(bz/c;q)_\infty}{}_3\phi_2\left[\begin{matrix}a,\frac{c}{b},0\\c,\frac{cq}{bz}\end{matrix};q,q\right], \tag{3.10}$$

where (3.10) holds, provided that the series terminates.

Let $f(z)$ be the left-hand side (3.9). Thus, we have the recurrence relation (3.8) and $\lim_{N\to\infty} f(zq^N)=1$. By using the q-Zeilberger algorithm, one can verify that the right-hand sides of (3.9) and (3.10) also satisfy the same recurrence relation. Moreover, for the summation (3.10), the terminating condition is required to ensure $\lim_{k\to\infty} g_{n,k}=0$ in (2.1). By considering the limit values, we get the transformation formulae (3.9) and (3.10).

A similar discussion implies the following transformation formula for terminating ${}_2\phi_1$ series:

$$ {}_2\phi_1\left[\begin{matrix}a,b\\c\end{matrix};q,z\right]=\frac{(c/b,c/a;q)_\infty}{(c/ab,c;q)_\infty}{}_3\phi_2\left[\begin{matrix}a,b,\frac{abz}{c}\\\frac{abq}{c},0\end{matrix};q,q\right],$$

provided that the right-hand side summation terminates.

Example 3.5. Transformations of ${}_3\phi_2$ series:

$$ {}_3\phi_2\left[\begin{matrix}a,b,c\\d,e\end{matrix};q,\frac{de}{abc}\right]=\frac{(e/a,de/bc;q)_\infty}{(e,de/abc;q)_\infty}{}_3\phi_2\left[\begin{matrix}a,\frac{d}{b},\frac{d}{c}\\d,\frac{de}{bc}\end{matrix};q,\frac{e}{a}\right] \tag{3.11}$$

$$=\frac{(b,de/ab,de/bc;q)_\infty}{(d,e,de/abc;q)_\infty}{}_3\phi_2\left[\begin{matrix}\frac{d}{b},\frac{e}{b},\frac{de}{abc}\\\frac{de}{ab},\frac{de}{bc}\end{matrix};q,b\right]. \tag{3.12}$$

We take d as the parameter. Let $f(d)$ be the series on the left-hand side of (3.11). We have $f(0)=1$ and

$$f(d)=-\frac{(1+q)ed^2+(-abc-eb-ea-ec)d+abc+abce/q}{(-ed+abc)(-1+d)}f(dq)\\ +\frac{e(-c+dq)(-dq+b)(-dq+a)}{q(-ed+abc)(-1+dq)(-1+d)}f(dq^2).$$

On the other hand, one can verify that both the series on the right-hand side of (3.11) and in (3.12) have the same limit value and satisfy the same recurrence relation as the left-hand side of (3.11).

Example 3.6. The Sears–Carlitz transformation:

$$
{}_3\phi_2\left[\begin{matrix} a,b,c \\ \frac{aq}{b},\frac{aq}{c} \end{matrix};q,\frac{aqz}{bc}\right] = \frac{(az;q)_\infty}{(z;q)_\infty}\,{}_5\phi_4\left[\begin{matrix} a^{1/2},-a^{1/2},(aq)^{1/2},-(aq)^{1/2},\frac{aq}{bc} \\ \frac{aq}{b},\frac{aq}{c},az,\frac{q}{z} \end{matrix};q,q\right],
$$

provided that the right-hand side terminates.

Let us take z as the parameter and denote the series by $f(z)$. One can verify that both sides have the same limit value $\lim_{N\to\infty} f(zq^N)=1$ and satisfy the following recurrence relation:

$$
f(z)=r_1(z)f(zq)+r_2(z)f(zq^2)+r_3(z)f(zq^3),
$$

where

$$
r_1(z)=\frac{ab+ac+bc}{bc}+O(z),\qquad r_2(z)=\frac{-a(b+a+c)}{bc}+O(z),\qquad r_3(z)=\frac{a^2}{bc}+O(z).
$$

Note that to comply with the conditions of Theorem 3.1, we only need the values $r_1(0)$, $r_2(0)$ and $r_3(0)$. Therefore, we do not give the explicit formulae for $r_1(z)$, $r_2(z)$ and $r_3(z)$.

Example 3.7. Transformations of very well-poised ${}_8\phi_7$ series:

$$
{}_8\phi_7\left[\begin{matrix} a,qa^{1/2},-qa^{1/2},b,c,d,e,f \\ a^{1/2},-a^{1/2},\frac{aq}{b},\frac{aq}{c},\frac{aq}{d},\frac{aq}{e},\frac{aq}{f} \end{matrix};q,\frac{a^2q^2}{bcdef}\right] \tag{3.13}
$$

$$
\begin{aligned}
&= \frac{(aq,aq/ef,\lambda q/e,\lambda q/f;q)_\infty}{(aq/e,aq/f,\lambda q,\lambda q/ef;q)_\infty} \\
&\quad\times {}_8\phi_7\left[\begin{matrix} \lambda,q\lambda^{1/2},-q\lambda^{1/2},\frac{\lambda b}{a},\frac{\lambda c}{a},\frac{\lambda d}{a},e,f \\ \lambda^{1/2},-\lambda^{1/2},\frac{aq}{b},\frac{aq}{c},\frac{aq}{d},\frac{\lambda q}{e},\frac{\lambda q}{f} \end{matrix};q,\frac{aq}{ef}\right]
\end{aligned} \tag{3.14}
$$

$$
\begin{aligned}
&= \frac{(aq,b,bc\mu/a,bd\mu/a,be\mu/a,bf\mu/a;q)_\infty}{(aq/c,aq/d,aq/e,aq/f,\mu q,b\mu/a;q)_\infty} \\
&\quad\times {}_8\phi_7\left[\begin{matrix} \mu,q\mu^{1/2},-q\mu^{1/2},\frac{aq}{bc},\frac{aq}{bd},\frac{aq}{be},\frac{aq}{bf},\frac{b\mu}{a} \\ \mu^{1/2},-\mu^{1/2},\frac{bc\mu}{a},\frac{bd\mu}{a},\frac{be\mu}{a},\frac{bf\mu}{a},\frac{aq}{b} \end{matrix};q,b\right],
\end{aligned} \tag{3.15}
$$

where $\lambda=qa^2/bcd$ and $\mu=q^2a^3/b^2cdef$.

We choose a, b, f as parameters for the series in (3.13) and (3.14) and denote the series by $H(a,b,f)$. It follows from (3.11) that both series have the same limit value $\lim_{N\to\infty} H(aq^N,bq^N,fq^N)$. By computation, one sees that they satisfy the following recurrence relation:

$$
H(a,b,f)=r_1(a,b,f)H(aq,bq,fq)+r_2(a,b,f)H(aq^2,bq^2,fq^2),
$$

where

$$
r_1(a,b,f)=1+O(a),\qquad r_2(a,b,f)=O(a).
$$

Thus, we have verified the first transformation formula. To prove the second transformation formula, we choose a, c, f as the parameters and denote the series by $H(a,c,f)$. By computation, the series in (3.13) and (3.15) satisfy the following recurrence relation:

$$H(a,c,f) = r_1(a,c,f)H(aq,cq,fq) + r_2(a,c,f)H(aq^2,cq^2,fq^2).$$

Using the transformation formula (3.12), one sees that both sides have the same limit value $\lim_{N\to\infty} H(aq^N,cq^N,fq^N)$. Thus, we have obtained the second transformation formula.

Example 3.8 (a limiting case of Watson's formula). Watson [**47**] used the following formula to prove the Rogers–Ramanujan identities [**28**] (see also [**25**, § 2.7]):

$$\sum_{k=0}^{\infty} \frac{(aq;q)_{k-1}(1-aq^{2k})}{(q;q)_k}(-1)^k a^{2k} q^{k(5k-1)/2} = (aq;q)_\infty \sum_{k=0}^{\infty} \frac{a^k q^{k^2}}{(q;q)_k}. \tag{3.16}$$

We choose a as the parameter. Then we can verify that both sides of (3.16) have the same limit value $f(0)=1$ and satisfy the same recurrence relation

$$f(a) = (1-aq)f(aq) + aq(1-aq)(1-aq^2)f(aq^2).$$

Setting $a=1$ and $a=q$ in (3.16), we obtain the Rogers–Ramanujan identities by Jacobi's triple product identity:

$$(q;q)_\infty \sum_{k=0}^{\infty} \frac{q^{k^2}}{(q;q)_k} = \sum_{k=-\infty}^{\infty} (-q^2)^k q^{5\binom{k}{2}} = (q^2,q^3,q^5;q^5)_\infty$$

and

$$(q;q)_\infty \sum_{k=0}^{\infty} \frac{q^{k^2+k}}{(q;q)_k} = \sum_{k=-\infty}^{\infty} (-q^4)^k q^{5\binom{k}{2}} = (q,q^4,q^5;q^5)_\infty.$$

Finite forms of the above identities have been proved by Paule [**40**] by using the q-Zeilberger algorithm.

Example 3.9 (a generalization of Lebesgue's identity). The following transformation formula is due to Carlitz [**19**] (see [**2**]):

$$\sum_{k=0}^{\infty} \frac{(x;q)_k q^{\binom{k}{2}}(-a)^k}{(q,bx;q)_k} = \frac{(a,x;q)_\infty}{(bx;q)_\infty} \sum_{k=0}^{\infty} \frac{(b;q)_k x^k}{(q,a;q)_k}. \tag{3.17}$$

We choose x as the parameter. Both sides of (3.17) have the same limit value $f(0) = (a;q)_\infty$ and satisfy the same recurrence relation:

$$f(x) = \left(\frac{q+a}{q} + O(x)\right) f(xq) + \left(-\frac{a}{q} + O(x)\right) f(xq^2).$$

Example 3.10 (three-term transformation formulae). Our approach also applies to certain three-term transformation formulae. It is sometimes the case that the left-hand side of the identity satisfies a homogenous recursion, and the two terms on the right-hand side satisfy non-homogenous recursions respectively but their sum leads to a homogenous recurrence relation.

The first example is

$$ {}_3\phi_2\left[\begin{matrix} a,b,c \\ d,e \end{matrix}; q, \frac{de}{abc}\right] = \frac{(e/b,e/c;q)_\infty}{(e,e/bc;q)_\infty}\,{}_3\phi_2\left[\begin{matrix} \dfrac{d}{a},b,c \\ d,\dfrac{bcq}{e} \end{matrix}; q,q\right] + \frac{(d/a,b,c,de/bc;q)_\infty}{(d,e,bc/e,de/abc;q)_\infty}\,{}_3\phi_2\left[\begin{matrix} \dfrac{e}{b},\dfrac{e}{c},\dfrac{de}{abc} \\ \dfrac{de}{bc},\dfrac{eq}{bc} \end{matrix}; q,q\right]. \tag{3.18} $$

Let us choose e as the parameter. Then both sides of (3.18) have the same limit value $\lim_{N\to\infty} f(eq^N) = 1$ and satisfy the same recurrence relation:

$$ f(e) = r_1(e)f(eq) + r_2(e)f(eq^2), $$

where

$$ r_1(e) = \frac{q+d}{q} + O(e), \qquad r_2(e) = -\frac{d}{q} + O(e). $$

The second example is

$$ {}_8\phi_7\left[\begin{matrix} a, qa^{1/2}, -qa^{1/2}, b, c, d, e, f \\ a^{1/2}, -a^{1/2}, \dfrac{aq}{b}, \dfrac{aq}{c}, \dfrac{aq}{d}, \dfrac{aq}{e}, \dfrac{aq}{f} \end{matrix}; q, \frac{a^2q^2}{bcdef}\right] $$

$$ = \frac{(aq,aq/de,aq/df,aq/ef;q)_\infty}{(aq/d,aq/e,aq/f,aq/def;q)_\infty}\,{}_4\phi_3\left[\begin{matrix} \dfrac{aq}{bc},d,e,f \\ \dfrac{aq}{b},\dfrac{aq}{c},\dfrac{def}{a} \end{matrix}; q,q\right] $$

$$ + \frac{(aq,aq/bc,d,e,f,a^2q^2/bdef,a^2q^2/cdef;q)_\infty}{(aq/b,aq/c,aq/d,aq/e,aq/f,a^2q^2/bcdef,def/aq;q)_\infty} $$

$$ \times\,{}_4\phi_3\left[\begin{matrix} \dfrac{aq}{de},\dfrac{aq}{df},\dfrac{aq}{ef},\dfrac{a^2q^2}{bcdef} \\ \dfrac{a^2q^2}{bdef},\dfrac{a^2q^2}{cdef},\dfrac{aq^2}{def} \end{matrix}; q,q\right]. \tag{3.19} $$

We take a, b, f as parameters and denote the series by $H(a,b,f)$. By the transformation formula (3.18), we see that both sides of (3.19) have the same limit value $\lim_{N\to\infty} H(aq^N, bq^N, fq^N)$. Moreover, they satisfy the same recurrence relation:

$$ H(a,b,f) = r_1(a,b,f)H(aq,bq,fq) + r_2(a,b,f)H(aq^2,bq^2,fq^2), $$

where

$$r_1(a,b,f) = 1 + O(a), \quad r_2(a,b,f) = O(a).$$

Example 3.11 (Rogers–Fine identity). To conclude this paper, we consider a transformation formula that can be justified by using non-homogeneous recurrence relations. This is the Rogers–Fine identity [**23**]:

$$\sum_{k=0}^{\infty} \frac{(a;q)_k}{(b;q)_k} z^k = \sum_{k=0}^{\infty} \frac{(a, azq/b;q)_k (1-azq^{2k}) q^{k^2-k} (bz)^k}{(b,z;q)_k (1-zq^k)}. \tag{3.20}$$

We choose z as the parameter. By computation, both sides of (3.20) satisfy the following recurrence relation:

$$(-b+azq)f(zq) + (-zq+q)f(z) = q-b.$$

The non-homogenous term $q-b$ occurs because $g_{0,0} = -q+b$ and $\lim_{k\to\infty} g_{0,k} = 0$ when one implements the q-Zeilberger algorithm. Let $d(z)$ be the difference of the two sides of (3.20). Then we have

$$d(z) = \frac{b}{q} \frac{1-azq/b}{1-z} d(zq).$$

Since $d(0) = 0$, equation (3.20) holds for $|z| < 1$ and $|b| < |q|$. By analytic continuation, it holds for $|z| < 1$.

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