Converging to Gosper's Algorithm

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Abstract

Given two polynomials, we find a convergence property of the GCD of the rising factorial and the falling factorial. Based on this property, we present a unified approach to computing the universal denominators as given by Gosper's algorithm and Abramov's algorithm for finding rational solutions to linear difference equations with polynomial coefficients.

Keywords: Gosper's algorithm, Abramov's algorithm, universal denominator, dispersion.

AMS Subject Classification: 33F10; 05A19

1 Introduction

Let $\mathbb{N}$ be the set of nonnegative integers, $\mathbb{K}$ be a field of characteristic zero, $\mathbb{K}(n)$ be the field of rational functions over $\mathbb{K}$, and $\mathbb{K}[n]$ be the ring of polynomials over $\mathbb{K}$. We assume that subject to normalization the gcd (greatest common divisor) of two polynomials always takes a value as a monic polynomial, namely, polynomials with the leading coefficient being 1. Recall that a nonzero term t_n is called a hypergeometric term over $\mathbb{K}$ if there exists a rational function $r \in \mathbb{K}(n)$ such that

$$\frac{t_{n+1}}{t_n} = r(n).$$

If $r(n) = a(n)/b(n)$, where $a(n), b(n) \in \mathbb{K}[n]$, then the function $a(n)/b(n)$ is called a rational representation of the rational function $r(n)$. If $\gcd(a(n), b(n)) = 1$ holds, then $a(n)/b(n)$ is called a reduced rational representation of $r(n)$.

Gosper's algorithm [7] (also see [6, 8, 9, 15, 18, 19, 20, 17]) has been extensively studied and widely used to prove hypergeometric identities. Given a hypergeometric term t_n, Gosper's algorithm is a procedure to find a hypergeometric term z_n satisfying

$$z_{n+1} - z_n = t_n, \tag{1.1}$$

if it exists, or confirm the nonexistence of any solution of (1.1). The key idea of Gosper's algorithm lies in a representation of rational functions called Gosper representation; i.e., writing the rational function $r(n)$ in the following form:

$$r(n) = \frac{a(n)}{b(n)} \frac{c(n+1)}{c(n)},$$

where a, b and c are polynomials over $\mathbb{K}$ and

$$\gcd(a(n), b(n+h)) = 1 \text{ for all } h \in \mathbb{N}.$$

Petkovšek [13] has realized that a Gosper representation becomes unique, which is called the Gosper-Petkovšek representation, or GP representation, for short, if we further require that b, c are monic polynomials such that

$$\gcd(a(n), c(n)) = \gcd(b(n), c(n+1)) = 1.$$

In the same paper, Petkovšek also gave an algorithm to compute GP representations; subsequently we will call it the "GP algorithm". In [12], Paule and Strehl gave a derivation of Gosper's algorithm by using the GP representation. In [11], equipped with the Greatest Factorial Factorization (GFF), Paule presented a new approach to indefinite hypergeometric summation which leads to the same algorithm as Gosper's, but in a new setting. In [10], Lisoněk and *et al.,* gave a detailed study of the degree setting for Gosper's algorithm.

Finding rational solutions is important in computer algebra because many problems can be reduced to rational solutions. For example, we may consider the generalization of Gosper's algorithm. Given a linear difference equation

$$\sum_{m=0}^{d} p_m(n) y(n+m) = p(n), \tag{1.2}$$

where $p_0(n), p_1(n), \ldots, p_d(n), p(n) \in \mathbb{K}[n]$ are given polynomials such that $p_0(n) \neq 0$, $p_d(n) \neq 0$, a polynomial $g(n) \in \mathbb{K}[n]$ is called a universal denominator for (1.2) if and only if for every solution $y(n) \in \mathbb{K}(n)$ to (1.2) there exists a $f(n) \in \mathbb{K}[n]$ such that $y(n) = f(n)/g(n)$. Once a universal denominator is found, then it is easy to find the rational solutions of the linear difference equation (1.2) by finding the polynomial solutions using the techniques in [2, 4, 13]. Abramov [2] developed an algorithm to find a universal denominator of (1.2) which relies on all the coefficients $p_0(n), p_1(n), \ldots, p_d(n), p(n)$. In [3], an improved version is given which requires only two coefficients $p_0(n)$ and $p_d(n)$. Compared with the simplicity of the output of Abramov's algorithm, the justification is quite involved. Recall that the dispersion $\mathrm{dis}(a(n), b(n))$ of the polynomials $a(n), b(n) \in \mathbb{K}[n]$ is the greatest nonnegative integer k (if it exists) such that $a(n)$ and $b(n+k)$ have a nontrivial common divisor, i.e.,

$$\mathrm{dis}(a, b) = \max\{k \in \mathbb{N} \mid \deg \gcd(a(n), b(n+k)) \geq 1\}.$$

If k does not exist then we set $\mathrm{dis}(a, b) = -1$. Observe that $\mathrm{dis}(a(n), b(n))$ can be computed as the largest nonnegative integer root of the polynomial $R(h) \in \mathbb{K}[h]$ where $R(h) = Res_n(a(n), b(n+h))$.

The main result of this paper is the discovery of a convergence property of the GCD of rising factorial of a polynomial $b(n)$ and the falling factorial of another polynomial $a(n)$. By using the limit of the GCD sequence, we may transform a rational difference equation into a polynomial difference equation. The convergence argument yields a new and streamlined approach to the explicit formula for Abramov's universal denominator. Note that this explicit formula can be used to compute rational solutions of a linear difference equation (1.2). In addition, we derive Abramov's universal denominator from Barkatou's explicit formula. The relation between Barkatou's approach and Abramov's algorithm has been discussed in detail by Weixlbaumer [16].

2 The Convergence Property

The main idea of this paper is the following convergence property of a GCD sequence. It turns out that this simple observation plays a fundamental role in finding rational solutions of linear difference equations, and it can be viewed as a unified approach to several well-known algorithms.

Theorem 2.1. *Let $a(n)$ and $b(n)$ be two nonzero polynomials in n and let*

$$k_0 = \mathrm{dis}(a(n-1), b(n)) = \max\{k \in \mathbb{N} \mid \deg\, \gcd(a(n-1), b(n+k)) \geq 1\}. \tag{2.1}$$

Define

$$G_k(n) = \gcd(b(n)b(n+1)\dots b(n+k-1), a(n-1)a(n-2)\dots a(n-k)). \tag{2.2}$$

Then the sequence $G_1(n), G_2(n), \dots$ converges to $G_{k_0+1}(n)$.

Proof. For all $k > k_0$ we have

$$\gcd(a(n-1), b(n+k)) = \cdots = \gcd(a(n-k-1), b(n+k)) = 1. \tag{2.3}$$

Note that

$$G_{k+1}(n) = \gcd(b(n)b(n+1)\dots b(n+k), a(n-1)a(n-2)\dots a(n-k-1)).$$

This implies that

$$G_k(n) = G_{k+1}(n),$$

for all $k > k_0$. Moreover, one sees that once (2.3) is satisfied for $k > k_0$, it is also satisfied for $k+1$. It follows that

$$G_k(n) = G_{k+1}(n) = G_{k+2}(n) = \cdots,$$

for all $k > k_0$, and this completes the proof. □

Using the above convergence property of the sequence $G_k(n)$, we are led to a simple approach to Gosper's algorithm without resorting to the Gosper representation or GP representation of rational functions. Given a hypergeometric term t_n and suppose that there exists a hypergeometric term z_n satisfying equation (1.1), then by using (1.1) we find

$$r(n)y(n+1) - y(n) = 1, \tag{2.4}$$

where $r(n) = t_{n+1}/t_n$ and $y(n) = z_n/t_n$ are rational functions of n, see [15].

Theorem 2.2. *Let $r(n)$ and $y(n)$ in equation (2.4) be in terms of their reduced rational representations*:

$$r(n) = \frac{a(n)}{b(n)}, \; y(n) = \frac{f(n)}{g(n)}. \tag{2.5}$$

Then

$$g(n) \mid G_{k_0+1}(n),$$

where k_0 and $G_k(n)$ are defined in Theorem 2.1.

Proof. Using (2.5) in (2.4) gives

$$a(n)g(n)f(n+1) - b(n)g(n+1)f(n) = b(n)g(n)g(n+1). \tag{2.6}$$

From the above relation, we immediately get that

$$g(n) \,|\, b(n)g(n+1) \quad \text{and that} \quad g(n+1) \mid a(n)g(n).$$

Using these two relations repeatedly we obtain

$$g(n) \mid b(n)b(n+1)\dots b(n+k-1)g(n+k),$$
$$g(n) \mid a(n-1)a(n-2)\dots a(n-k)g(n-k),$$

for all $k \in \mathbb{N}$. Since $\mathbb{K}$ has characteristic zero,

$$\gcd(g(n), g(n+k)) = \gcd(g(n), g(n-k)) = 1,$$

for all large enough k. It follows that

$$g(n) \mid b(n)b(n+1)\dots b(n+k-1), \tag{2.7}$$
$$g(n) \mid a(n-1)a(n-2)\dots a(n-k), \tag{2.8}$$

for all large enough k. Therefore

$$g(n) \mid G_k(n),$$

for all large enough k. The rest of the proof follows when k goes to infinity in this equation and by Theorem 2.1. □

The next step is simply to set

$$g(n) = G_{k_o+1}(n) \tag{2.9}$$

in equation (2.6) as in the GFF algorithm of Paule. If equation (2.6) can be solved for $f \in \mathbb{K}[n]$, then

$$z_n = \frac{f(n)}{g(n)}\, t_n$$

is a hypergeometric solution of (1.1); Otherwise no hypergeometric solution of (1.1) exists. Note that the solution $f(n)$ may not be coprime to $g(n)$. However, it is clear that this does not affect the solution of $y(n)$. Indeed, the polynomials $f(n)$ and $g(n)$ can be recovered from the solution of $y(n)$ after dividing the greatest common factors.

Algorithm 2.3.
INPUT: $r(n) \in \mathbb{K}(n)$ *such that* $t_{n+1}/t_n = r(n)$ *for large enough* n *in* $\mathbb{N}$.
OUTPUT: a hypergeometric solution z_n *of* (1.1) *if it exists, otherwise "no hypergeometric solution of* (1.1) *exists".*

(1) *Decompose* $r(n)$ *into* a/b *where* a, b *are two relatively prime polynomials.*

(2) *Compute* k_0 *as in* (2.1).

(3) *If* $k_0 \geq 0$ *then compute* $g(n) = G_{k_o+1}(n)$, *where* $G_k(n)$ *is defined as in* (2.2), *otherwise* $g(n) = 1$.

(4) *If equation* (2.6) *can be solved for* $f \in \mathbb{K}[n]$ *then return* $z_n = \frac{f(n)}{g(n)}\, t_n$*; Otherwise return "no hypergeometric solution of* (1.1) *exists".*

Let us take an example from [15]:

Example 2.4. *Let* $t_n = (4n+1) \cdot \frac{n!}{(2n+1)!}$, *then*

$$r(n) = \frac{t_{n+1}}{t_n} = \frac{4n+5}{2(4n+1)(2n+3)}.$$

Hence $a(n) = 4n+5$, $b(n) = 2(4n+1)(2n+3)$ *and then* $k_0 = 0$. *Note that for all* $k > k_0$, *equation* (2.3) *is satisfied. From* (2.9), $g(n) = n + \frac{1}{4}$. *By* (2.6), $f(n)$ *is a polynomial which satisfies*

$$2f(n+1) - 4(2n+3)f(n) = (2n+3)(4n+1).$$

The polynomial $f(n) = -\frac{1}{2}(2n+1)$ *is a solution of this equation. Therefore,*

$$z_n = \frac{f(n)}{g(n)}\; t_n = -2\; \frac{n!}{(2n)!}.$$

We remark that the argument for the relations (2.7) and (2.8) is used by Petkovšek [13]. Moreover, the products on the right hand sides of (2.7) and (2.8) can be written in the notation of rising or falling factorials as introduced by Paule [11].

3 Connections to Gosper's and Abramov's Algorithms

We will show how Theorem 2.2 is related to Gosper's algorithm and Abramov's algorithm for finding rational solutions of linear difference equations with polynomial coefficients [3].

Abramov's Algorithm (general order d): Consider the difference equation

$$p_d(n)y(n+d) + \ldots + p_0(n)y(n) = p(n) \tag{3.1}$$

with given $p_0(n), p_1(n), \ldots, p_d(n), p(n) \in \mathbb{K}[n]$ such that p_0 and p_d are nonzero. Abramov gave the following algorithm to compute a universal denominator $G(n)$ for (3.1): Define

$$N = \text{dis}(p_d(n-d), p_0(n)) = \max\{k \in \mathbb{N} \mid \deg \gcd(p_d(n-d), p_0(n+k)) \geq 1\}. \tag{3.2}$$

If $N=-1$ set $G(n)=1$, i.e., in this case all rational solutions are polynomials. If $N \geq 0$, define

$$A_{N+1}(n)=p_d(n-d), \quad B_{N+1}(n)=p_0(n), \tag{3.3}$$

and for $i=N$ down to $i=0$ do:

$$d_i(n) \;=\; \gcd(A_{i+1}(n), B_{i+1}(n+i)), \tag{3.4}$$

$$A_i(n) \;=\; \frac{A_{i+1}(n)}{d_i(n)} \quad \text{and} \quad B_i(n)=\frac{B_{i+1}(n)}{d_i(n-i)}. \tag{3.5}$$

If we use the notation of the falling factorial of a polynomial introduced in [11] by

$$[f(x)]^{\underline{k}} = f(x)f(x-1)\cdots f(x-k+1),$$

then Abramov's universal denominator of (3.1) can be written as

$$G(n)=[d_0(n)]^{\underline{1}}[d_1(n)]^{\underline{2}}\cdots[d_N(n)]^{\underline{N+1}}. \tag{3.6}$$

There is also an explicit formula for Abramov's universal denominator (3.6), namely,

$$G(n)=\gcd([p_0(n+N)]^{\underline{N+1}}, [p_d(n-d)]^{\underline{N+1}}). \tag{3.7}$$

It can be seen that (3.6) is equivalent to Theorem 3 in Abramov-Petkovšek-Ryabenko [1]. A generalized form of (3.7) can be found in Barkatou [5]. For completeness, we give a proof of the fact that the presentations (3.6) and (3.7) indeed coincide with each other. To this end we will follow the survey of Weixlbaumer [16].

First of all, based on the definition of N and the fact that $A_i \mid A_{i+1}$ and $B_i \mid B_{i+1}$, it follows that for $0 \leq i \leq N+1$, we have

$$\gcd(A_i(n), B_i(n+k)) = 1 \quad \text{for all} \quad k > N. \tag{3.8}$$

Moreover, from (3.4) and (3.5), for $0 \leq i \leq N$ it follows that

$$\gcd(A_j(n), B_j(n+i)) = 1 \quad \text{for} \quad 0 \leq j \leq i. \tag{3.9}$$

Therefore,

$$\begin{aligned}
G(n) &= \gcd([p_0(n+N)]^{\underline{N+1}}, [p_d(n-d)]^{\underline{N+1}}) \\
&= \gcd([B_{N+1}(n+N)]^{\underline{N+1}}, [A_{N+1}(n)]^{\underline{N+1}}) \\
&= \gcd\left(\left[\frac{B_{N+1}(n+N)}{d_N(n)}\right]^{\underline{N+1}}, \left[\frac{A_{N+1}(n)}{d_N(n)}\right]^{\underline{N+1}}\right)\cdot [d_N(n)]^{\underline{N+1}} \\
&= [d_N(n)]^{\underline{N+1}}\cdot \gcd([B_N(n+N)]^{\underline{N+1}}, [A_N(n)]^{\underline{N+1}}).
\end{aligned}$$

Observe that

$$\begin{aligned}
&\gcd(B_N(n+N), [A_N(n)]^{\underline{N+1}}) \\
&\quad = \gcd(B_N(n+N), A_N(n)) && \text{[by (3.8)]} \\
&\quad = 1, && \text{[by (3.9)]}
\end{aligned}$$

and similarly,

$$\gcd([B_N(n+N-1)]^{\underline{N}}, A_N(n-N)) = 1.$$

Consequently,

$$\gcd([B_N(n+N)]^{\underline{N+1}}, [A_N(n)]^{\underline{N+1}}) = \gcd([B_N(n+N-1)]^{\underline{N}}, [A_N(n)]^{\underline{N}}).$$

In the same manner one can successively split off the factors

$$[d_{N-1}(n)]^{\underline{N}}, \ldots, [d_0(n)]^{\underline{1}}$$

until one arrives at (3.6), which completes the proof of the equality of (3.6) and (3.7).

Next we remark that Theorem 2.2 follows from Abramov's algorithm, strictly speaking, the universal denominator given by Abbramov's algorithm. Equation (2.4) is equivalent to

$$a(n)y(n+1) - b(n)y(n) = b(n), \tag{3.10}$$

which is (3.1) with $d = 1$, $p_1(n) = a(n)$, $p_0(n) = -b(n)$, and $p(n) = b(n)$. From (3.7), Abramov's algorithm gives the following universal denominator of (3.10)

$$G(n) = \gcd([a(n-1)]^{\underline{N+1}}, [b(n+N)]^{\underline{N+1}}). \tag{3.11}$$

where $N = k_0$ by (2.1). Using (2.2) we have that $G(n) = G_{k_0+1}(n)$, hence Theorem 2.2 determines the same universal denominator as Abramov's algorithm.

Next we show that Abramov's algorithm delivers a Gosper representation for $r(n) = t_{n+1}/t_n$ if $r(n) = a(n)/b(n)$ is the reduced rational representation of $r(n)$. From (3.5) we obtain that $a(n-1) = A_{N+1}(n) = d_N(n)A_N(n)$, and by iteration,

$$a(n) = d_0(n+1)\ d_1(n+1)\ldots d_N(n+1)\ A_0(n+1). \tag{3.12}$$

Analogously, (3.5) implies that

$$b(n) = -d_0(n)\ d_1(n-1)\ldots d_N(n-N)\ B_0(n). \tag{3.13}$$

Consequently, in view of representation (3.6) for $G(n)$, one obtains that

$$\frac{a(n)}{b(n)} = -\frac{G(n+1)}{G(n)}\ \frac{A_0(n+1)}{B_0(n)}. \tag{3.14}$$

Note that in (3.10) w.l.o.g. we can assume that $\gcd(a(n), b(n)) = 1$, which then implies

$$\gcd(A_0(n+1), B_0(n)) = 1. \tag{3.15}$$

But more is true. Namely, by (3.9)

$$\gcd(A_0(n+1), B_0(n+i+1)) = 1 \quad \text{for } 0 \le i \le N,$$

i.e.,

$$\gcd(A_0(n+1), B_0(n+i)) = 1 \quad \text{for } 1 \le i \le N+1, \tag{3.16}$$

and by (3.8),

$$\gcd(A_0(n+1), B_0(n+k+1)) = 1 \quad \text{for all } k > N,$$

i.e.,

$$\gcd(A_0(n+1), B_0(n+k)) = 1 \quad \text{for all } k > N+1. \tag{3.17}$$

Finally, combining (3.15), (3.16) and (3.17) into one condition results in

$$\gcd(A_0(n+1), B_0(n+h)) = 1 \quad \text{for all } h \geq 0. \tag{3.18}$$

Hence the right hand side of (3.14) is a Gosper representation for $a(n)/b(n)$.

Example 3.1. *Let* $t_n = \frac{(n+2)}{n!}$, *then*

$$r(n) = \frac{t_{n+1}}{t_n} = \frac{a(n)}{b(n)},$$

where $a(n) = n+3$, $b(n) = (n+1)(n+2)$. *From Abramov's algorithm, we have* $N = 1$. *By using Abramov's algorithm with* $A_{N+1}(n) = A_2(n) = a(n-1) = n+2$ *and* $B_{N+1}(n) = B_2(n) = b(n) = (n+1)(n+2)$, *we obtain that* $A_0(n) = 1$, $B_0(n) = n+2$, *and the universal denominator* $G(n) = (n+1)(n+2)$. *Note that*

$$\gcd(G(n+1), B_0(n)) = n+2.$$

This means that the Gosper representation (3.14) *in general is not the* GP *representation for* $a(n)/b(n)$.

As explained in Paule [11], also the GP algorithm [13] for finding a GP representation of $r(n) = a(n)/b(n)$ computes a universal denominator for (3.10), namely as follows.

Petkovšek's GP Algorithm: Compute N as in Abramov's algorithm; i.e., $N = k_0$ as in (3.11). If $N = -1$ set $u(n) = 1$. If $N \geq 0$ define

$$a_0(n) = a(n), \quad b_0(n) = b(n), \tag{3.19}$$

and for $i = 1$ up to $i = N+1$ do:

$$\delta_i(n) = \gcd(a_{i-1}(n), b_{i-1}(n+i)), \tag{3.20}$$

$$a_i(n) = \frac{a_{i-1}(n)}{\delta_i(n)} \quad \text{and} \quad b_i(n) = \frac{b_{i-1}(n)}{\delta_i(n-i)}. \tag{3.21}$$

This determines a universal denominator by setting

$$u(n) = [\delta_1(n-1)]^{\underline{1}}[\delta_2(n-1)]^{\underline{2}} \cdots [\delta_{N+1}(n-1)]^{\underline{N+1}}. \tag{3.22}$$

Note that this algorithm essentially consists in running the loop in Abramov's algorithm in the REVERSE direction. (Note that also its initialization is slightly different, namely starting with the pair $(a(n), b(n))$ instead of $(a(n-1), b(n))$).

Finally, as above, from (3.21) and by using (3.22) one obtains that

$$\frac{a(n)}{b(n)} = \frac{u(n+1)}{u(n)} \frac{a_{N+1}(n)}{b_{N+1}(n)}. \tag{3.23}$$

Petkovšek's algorithm is designed in such a way that (3.23) is not only a Gosper representation, but also a GP representation for $a(n)/b(n)$; see [13], [15], and also [11]. This means that besides

$$\gcd(a_{N+1}(n), b_{N+1}(n+h)) = 1, \quad \text{for all } h \geq 0. \tag{3.24}$$

we also have that

$$\gcd(u(n+1), b_{N+1}(n)) = 1, \tag{3.25}$$

and

$$\gcd(u(n), a_{N+1}(n)) = 1. \tag{3.26}$$

From (3.14) and (3.23) we get

$$\frac{u(n+1)}{u(n)} \frac{a_{N+1}(n)}{b_{N+1}(n)} = -\frac{G(n+1)}{G(n)} \frac{A_0(n+1)}{B_0(n)}. \tag{3.27}$$

By Lemma 5.3.1 (see [15], p.82), we obtain that

$$u(n) \mid G(n),$$

which implies that the universal denominator given by GP algorithm is a factor of the universal denominator given by Abramov's algorithm.

4 Rational Solutions of Linear Difference Equations

In this section we show how to deduce the explicit formula (3.7) for Abramov's universal denominator by using the convergence argument.

Theorem 4.1. *Let $p_0(n)$ and $p_d(n)$ be two nonzero polynomials in n, and let N be defined as in (3.2). Put*

$$G_k(n) = \gcd\left(\prod_{j=0}^{k-1} p_0(n+j), \prod_{j=0}^{k-1} p_d(n-d-j)\right). \tag{4.1}$$

Then the sequence $G_1(n), G_2(n), \ldots$ converges to $G_{N+1}(n)$.

Proof. For all $k > N$ we have

$$\gcd(p_0(n), p_d(n-d-k)) = \cdots = \gcd(p_0(n+k), p_d(n-d-k)) = 1. \tag{4.2}$$

Note that

$$G_{k+1}(n) = \gcd\left(\prod_{j=0}^{k} p_0(n+j), \prod_{j=0}^{k} p_d(n-d-j)\right).$$

This implies that

$$G_k(n) = G_{k+1}(n),$$

for all $k > N$. Moreover, one sees that once (4.2) is satisfied for $k > N$, it is also satisfied for $k+1$. It follows that

$$G_k(n) = G_{k+1}(n) = G_{k+2}(n) = \cdots,$$

for all $k > N$, and this completes the proof. $\square$

By using the convergence property of the sequence $G_k(n)$, we can obtain the explicit formula (3.7) for Abramov's universal denominator. Our proof requires only lcm and gcd computations.

Theorem 4.2. *Given a linear difference equation*

$$\sum_{m=0}^{d} p_m(n)y(n+m) = p(n), \tag{4.3}$$

where $p_0(n), p_1(n), \ldots, p_d(n), p(n) \in \mathbb{K}[n]$ *are given polynomials such that* $p_0(n) \neq 0$, $p_d(n) \neq 0$. *Let* $f(n)/g(n)$ *be the reduced rational representation of* $y(n)$, *and* N *be the dispersion of* $p_d(n-d)$ *and* $p_0(n)$. *Then*

$$G(n) = \gcd([p_0(n+N)]^{\underline{N+1}}, [p_d(n-d)]^{\underline{N+1}}),$$

as given by (3.7), *is a universal denominator of rational solutions of* (4.3).

Proof. From (4.3) it follows that

$$\sum_{m=0}^{d} p_m(n)\frac{f(n+m)}{g(n+m)} = p(n). \tag{4.4}$$

Letting

$$l(n) = \operatorname{lcm}(g(n+1), g(n+2), \ldots, g(n+d)), \tag{4.5}$$

and multiplying equation (4.4) by $l(n)g(n)$, we obtain

$$\sum_{m=0}^{d} p_m(n)f(n+m)\frac{l(n)}{g(n+m)}\; g(n) = p(n)l(n)g(n). \tag{4.6}$$

From (4.5), we have the following divisibility conditions:

$$g(n+m) \mid l(n) \quad \text{for } \; m = 1, 2, \ldots, d.$$

Thus $l(n)/g(n+m)$ are polynomials for $m = 1, 2, \ldots, d$. From (4.6) we obtain

$$g(n) \mid p_0(n) \cdot \operatorname{lcm}(g(n+1), g(n+2), \ldots, g(n+d)). \tag{4.7}$$

Similarly, multiplying equation (4.4) by $l(n-1)g(n+d)$ and then substituting $n-d$ for n, we obtain that

$$g(n) \mid p_d(n-d) \cdot \operatorname{lcm}(g(n-1), g(n-2), \ldots, g(n-d)). \tag{4.8}$$

Shifting n by 1 in (4.7) yields

$$g(n+1) \mid p_0(n+1) \cdot \operatorname{lcm}(g(n+2), g(n+3), \ldots, g(n+d+1)). \tag{4.9}$$

Substituting (4.9) into (4.7) we see that $g(n)$ divides

$$p_0(n) \cdot \operatorname{lcm}(p_0(n+1) \cdot \operatorname{lcm}(g(n+2), g(n+3), \ldots, g(n+d+1)), g(n+2), \ldots, g(n+d)).$$

So we can write

$$g(n) \mid p_0(n)p_0(n+1)\cdot \mathrm{lcm}(g(n+2), g(n+3), \ldots, g(n+d+1)).$$

By induction we may derive for $k \geq 1$,

$$g(n)\Big|\prod_{j=0}^{k-1} p_0(n+j)\cdot \mathrm{lcm}(g(n+k), g(n+k+1), \ldots, g(n+k+d-1)).$$

It follows that

$$g(n)\Big|\prod_{j=0}^{k-1} p_0(n+j)\cdot g(n+k)g(n+k+1)\ldots g(n+k+d-1).$$

Since $\mathbb{K}$ has characteristic zero, there is a large enough k such that for any $j \geq k$

$$\gcd(g(n), g(n+j)) = 1.$$

It follows that

$$g(n)\Big|\prod_{j=0}^{k-1} p_0(n+j),$$

for all large enough k. Analogously, from (4.8) we get

$$g(n)\Big|\prod_{j=0}^{k-1} p_d(n-d-j),$$

for all large enough k. Therefore

$$g(n) \mid G_k(n),$$

for all large enough k, where $G_k(n)$ is defined as in (4.1). Setting k to infinity in this equation, by Theorem 4.1 we get

$$g(n) \mid G_{N+1}(n) = \gcd\left(\prod_{j=0}^{N} p_0(n+j), \prod_{j=0}^{N} p_d(n-d-j)\right) = G(n),$$

as desired. □

From equation (4.4) we get

$$\sum_{m=0}^{d} p_m(n)\cdot f(n+m)\cdot \prod_{\substack{j=0\\ j\neq m}}^{d} g(n+j) = p(n)\cdot \prod_{j=0}^{d} g(n+j). \tag{4.10}$$

The next step is simply to set

$$g(n) = G(n),$$

in equation (4.10). If equation (4.10) can be solved for $f \in \mathbb{K}[n]$ then $y(n) = f(n)/g(n)$ is a solution of (4.3); otherwise (4.3) has no rational solutions.

Algorithm 4.3.
INPUT: nonzero polynomials $p_0(n), p_d(n)$.
OUTPUT: a universal denominator $g(n)$ *of* (4.3).

(1) *Compute* $N = \mathrm{dis}(p_d(n-d), p_0(n)) = \max\{k \in \mathbb{N} \mid \deg \gcd(p_d(n-d), p_0(n+k)) \geq 1\}$.

(2) *If* $N \geq 0$ *then compute* $g(n) = G(n)$, *where* $G(n)$ *is defined as in* (3.7), *otherwise* $g(n) = 1$.

Example 4.4. *Find a rational solution of the equation*

$$\begin{aligned}&(n+4)(2n+1)(n+2)y(n+3) - (2n+3)(n+3)(n+1)y(n+2)\\&+n(n+2)(2n-3)y(n+1) - (n-1)(2n-1)(n+1)y(n) = 0.\end{aligned} \tag{4.11}$$

We have $p_0(n) = -(n-1)(2n-1)(n+1)$, $p_d(n) = (n+4)(2n+1)(n+2)$, *then* $N = 2$ *and then* $g(n) = (n-1)(n+1)n$. *By* (4.10), $f(n)$ *is a polynomial which satisfies*

$$\begin{aligned}&n(2n+1)(n+2)(n+1)f(n+3) - n(2n+3)(n+3)(n+1)f(n+2) + n(2n-3)\\&\cdot(n+3)(n+2)f(n+1) - (2n-1)(n+3)(n+1)(n+2)f(n) = 0.\end{aligned}$$

The polynomial $f(n) = Cn(2n-3)$ *is a solution of this equation. Thus*

$$y(n) = \frac{f(n)}{g(n)} = C\frac{(2n+1)}{(n^2-1)}$$

is a rational solution of (4.11).

Acknowledgments. We would like to thank Qing-Hu Hou, Bruno Salvy, Lisa H. Sun and Doron Zeilberger for helpful comments. This work was supported by the 973 Project, the PCSIRT Project of the Ministry of Education, the Ministry of Science and Technology, and the National Science Foundation of China. Peter Paule was partially supported by the SFB grant F1305 of the Austrian Science Foundation FWF.

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