Stanley's Zrank Problem on Skew Partitions

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Abstract. We present an affirmative answer to Stanley's zrank problem, namely, the zrank and rank are equal for any skew partition. We show that certain classes of restricted Cauchy matrices are nonsingular and furthermore, the signs depend on the number of zero entries. Similar to notion of the jrank of a skew partition, we give a characterization of the rank in terms of the Giambelli type matrices of the corresponding skew Schur functions. We also show that the sign of the determinant of a factorial Cauchy matrix is uniquely determined by the number of its zero entries, which implies the nonsingularity of the inverse binomial coefficient matrix.

Keywords: zrank, rank, grank, restricted Cauchy matrix, factorial Cauchy matrix, inverse binomial coefficient matrix.

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1 Introduction

In the study of tensor products of Yangian modules, Nazarov and Tarasov [12] gave a generalization of the rank of an ordinary partition to a skew partition. Stanley [15] obtained several characterizations of the rank of a skew partition in terms of the reduced partition code, the Jacobi-Trudi matrix, and the minimal border strip decomposition. Stanley also introduced the notion of the zrank of a skew partition in terms of the specialization of the skew Schur function, and proposed the problem whether the zrank and the rank are always equal.

Yan, Yang and Zhou [16] gave an equivalent characterization of Stanley's problem in terms of the restricted Cauchy matrix based on two integer sequences. In this paper, we extend the definition of a restricted Cauchy matrix to two sequences of real numbers subject to certain conditions. We prove that every restricted Cauchy matrix is nonsingular, and thus give an affirmative answer to Stanley's problem.

In the spirit of Stanley's notion of the jrank of a skew partition which is defined as the number of rows in the Jacobi-Trudi matrix in which one does not appear, we introduce the notion of *grank* in terms of the Giambelli type matrix defined by Hamel and Goulden for a skew Schur function [9]. Given any outside decomposition of a skew partition, the grank is defined by the number of rows in which there are no entries equal to one. It turns out that the grank is well-defined, namely, it does not depend on the outside decomposition of the skew partition. We show that the grank is always equal to the rank for any skew partition.

This paper is also concerned with the nonsingularity of the factorial Cauchy matrices. Given a sequence A of real numbers and a sequence B of integers, we define the factorial Cauchy matrix to be a matrix with each entry being either the inverse of the falling factorial or zero, similar to the definition of the restricted Cauchy matrix. We prove that the determinant of the factorial Cauchy matrix has the same property as the restricted Cauchy matrix. A special case of the factorial Cauchy matrix falls into the framework of the calculation of some determinants involving the s-shifted factorial by Normand [13] in the study of the probability density of the determinant of random matrices [7, 11].

The double Schur functions serve as a tool for proving the nonsingularity of the factorial Cauchy matrix without zero entries. The double Schur functions are a natural extension of the factorial Schur function introduced by Biedenharn and Louck [3], and further studied by Chen and Louck [4], Goulden and Greene [8], Macdonald [10], Chen, Li and Louck [5].

As a direct application of the nonsingularity of factorial Cauchy matrices, we prove the nonsingularity of inverse binomial coefficient matrices, which are defined as matrices with entries being either zeros or the inverses of the binomial coefficients.

2 The restricted Cauchy matrices

Let $A = (a_1, \ldots, a_n)$ and $B = (b_1, \ldots, b_n)$ be two sequences of real numbers. Suppose that A is strictly decreasing, B is strictly increasing, and $a_i > b_{n+1-i}$ and $a_i \neq b_j$ for any i, j. We define a matrix $\mathbf{C}(A, B) = (c_{ij})_{i,j=1}^n$ by setting

$$c_{ij} = \begin{cases} \dfrac{1}{a_i - b_j}, & \text{if } a_i > b_j, \\ 0, & \text{if } a_i < b_j. \end{cases} \tag{2.1}$$

Similar to the definition in [16], a matrix M is called a *restricted real Cauchy matrix* if there exist two sequences A and B satisfying the above conditions such that $M = \mathbf{C}(A, B)$. If A and B are restricted to integer sequences, we call M a *restricted integer Cauchy matrix*.

Let $\omega(M)$ be the number of zero entries of M. We have the following criterion for the sign of the restricted Cauchy determinant.

Theorem 2.1 *Any restricted real Cauchy matrix $M = \boldsymbol{C}(A, B)$ is nonsingular. Furthermore, the determinant $\det(M)$ is positive if $\omega(M)$ is even; or negative if $\omega(M)$ is odd.*

Before giving the proof of the above theorem, let us recall some definitions on matrices. A matrix $M = (m_{ij})_{i,j=1}^n$ is called *reducible* if the indices $1, 2, ..., n$ can be divided into two disjoint nonempty subsets $\{i_1, i_2, ..., i_s\}$ and $\{j_1, j_2, ..., j_t\}$ (with $s + t = n$) such that

$$m_{i_\alpha, j_\beta} = 0, \text{ for } \alpha = 1, 2, ..., s \text{ and } \beta = 1, 2, ..., t.$$

Otherwise, M is said to be an irreducible matrix. Clearly, a restricted real Cauchy matrix $M = \mathbf{C}(A, B)$ is irreducible if $a_i > b_{n+2-i}$ for $i = 2, 3, \ldots, n$. Given a square matrix $M = (m_{ij})_{i,j=1}^n$, let M_{ij} denote the (i, j)-th *minor* of M which is the matrix obtained from M by deleting the i-th row and the j-th column. We have the following lemma:

Lemma 2.2 *If $M = \boldsymbol{C}(A, B)$ is an irreducible restricted Cauchy matrix, then each minor M_{ij} is a restricted Cauchy matrix.*

Proof. Let

$$\begin{aligned} A' &= (a'_1, \ldots, a'_{n-1}) = (a_1, \ldots, \hat{a}_i, \ldots, a_n) \\ B' &= (b'_1, \ldots, b'_{n-1}) = (b_1, \ldots, \hat{b}_j, \ldots, b_n) \end{aligned}$$

where $\hat{}$ stands for the missing entry from the sequence. It suffices to prove that $M_{ij} = \mathbf{C}(A', B')$. Clearly, each entry of M_{ij} is defined by (2.1). We only need to show that $a'_k > b'_{n-k}$ for any $k : 1 \leq k \leq n - 1$. There are four cases:

(a) If $k < i$ and $n - k < j$, then $a'_k = a_k > b_{n+1-k} > b_{n-k} = b'_{n-k}$ since B is strictly increasing.

(b) If $k < i$ and $n - k \geq j$, then $a'_k = a_k > b_{n+1-k} = b'_{n-k}$.

(c) If $k \geq i$ and $n - k < j$, then $a'_k = a_{k+1} > b_{n-k} = b'_{n-k}$.

(d) If $k \geq i$ and $n - k \geq j$, then $a'_k = a_{k+1} > b_{n-k+1} = b'_{n-k}$ since $M = \mathbf{C}(A, B)$ is irreducible.

This completes the proof. ∎

The *adjoint matrix* of M is defined to be the matrix $((-1)^{i+j}\det(M_{ji}))_{i,j=1}^{n}$, denoted M^*. The *rank* of a matrix M is the maximum number of linearly independent rows or columns of the matrix, denoted $\mathrm{rk}(M)$. For an $n\times n$ square matrix M, we have the following relationship between $\mathrm{rk}(M)$ and $\mathrm{rk}(M^*)$:

$$\mathrm{rk}(M^*) = \begin{cases} n, & \text{if } \mathrm{rk}(M) = n, \\ 1, & \text{if } \mathrm{rk}(M) = n-1, \\ 0, & \text{if } \mathrm{rk}(M) < n-1. \end{cases} \tag{2.2}$$

The restricted Cauchy matrix $M = \mathbf{C}(A,B)$ reduces to the classical Cauchy matrix when $a_n > b_n$. In this case, the determinant $\det(M)$ is given by the following well known formula

$$\det\left(\frac{1}{a_i-b_j}\right)_{i,j=1}^{n} = \prod_{i<j}(a_i-a_j)\prod_{i<j}(b_j-b_i)\prod_{i,j}\frac{1}{a_i-b_j}. \tag{2.3}$$

Since A is strictly decreasing and B is strictly increasing, the above determinant is positive.

We now proceed to give the proof of Theorem 2.1.

Proof of Theorem 2.1. We apply induction on n. The cases of $n=1,2$ are clear. Suppose that Theorem 2.1 holds for matrices of order less than n. We will prove that the theorem is also true for matrices of order n.

If M is a reducible Cauchy matrix, then there exists an integer k greater than or equal to 2 such that $a_k < b_{n+2-k}$. Now M has the following block decomposition

$$\begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix},$$

where M_1 is a $(k-1)\times(n-k+1)$ matrix, M_2 is the $(k-1)\times(k-1)$ restricted Cauchy matrix $\mathbf{C}((a_1,\ldots,a_k),(b_{n-k+1},\ldots,b_n))$, M_3 is the $(n-k+1)\times(n-k+1)$ restricted Cauchy matrix $\mathbf{C}((a_{k+1},\ldots,a_n),(b_1,\ldots,b_{n-k}))$, and M_4 is an $(n-k+1)\times(k-1)$ zero block. Thus we get

$$\det(M) = (-1)^{\omega(M_4)}\det(M_2)\det(M_3).$$

By induction the sign of $\det(M_2)$ is $(-1)^{\omega(M_2)}$, and the sign of $\det(M_3)$ is $(-1)^{\omega(M_3)}$. Since $\omega(M_1)=0$, the sign of $\det(M)$ equals

$$(-1)^{\omega(M_4)+\omega(M_3)+\omega(M_2)} = (-1)^{\omega(M)}.$$

We now suppose that $M = (m_{ij})_{i,j=1}^n$ is an irreducible Cauchy matrix. If M has no zero entry, then the theorem is true because of (2.3). If $\omega(M) > 0$, then we consider the following block decomposition of M

$$\begin{pmatrix} M_1' & M_2' \\ M_3' & 0 \end{pmatrix},$$

where M_1' is an $(n-1)\times(n-1)$ restricted Cauchy matrix, M_2' is an $(n-1)\times 1$ column vector, M_3' is a $1 \times (n-1)$ row vector. By Lemma 2.2, we see that the minors $M_{11}, M_{nn}, M_{1n}, M_{n1}$ are also restricted Cauchy matrices. Consider the submatrix

$$\begin{pmatrix} \det(M_{11}) & (-1)^{n+1}\det(M_{n1}) \\ (-1)^{n+1}\det(M_{1n}) & \det(M_{nn}) \end{pmatrix}$$

of the adjoint matrix M^*. Note that the signs of $\det(M_{11}), \det(M_{n1}), \det(M_{1n})$ and $\det(M_{nn})$ are respectively $(-1)^{\omega(M_1')+\omega(M_2')+\omega(M_3')+1}$, $(-1)^{\omega(M_1')+\omega(M_2')}$, $(-1)^{\omega(M_1')+\omega(M_3')}$ and $(-1)^{\omega(M_1')}$. It follows that

$$\det\begin{pmatrix} \det(M_{11}) & (-1)^{n+1}\det(M_{n1}) \\ (-1)^{n+1}\det(M_{1n}) & \det(M_{nn}) \end{pmatrix} \neq 0.$$

Therefore $\mathrm{rk}(M^*) \geq 2$. Owing to the relation (2.2) between $\mathrm{rk}(M^*)$ and $\mathrm{rk}(M)$, we have $\mathrm{rk}(M^*) = n$, that is, M is nonsingular.

It remains to show that the sign of $\det(M)$ coincides with the number of zero entries in M. Without loss of generality, we may assume that M is irreducible. If M does not contain any zero entry, then $\det(M)$ is clearly positive. We now assume that M contain at least one zero entry. Note that the conditions on A and B, for any row in $\mathbf{C}(A,B)$, if there is a zero in the j-column, then the entry in any column k $(k > j)$ must be zero. The same property also holds for the columns of $\mathbf{C}(A,B)$. Thus, the (n,n)-entry in $\mathbf{C}(A,B)$ must be zero. Since M is irreducible, there exists an integer $j : 2 \leq j \leq n-1$ such that $m_{nj} \neq 0$, but $m_{n,j+1} = m_{n,j+2} = \cdots = m_{n,n} = 0$. Let $\alpha = b_j$ and $\beta = \min(a_{n-1}, b_{j+1})$. Then the determinant $\det(M)$ can be regarded as a continuous function of a_n on the open interval (α,β). Note that when a_n varies in the open interval (α,β), the restricted Cauchy matrix M keeps the same shape, which means that the positions of zero entries are fixed. If $a_n = \eta$ for some $\eta \in (\alpha,\beta)$, denote the corresponding matrix M by M_η. When a_n tends to b_j from above, m_{nj} tends to $+\infty$, and for $k < j$ the entry m_{nk} tends to $\frac{1}{b_j - b_k}$.

Since the minor M_{nj} is a restricted Cauchy matrix of order $n-1$, by Lemma 2.2, the induction hypothesis implies that $\det(M_{nj}) \neq 0$. Therefore, the sign of $\det(M)$ coincides with the sign of $(-1)^{n+j}\det(M_{nj})$ when a_n tends into b_j from above. It follows that there exists $\xi \in (\alpha,\beta)$ such that the sign of $\det(M_\xi)$ coincides with the sign of $(-1)^{n+j}\det(M_{nj})$. By induction, the sign of $\det(M_{nj})$ equals $(-1)^{\omega(M_{nj})}$, thus the sign of $\det(M_\xi)$ equals

$$(-1)^{n+j+\omega(M_{nj})} = (-1)^{(n-j)+\omega(M_{nj})} = (-1)^{\omega(M_\xi)}.$$

For any $\eta \in (\alpha, \beta)$, the sign of $\det(M_\eta)$ coincides with the sign of $\det(M_\xi)$. Otherwise, there exists a number ζ between ξ and η such that $\det(M_\zeta) = 0$, which is a contradiction. Since $\omega(M_\xi) = \omega(M_\eta)$, we have completed the proof. ∎

3 The zrank problem

We assume that the reader is familiar with the notation and terminology on partitions and symmetric functions in [14]. Given a partition λ with decreasing components $\lambda_1, \lambda_2, \ldots$, the rank of λ, denoted $\mathrm{rank}(\lambda)$, is the number of i's such that $\lambda_i \geq i$. Clearly, $\mathrm{rank}(\lambda)$ counts the number of diagonal boxes in the Young diagram of λ, where the Young diagram is an array of squares in the plane justified from the top and left corner with $\ell(\lambda)$ rows and λ_i squares in row i. A square (i, j) in the diagram is the square in row i from the top and column j from the left. The content of (i, j), denoted $\tau(i, j)$, is given by $j - i$.

Given two partitions λ and μ, if for each i we have $\lambda_i \geq \mu_i$, then the skew partition λ/μ is defined to be the diagram obtained from the diagram of λ by removing the diagram of μ at the top-left corner. A border strip is a connected skew partition with no 2×2 squares. Nazarov and Tarasov [12] generalized rank of ordinary partitions to skew partitions in the following way: A square (i, j) is called an *inner corner* of λ/μ, if $(i, j), (i, j-1), (i-1, j) \in \lambda/\mu$ but $(i-1, j-1) \notin \lambda/\mu$; a square (i, j) is called an *outer corner box* of λ/μ, if $(i, j) \in \lambda/\mu$ but $(i-1, j-1), (i, j-1), (i-1, j) \notin \lambda/\mu$; the *inner diagonal* is composed of all the boxes $(i+p, j+p) \in \lambda/\mu$ if (i, j) is an inner corner; the *outer diagonal* is composed of all the boxes $(i+p, j+p) \in \lambda/\mu$ if (i, j) is an outer corner; let d^+ be the number of boxes on all outer diagonals, and let d^- be the number of boxes on all inner diagonals; then the *rank* of λ/μ, denoted $\mathrm{rank}(\lambda/\mu)$, is the difference $d^+ - d^-$. For example, $\mathrm{rank}((6, 5, 5, 3)/(2, 1, 1)) = 3$, as illustrated in Figure 1.

Stanley [15] gave several characterizations of $\mathrm{rank}(\lambda/\mu)$. The first characterization is based on the border strip decomposition of the skew diagram. Stanley proved that $\mathrm{rank}(\lambda/\mu)$ is the smallest number r such that λ/μ is a disjoint union of r border strips. As we see from Figure 2, $\mathrm{rank}((5, 4, 3, 2)/(2, 1, 1)) = 3$.

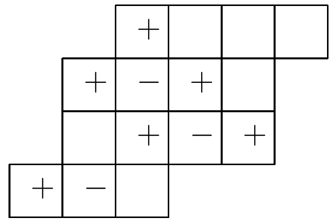

Figure 1: Outside and inside diagonals of $(6,5,5,3)/(2,1,1)$

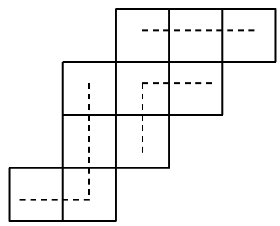

Figure 2: A minimal border strip decomposition of $(5,4,3,2)/(2,1,1)$

Recall that the Jacobi-Trudi identity for the skew Schur function states that

$$s_{\lambda/\mu} = \det \left(h_{\lambda_i - \mu_j - i + j} \right)_{i,j=1}^{\ell(\lambda)}, \tag{3.4}$$

where h_k denotes the k-th complete symmetric function, $h_0 = 1$ and $h_k = 0$ for $k < 0$. Let $J_{\lambda/\mu}$ be the matrix which appears in the Jacobi-Trudi identity. Stanley defined the jrank of λ/μ, denoted $\mathrm{jrank}(\lambda/\mu)$, by the number of rows of $J_{\lambda/\mu}$ which do not contain 1's, and proved that $\mathrm{jrank}(\lambda/\mu) = \mathrm{rank}(\lambda/\mu)$. For example,

$$J_{(6,5,5,3)/(2,1,1)} = \begin{pmatrix} h_4 & h_6 & h_7 & h_9 \\ h_2 & h_4 & h_5 & h_7 \\ h_1 & h_3 & h_4 & h_6 \\ 0 & 1 & h_1 & h_3 \end{pmatrix}.$$

The third characterization of $\mathrm{rank}(\lambda/\mu)$ involves the reduced code of λ/μ, denoted $\mathrm{c}(\lambda/\mu)$. The reduced code $\mathrm{c}(\lambda/\mu)$ is also known as the *partition sequence* of λ/μ [1, 2]. Consider the two boundary lattice paths of the diagram of λ/μ with steps $(0,1)$ or $(1,0)$ from the bottom-leftmost point to the top-rightmost point. Replacing each step $(0,1)$ by 1 and each step $(1,0)$ by 0, we obtain two binary sequences by reading the lattice paths from the bottom-left corner to the top-right corner. Denote the top-left binary sequence by $g_1, g_2, \ldots, g_k$, and the bottom-right binary sequence by $g'_1, g'_2, \ldots, g'_k$. The reduced code c(λ/μ) is defined by the two-row array

$$\begin{array}{cccc} g_1 & g_2 & \cdots & g_k \\ g'_1 & g'_2 & \cdots & g'_k \end{array}.$$

For example, the reduced code of the skew partition $(5,4,3,2)/(2,1,1)$ in Figure 3 is

$$\begin{array}{ccccccccc} 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{array}.$$

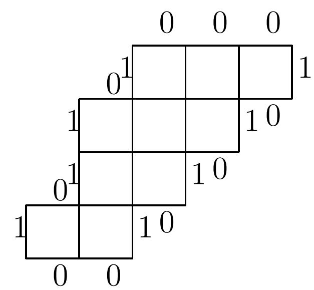

Figure 3: The reduced code of $(5,4,3,2)/(2,1,1)$

Stanley proved that the rank of a skew partition is also equal to the number of columns $\substack{1\\0}$ of c(λ/μ), as we see from Figure 3.

In [15] Stanley introduced the notion of *zrank* of a skew partition, and proposed the problem whether the zrank is always equal to the rank for any skew partition. Let $s_{\lambda/\mu}(1^t)$ denote the skew Schur function $s_{\lambda/\mu}$ evaluated at $x_1 = \cdots = x_t = 1$, $x_i = 0$ for $i > t$. The zrank of λ/μ, denoted zrank(λ/μ), is the exponent of the largest power of t dividing $s_{\lambda/\mu}(1^t)$.

The following equivalence was established by Yan, Yang and Zhou [16]:

Theorem 3.1 *The following two statements are equivalent:*

(i) The zrank and rank are equal for any skew partition.

(ii) Any restricted integer Cauchy matrix is nonsingular.

As an immediate consequence of Theorem 2.1 and Theorem 3.1, we have the following conclusion.

Theorem 3.2 *For all skew partitions λ/μ, we have* $\mathrm{zrank}(\lambda/\mu) = \mathrm{rank}(\lambda/\mu)$.

The above theorem allows us to give other characterizations of rank in terms of the Giambelli type matrix, which is related to the planar decomposition of skew diagrams introduced by Hamel and Goulden [9].

Let us first recall the Giambelli type determinant formulas of the skew Schur function. A border strip decomposition of λ/μ is said to be an *outside decomposition* if every strip in the decomposition has an initial square on the left or bottom perimeter of the diagram and a terminal square on the right or top perimeter. For an outside decomposition, Chen, Yan and Yang [6] introduced the notion of the cutting strip and obtained a transformation theorem on the Giambelli-type determinantal formulas for the skew Schur function.

Suppose that λ/μ has k diagonals, each of which is composed of the squares having the same content. The cutting strip of an outside decomposition is defined to be a border strip of length k. Given an outside decomposition of λ/μ, we see that each square in the diagram can be assigned a direction in the following way: starting with the bottom-left corner of a strip, we say that a square of a strip has the up direction (resp. right direction) if the next square in the strip lies on its top (resp. to its right). Then the squares on the same diagonal of λ/μ have the same direction. Based on this property, the cutting strip ϕ of an outside decomposition $\mathbf{D}$ of λ/μ is defined as follows: for $i = 1, 2, \ldots, k-1$ the i-th square in ϕ keeps the same direction as the i-th diagonal of λ/μ with respect to $\mathbf{D}$. Given a border strip θ of $\mathbf{D}$, let $p(\theta)$ denote the lower left-hand square of θ, and let $q(\theta)$ denote the upper right-hand square. Hamel and Goulden [9] derived the following determinantal formula.

Theorem 3.3 ([9, Theorem 3.1]) *For an outside decomposition* $\mathbf{D}$ *with* k *border strips* $\theta_1, \theta_2, \ldots, \theta_k$*, we have*

$$s_{\lambda/\mu} = \det\left(s_{[\tau(p(\theta_i)),\tau(q(\theta_j))]}\right)_{i,j=1}^{k}, \tag{3.5}$$

where for any two integers α, β*, a strip* $[\alpha, \beta]$ *is defined by the following rule: if* $\alpha \leq \beta$*, then let* $[\alpha, \beta]$ *be the segment of* ϕ *from the square with content* α *to the square with content* β*; if* $\alpha = \beta + 1$*, then let* $[\alpha, \beta]$ *be the empty strip and* $s_{[\alpha,\beta]} = 1$*; if* $\alpha > \beta + 1$*, then* $[\alpha, \beta]$ *is undefined and* $s_{[\alpha,\beta]} = 0$*. The content function* τ *is defined on the original skew diagram.*

Denote the matrix appearing in (3.5) by $G^{\mathbf{D}}_{\lambda/\mu}$. Given an outside decomposition $\mathbf{D}$ of λ/μ, let $\mathrm{grank}_{\mathbf{D}}(\lambda/\mu)$ be the number of rows in $G^{\mathbf{D}}_{\lambda/\mu}$ that do not contain 1's. Then we have the following theorem:

Theorem 3.4 *For any skew partition λ/μ and any outside decomposition* $\mathbf{D}$ *of λ/μ, we have* $\mathrm{grank}_{\mathbf{D}}(\lambda/\mu) = \mathrm{rank}(\lambda/\mu)$.

Proof. By Theorem 2.1, we have $\mathrm{rank}(\lambda/\mu) = \mathrm{zrank}(\lambda/\mu)$. So it suffices to prove $\mathrm{grank}_{\mathbf{D}}(\lambda/\mu) = \mathrm{zrank}(\lambda/\mu)$. According to the definition of $\mathrm{zrank}(\lambda/\mu)$, we need to consider the terms with the lowest degree in the expansion of $\det(G^{\mathbf{D}}_{\lambda/\mu}(1^t))$. Suppose that the square with content $\tau(p(\theta_i))$ lies in the p_i-th row of the cutting strip ϕ of $\mathbf{D}$, and the square with content $\tau(q(\theta_j))$ lies in the q_j-th row. For the nonempty border strip $[\tau(p(\theta_i)), \tau(q(\theta_j))]$, it is easy to show that

$$(t^{-1} s_{[\tau(p(\theta_i)),\tau(q(\theta_j))]}(1^t))_{t=0} = \frac{(-1)^{p_i - q_j}}{\tau(q(\theta_j)) + 1 - \tau(p(\theta_i))}. \tag{3.6}$$

Note that for any $i \neq j$ we have

$$\tau(q(\theta_i)) \neq \tau(q(\theta_j)), \quad \tau(p(\theta_i)) \neq \tau(p(\theta_j))$$

subject to the definition of the outside decomposition $\mathbf{D}$. By removing the rows and columns with 1's from $G^{\mathbf{D}}_{\lambda/\mu}$, extracting t from each row without 1's, and putting $t = 0$, we obtain a restricted Cauchy matrices up to permutations of rows and columns. From Theorem 2.1, we get the desired equality $\mathrm{grank}_{\mathbf{D}}(\lambda/\mu) = \mathrm{zrank}(\lambda/\mu)$. ∎

In fact, the above theorem can be proved in another way. Given a border strip decomposition $\mathbf{D} = \{\theta_1, \theta_2, \ldots, \theta_m\}$ of λ/μ, let

$$P_{\mathbf{D}} = \{\tau(p(\theta_1)), \tau(p(\theta_2)), \ldots, \tau(p(\theta_m))\}$$

and

$$Q_{\mathbf{D}} = \{\tau(q(\theta_1)) + 1, \tau(q(\theta_2)) + 1, \ldots, \tau(q(\theta_m)) + 1\}.$$

The following theorem is implicit in [16]:

Theorem 3.5 *For any border strip decomposition* $\mathbf{D}$*, the two sets* $P_{\mathbf{D}} - Q_{\mathbf{D}}$ *and* $Q_{\mathbf{D}} - P_{\mathbf{D}}$ *are independent of the border strip decomposition* $\mathbf{D}$*, hence uniquely determined by the skew shape λ/μ.*

Remark. We omit the proof here, since it is similar to the proof of [16, Proposition 3.1]. The border strip decomposition $\mathbf{D}$ may not be an outside decomposition. As shown by Yan, Yang and Zhou [16], these two sets are related to the noncrossing interval sets of a given skew partition. If $\mathbf{D}$ is a minimal border strip decomposition, the intersection $P_{\mathbf{D}} \cap Q_{\mathbf{D}}$ is the empty set. Otherwise, the cardinality of the intersection $P_{\mathbf{D}} \cap Q_{\mathbf{D}}$ is equal to the number of rows containing 1's in $G^{\mathbf{D}}_{\lambda/\mu}$. From this point of view, Theorem 3.5 is more general than Theorem 3.4.

4 The factorial Cauchy matrices

Before defining factorial Cauchy matrices and inverse binomial matrices, let us review some background on double Schur functions. Let $X = \{x_1, x_2, \ldots\}$ and $Y = \{y_1, y_2, \ldots\}$ be two sets of variables. For a positive integer k, we set

$$(x_i|Y)_k = \prod_{1\leq j\leq k} (x_i - y_j), \tag{4.7}$$

and define $(x_i|Y)_0 = 1$. Taking $y_i = i-1$, we obtain the falling factorial $(x_i)_k = x_i(x_i-1)\cdots(x_i-k+1)$. Taking $y_i = 1-i$, we get the rising factorial $(x_i)^k = x_i(x_i+1)\cdots(x_i+k-1)$.

Now we give the two equivalent definitions of the double Schur function $S_\lambda(X,Y)$. The first definition of $S_\lambda(X,Y)$ is a determinantal form. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$. We have

$$S_\lambda(X,Y) = \frac{\det\left((x_i|Y)_{\lambda_j+n-j}\right)_{i,j=1}^{n}}{\Delta(X)}, \tag{4.8}$$

where $\Delta(X)$ is the Vandermonde determinant in $x_1, x_2, \ldots, x_n$:

$$\Delta(X) = \prod_{i<j}(x_i - x_j).$$

The second definition of $S_\lambda(X,Y)$ is obtained by Macdonald [10], and Goulden and Green [8]. Chen, Li and Louck [5] obtained this combinatorial definition using the lattice path methodology. Recall that a *semistandard Young tableau* T of shape λ is a configuration of the Young diagram of λ with positive integers such that each row is weakly increasing and each column is strictly increasing. Given a Young tableau T and a cell α of T, let $T(\alpha)$ be the number filled in the cell α. The combinatorial definition of $S_\lambda(X,Y)$ is as follows.

Theorem 4.1 *Let λ be a partition of length n. Then*

$$S_\lambda(X,Y) = \sum_T \prod_{\alpha\in T} (x_{T(\alpha)} - y_{T(\alpha)+\tau(\alpha)}),$$

summing over all column strict tableaux T on $\{1,2,\ldots,n\}$ of shape λ.

We now define the factorial Cauchy matrix and the inverse binomial matrix. Let $A = (a_1, \ldots, a_n)$ be a strictly decreasing sequence of real numbers, and let $B = (b_1, \ldots, b_n)$ be a strictly increasing sequence of positive integers. Suppose that for any i, j we have $a_i > b_{n+1-i} - 1$ and $a_i \neq b_j - 1$. Then we define a matrix $\mathbf{F}(A, B) = (f_{ij})_{i,j=1}^n$ by setting

$$f_{ij} = \begin{cases} \dfrac{1}{(a_i)_{b_j}}, & \text{if } a_i > b_j - 1, \\ 0, & \text{if } a_i < b_j - 1. \end{cases}$$

A matrix M is called a *factorial Cauchy matrix* if there exist two sequences A and B satisfying the above conditions such that $M = \mathbf{F}(A, B)$.

When A is also a sequence of positive integers, we can define a matrix $\mathbf{R}(A, B) = (r_{ij})_{i,j=1}^n$ by setting

$$r_{ij} = \begin{cases} \dbinom{a_i}{b_j}^{-1}, & \text{if } a_i \geq b_j, \\ 0, & \text{if } a_i < b_j. \end{cases}$$

A matrix M is called an *inverse binomial matrix* if there exist two sequences A and B satisfying the above conditions such that $M = \mathbf{R}(A, B)$.

Suppose that the factorial Cauchy matrix $\mathbf{F}(A, B)$ has no zero entries, i.e., the two sequences A and B satisfy that $a_i > b_j - 1$ for any i, j. In this case, we have the following lemma:

Lemma 4.2 *Let $\boldsymbol{F}(A, B)$ be the factorial Cauchy matrix with $a_i > b_j - 1$ for any i, j. Then*

$$\det(\boldsymbol{F}(A, B)) = \frac{\Delta(X) S_\lambda(X, Y)}{\Pi_{k=1}^n (a_k)_{b_n}}, \tag{4.9}$$

where $\lambda_j = b_n - b_j + j - n$, $x_i = a_i - b_n + 1$, and $y_j = -j + 1$. In particular, we have $\det(\boldsymbol{F}(A, B)) > 0$.

Proof. Since $a_i > b_j$ for any i, j, then we have

$$\mathbf{F}(A, B) = \left(\frac{1}{(a_i)_{b_j}} \right)_{i,j=1}^n .$$

Thus

$$\begin{aligned} \det(\mathbf{F}(A, B)) &= \det \left(\frac{1}{(a_i)_{b_j}} \right)_{i,j=1}^n \\ &= \frac{\det \left((a_i - b_n + 1)^{b_n - b_j} \right)_{i,j=1}^n}{\Pi_{k=1}^n (a_k)_{b_n}} \\ &= \frac{\Delta(X) S_\lambda(X, Y)}{\Pi_{k=1}^n (a_k)_{b_n}}, \end{aligned}$$

where the last equality follows from the algebraic definition (4.8) of $S_\lambda(X, Y)$.

Applying Theorem 4.1, we have $\det(\mathbf{F}(A, B)) > 0$. ∎

The following theorem is the main result of this section.

Theorem 4.3 *Any factorial Cauchy matrix $M = \boldsymbol{F}(A, B)$ is nonsingular. Furthermore, the determinant $\det(M)$ is positive if $\omega(M)$ is even; or negative if $\omega(M)$ is odd.*

Proof. We use induction on n. Clearly, the theorem holds when $n = 1$ or 2. Suppose that Theorem 4.3 holds for matrices of order less than n. We proceed to prove that it is also true for matrices of order n.

If $\mathbf{F}(A, B)$ is a reducible factorial Cauchy matrix, then there exists an integer k greater than or equal to 2 such that $a_k < b_{n+2-k} - 1$. Now $\mathbf{F}(A, B)$ has the following block decomposition

$$\begin{pmatrix} F_1 & F_2 \\ F_3 & F_4 \end{pmatrix},$$

where F_1 is a $(k-1) \times (n-k+1)$ matrix, F_2 is a $(k-1) \times (k-1)$ factorial Cauchy matrix, F_3 is an $(n-k+1) \times (n-k+1)$ factorial Cauchy matrix, and F_4 is an $(n-k+1) \times (k-1)$ zero block. So we have

$$\det(\mathbf{F}(A, B)) = (-1)^{\omega(F_4)} \det(F_2) \det(F_3).$$

By the assumption, the sign of $\det(F_2)$ is $(-1)^{\omega(F_2)}$, and the sign of $\det(F_3)$ is $(-1)^{\omega(F_3)}$. Since $\omega(F_1) = 0$, the sign of $\det(\mathbf{F}(A, B))$ equals

$$(-1)^{\omega(F_4)+\omega(F_3)+\omega(F_2)} = (-1)^{\omega(\mathbf{F}(A,B))}.$$

Now suppose that $\mathbf{F}(A, B) = (f_{ij})_{i,j=1}^{n}$ is an irreducible factorial Cauchy matrix. If $\mathbf{F}(A, B)$ has no zero entry, then the theorem is true according to Lemma 4.2. If $\omega(\mathbf{F}(A, B)) > 0$, we consider the following block decomposition of M

$$\begin{pmatrix} F_1' & F_2' \\ F_3' & 0 \end{pmatrix},$$

where F_1' is an $(n-1) \times (n-1)$ factorial Cauchy matrix, F_2' is an $(n-1) \times 1$ column vector, F_3' is an $1 \times (n-1)$ row vector. It is easy to see that the minors $M_{11}, M_{nn}, M_{1n}, M_{n1}$ of $M = \mathbf{F}(A, B)$ are also factorial Cauchy matrices. Consider the submatrix

$$\begin{pmatrix} \det(M_{11}) & (-1)^{n+1} \det(M_{n1}) \\ (-1)^{n+1} \det(M_{1n}) & \det(M_{nn}) \end{pmatrix}$$

of the adjoint matrix M^*. Note that the signs of $\det(M_{11}), \det(M_{n1}), \det(M_{1n})$ and $\det(M_{nn})$ are given by $(-1)^{\omega(F_1')+\omega(F_2')+\omega(F_3')+1}$, $(-1)^{\omega(F_1')+\omega(F_2')}$, $(-1)^{\omega(F_1')+\omega(F_3')}$ and $(-1)^{\omega(F_1')}$. Thus $\det(M_{11})/((-1)^{n+1}\det(M_{1n}))$ and $(-1)^{n+1}\det(M_{n1})/\det(M_{nn})$ must have different signs. Therefore, we obtain $\mathrm{rk}(M^*) \geq 2$. By the relation (2.2), we have $\mathrm{rk}(M^*) = n$, that is, M is nonsingular.

It remains to show that the sign of $\det(M)$ coincides with the number of zero entries in M. Without loss of generality, we may assume that M is irreducible. If M does not contain any zero entry, then $\det(M)$ is clearly positive. We now assume that $M = \mathbf{F}(A,B)$ contain at least one zero entry. Note that the matrix $\mathbf{F}(A,B)$ has the same distribution of zeros as the restricted Cauchy matrix. Thus, the (n,n)-entry in $\mathbf{F}(A,B)$ must be zero. Since M is irreducible, there exists an integer $j : 2 \leq j \leq n-1$ such that $m_{nj} \neq 0$, but $m_{n,j+1} = m_{n,j+2} = \cdots = m_{n,n} = 0$. Let $\alpha = b_j - 1$ and $\beta = \min(a_{n-1}, b_{j+1} - 1)$. Then the determinant $\det(M)$ can be regarded as a continuous function of a_n on the open interval (α, β). Note that when a_n varies in the open interval (α,β), the factorial Cauchy matrix M keeps the same shape. If $a_n = \eta$ for some $\eta \in (\alpha,\beta)$, denote the corresponding factorial Cauchy matrix M by M_η. When a_n tends to $\alpha = b_j - 1$ from above, m_{nj} tends to $+\infty$, and for $k < j$ the entry m_{nk} tends to $\frac{1}{(b_j-1)_{b_k}}$.

Since the minor M_{nj} is a factorial Cauchy matrix of order $n-1$, the induction hypothesis implies that $\det(M_{nj}) \neq 0$. Therefore, the sign of $\det(M)$ coincides with the sign of $(-1)^{n+j}\det(M_{nj})$ when a_n tends to $b_j - 1$ from above. It follows that there exists $\xi \in (\alpha,\beta)$ such that the sign of $\det(M_\xi)$ coincides with the sign of $(-1)^{n+j}\det(M_{nj})$. By induction, the sign of $\det(M_{nj})$ equals $(-1)^{\omega(M_{nj})}$, thus the sign of $\det(M_\xi)$ equals

$$(-1)^{n+j+\omega(M_{nj})} = (-1)^{(n-j)+\omega(M_{nj})} = (-1)^{\omega(M_\xi)}.$$

For any $\eta \in (\alpha,\beta)$, the sign of $\det(M_\eta)$ coincides with the sign of $\det(M_\xi)$. Otherwise, there exists a number ζ between ξ and η such that $\det(M_\zeta) = 0$, which is a contradiction. Since $\omega(M_\xi) = \omega(M_\eta)$, we have completed the proof. ∎

Corollary 4.4 *Any inverse binomial matrix $M = \boldsymbol{R}(A,B)$ is nonsingular. Furthermore, the determinant $\det(M)$ is positive if $\omega(M)$ is even; or negative if $\omega(M)$ is odd.*

Proof. Note that

$$\det(\mathbf{R}(A,B)) = \det(r_{ij})_{i,j=1}^n = \det(\mathbf{F}(A,B)) \prod_{i=1}^n b_j!.$$

Applying Theorem 4.3, we get the desired assertion. ∎

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