Weighted Forms of Euler's Theorem

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Abstract. In answer to a question of Andrews about finding combinatorial proofs of two identities in Ramanujan's "lost" notebook, we obtain weighted forms of Euler's theorem on partitions with odd parts and distinct parts. This work is inspired by the insight of Andrews on the connection between Ramanujan's identities and Euler's theorem. Our combinatorial formulations of Ramanujan's identities rely on the notion of rooted partitions. Pak's iterated Dyson's map and Sylvester's fish-hook bijection are the main ingredients in the weighted forms of Euler's theorem.

Keywords: partition, rooted partition, Euler's theorem, Ramanujan's identities, Pak's iterated Dyson's map, Sylvester's fish-hook bijection

AMS Classifications: 05A17, 11P81

1 Introduction

This paper is concerned with the combinatorial treatments of the following two identities from Ramanujan's "lost" notebook:

$$\sum_{n=0}^{\infty}[(-q;q)_\infty-(-q;q)_n]=(-q;q)_\infty\left[-\frac{1}{2}+\sum_{d=1}^{\infty}\frac{q^d}{1-q^d}\right]+\frac{1}{2}\left[1+\sum_{n=1}^{\infty}\frac{q^{\binom{n+1}{2}}}{(-q;q)_n}\right], \quad (1.1)$$

$$\sum_{n=0}^{\infty}\left[\frac{1}{(q;q^2)_\infty}-\frac{1}{(q;q^2)_n}\right]=(-q;q)_\infty\left[-\frac{1}{2}+\sum_{d=1}^{\infty}\frac{q^{2d}}{1-q^{2d}}\right]+\frac{1}{2}\left[1+\sum_{n=1}^{\infty}\frac{q^{\binom{n+1}{2}}}{(-q;q)_n}\right], \quad (1.2)$$

where the q-shifted factorial is defined by $(x;q)_0=1$ and for $n\geq 1$,

$$(x;q)_n=(1-x)(1-qx)\cdots(1-q^{n-1}x).$$

Andrews [4] has obtained algebraic proofs of the above identities. Furthermore he asked "Can a 'near bijection' be provided between the weighted counts of partitions given by the left sides of (1.1) and (1.2) and the convolution of partition functions generated by the first summation of the right sides of (1.1) and (1.2)?" Andrews also gave an insightful remark that these two identities may be seen as closely related to Euler's result although not strictly generalizations of it, and pointed out the combinatorial possibilities of studying weighted counts of partitions related to these two identities. Our work is indeed inspired by the ideas of Andrews.

Recently, Andrews, Jiménez-Urroz and Ono proved several identities related to the Dedekind eta-function in [5], including the above two identities. Chapman [10] found a combinatorial formulation of (1.1). But he did not give a combinatorial correspondence and remarked that it would be interesting to find one. In fact, the left hand sides and the second sums on the right sides of (1.1) and (1.2) are easily seen to be weighted sums over partitions into distinct parts or odd parts. The difficult parts are the first sums on the right hand sides of (1.1) and (1.2). In order to give combinatorial interpretations of the these two terms, we introduce the notion of rooted partitions and obtain generating functions for rooted partitions as well as identities on rooted partitions. Thus these two terms can be expressed as a weighted sum over partitions into odd parts minus a weighted sum over partitions into distinct parts.

Let us recall some common notation and terminology on partitions as used in [1, Chapter 1]. *A partition* λ of a positive integer n is a finite nonincreasing sequence of positive integers $(\lambda_1, \lambda_2, \ldots, \lambda_r)$ such that $\sum_{i=1}^{r} \lambda_i = n$, where the λ_i are called the parts of λ. The number of parts of λ is called the length of λ, denoted by $l(\lambda)$. The weight of λ is the sum of parts, denoted $|\lambda|$. *The rank of a partition* λ introduced by Dyson [12] is defined as the largest part minus the number of parts, which is usually denoted by $r(\lambda)$. As a convention, we shall assume that the empty partition has rank zero. Let D denote the set of all partitions into distinct parts and O denote the set of all partitions into odd parts.

We are now ready to present the combinatorial interpretations of the two terms occurring in (1.1) and (1.2).

Lemma 1.1 *The following relation holds*

$$(-q;q)_\infty \sum_{d=1}^{\infty} \frac{q^d}{1-q^d} = \sum_{\lambda \in O} 2l(\lambda) q^{|\lambda|} - \sum_{\mu \in D} l(\mu) q^{|\mu|}. \tag{1.3}$$

Lemma 1.2 *The following relation holds*

$$(-q;q)_\infty \sum_{d=1}^{\infty} \frac{q^{2d}}{1-q^{2d}} = \sum_{\lambda \in O} l(\lambda) q^{|\lambda|} - \sum_{\mu \in D} l(\mu) q^{|\mu|}. \tag{1.4}$$

Based on the above two lemmas, we may reformulate Ramanujan's identities (1.1) and (1.2) by the following two weighted forms of Euler's theorem, just as anticipated by Andrews [4].

Theorem 1.3 *We have*

$$\sum_{\mu \in D} \left(l(\mu) + \mu_1 + \frac{1-(-1)^{r(\mu)}}{2} \right) q^{|\mu|} = \sum_{\lambda \in O} 2l(\lambda) q^{|\lambda|}. \tag{1.5}$$

Theorem 1.4 *We have*

$$\sum_{\mu\in D}\left(l(\mu)+\frac{1-(-1)^{r(\mu)}}{2}\right)q^{|\mu|}=\sum_{\lambda\in O}\left(l(\lambda)-\frac{\lambda_1-1}{2}\right)q^{|\lambda|}, \tag{1.6}$$

It appears that none of the existing bijective proofs of Euler's theorem can establish the above two weighted forms. Luckily, they can be deduced from weighted forms (2.7), (2.8) and (2.9) of Euler's theorem coming from Sylvester's fish-hook bijection and Pak's iterated Dyson's map respectively. To be specific, Theorem 1.3 follows from Lemma 2.1 and Lemma 2.3, and Theorem 1.4 follows from Lemmas 2.2 2.3 and Euler's theorem. Therefore, we have reached our goal to give the combinatorial treatments of Ramanujan's identities (1.1) and (1.2).

This paper is organized as follows. We give a brief review of Sylvester's fish-hook bijection and Pak's iterated Dyson's map in Section 2, and give the proofs of the weighted forms (1.5) and (1.6) of Euler's theorem. In Section 3, we introduce the notion of rooted partitions and obtain generating functions for rooted partitions as well as identities on rooted partitions. In Section 4, we establish the connections between weighted forms (1.5) and (1.6) of Euler's theorem and Ramanujan's identities (1.1) and (1.2) via identities on rooted partitions.

2 Weighted Forms of Euler's Theorem

In this section, we give the proofs of weighted forms (1.5) and (1.6) of Euler's theorem from Sylvester's fish-hook bijection and Pak's iterated Dyson's map. Euler's theorem states that the number of partitions of n into distinct parts equals to the number of partitions of n into odd parts for $n\geq 1$ which follows from the following generating function identity:

$$\sum_{\mu\in D}q^{|\mu|}=(-q;q)_\infty=\frac{1}{(q;q^2)_\infty}=\sum_{\lambda\in O}q^{|\lambda|}.$$

Sylvester's fish-hook bijection [20], also referred to as Sylvester's bijection, and Pak's iterated Dyson's map [19] are two correspondences between D and O. As we will see, they are the basic ingredients in the proofs of the weighted forms of Euler's theorem.

There are several ways to describe Sylvester's bijection [17, p. 13, 249] [9, p. 44–45] [3, 8, 16, 18]. Here we give a description by using 2-modular diagrams as given by Bessenrodt [8].

Sylvester's bijection φ: Given a partition λ of n with odd parts, represent each part $2m+1$ by a row of m 2's and a 1 at the end. This diagram is called the 2-modular diagram of λ. Decompose the 2-modular diagram into hooks $H_1, H_2, \ldots$ with the diagonal boxes as corners. Let μ_1 be the number of squares in H_1, let μ_2 be the number of 2's in H_1, let μ_3 be the number of squares in H_2, let μ_4 be the number of 2's in H_2, and so on. Set $\varphi(\lambda) = \mu = (\mu_1, \mu_2, \mu_3, \ldots)$. Then $\varphi(\lambda)$ is clearly a partition with distinct parts, see Figure 1.

The inverse map φ^{-1}: Let $\mu = (\mu_1, \mu_2, \ldots, \mu_{2k-1}, \mu_{2k})$ be a partition of n into distinct parts, where $\mu_i > 0$ for $1 \leq i \leq 2k-1$ and $\mu_{2k} \geq 0$. First we consider the part μ_{2k}, and write down μ_{2k} 2's in a row and add a 1 to the end, then add $(\mu_{2k-1} - \mu_{2k} - 1)$ 1's to the first column. Let us denote this hook by H_k. Note that the 2's can only appear in the first row in this hook. Let us continue to consider the parts μ_{2k-3}, μ_{2k-2}. The hook H_{k-1} is constructed as follows. There will be μ_{2k-2} 2's and $\mu_{2k-3} - \mu_{2k-2}$ 1's in H_{k-1}. If there is a 1 in the i-th row of H_k, then there must be a 2 on the left of the 1 in H_k. The rest of the 2's will have to be put in the first row of H_{k-1}. Then the 1's are easily dispatched in the first row and the first column. Now we may repeat the above procedure to construct a partition with odd parts.

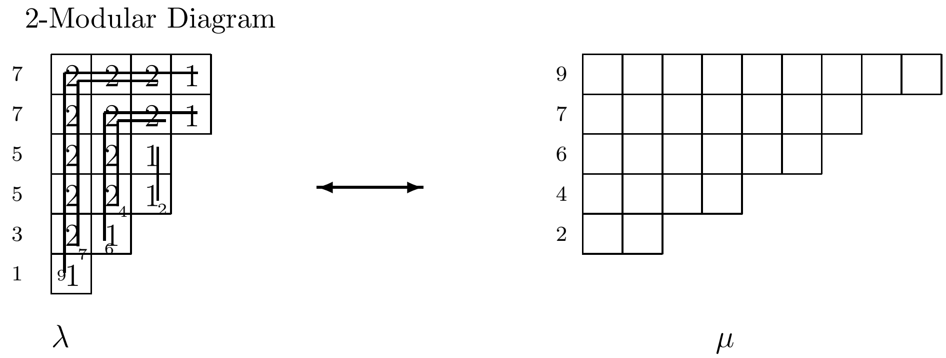

Figure 1: Sylvester's bijection $\varphi:\ (7, 7, 5, 5, 3, 1) \mapsto (9, 7, 6, 4, 2)$.

We now give a brief description of the bijection due to Pak [19], which we call Pak's iterated Dyson's map. This correspondence leads to a combinatorial proof of a partition theorem of Fine in [14] (see also [15, p. 47, (24.6)]). Andrews gives an inductive proof in [2].

We review Dyson's map [13], sometimes called Dyson's adjoint [7]. Denote by $H_{n,r}$ and $G_{n,r}$ the sets of partitions of n with rank at most r and at least r, respectively. Dyson's map ψ_r is a bijection between $H_{n,r+1}$ and $G_{n+r,r-1}$.

Dyson's map ψ_r: Start with a Young diagram corresponding to a partition $\lambda \in H_{n,r+1}$. Note that λ has $l = l(\lambda)$ parts, where $l(\lambda)$ is the length. Remove the first column, add $l + r$ squares to the top row to obtain a Young diagram, it follows that the resulting Young diagram is a partition $\mu \in G_{n+r,r-1}$. It is easy to see that the above procedure is reversible. Hence, Dyson's map ψ_r is a bijection. An example is illustrated in Figure 2.

We are ready to describe Pak's iterated Dyson's map ϕ: $O \mapsto D$.

Pak's iterated Dyson's map ϕ: Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ be a partition of n into odd parts. We construct a partition μ of n from λ by the following process. Let $\nu^{\,l} = (\lambda_l)$ and let $\nu^{\,i}$ denote the partition obtained by applying Dyson's map ψ_{λ_i} to $\nu^{\,i+1}$, i.e.

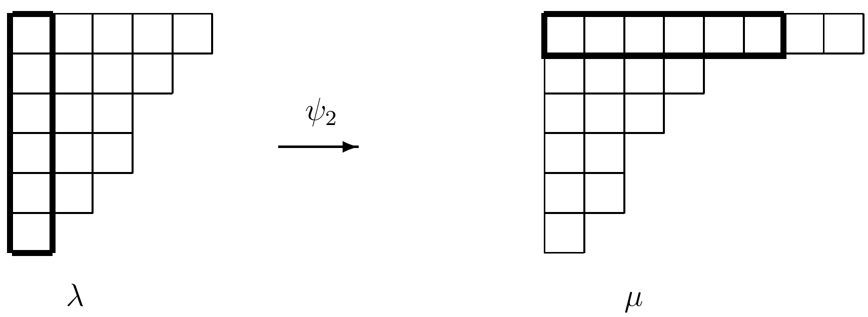

Figure 2: $\lambda = (5,\,4,\,3,\,3,\,2,\,1)$ and $\mu = (8,\,4,\,3,\,2,\,2,\,1)$.

$\nu^{\,i} = \psi_{\lambda_i}(\nu^{\,i+1})$. Finally, set $\mu = \nu^{\,1}$. Since $\nu^{\,i} = \lambda_i + \lambda_{i+1} + \cdots + \lambda_l$, one sees that $|\mu| = |\lambda|$. Furthermore μ is a partition into distinct parts and Pak's iterated Dyson's map ϕ is a bijection [19].

The inverse of Pak's map is described as a recursive procedure. Let $\mu = (\mu_1,\,\mu_2,\ldots,\,\mu_l)$ be a partition of n into distinct parts. Set $\lambda_1 = r(\mu) = \mu_1 - l(\mu)$ if $r(\mu)$ is odd; otherwise set $\lambda_1 = r(\mu)+1 = \mu_1 - l(\mu) + 1$. Applying the inverse of Dyson's map $\psi^{-1}_{\lambda_1}$ to μ, we get a partition $\nu^{\,2} = \psi^{-1}_{\lambda_1}(\mu)$. Iterating the above procedure to $\nu^{\,j}$ $(j = 2,\,3,\,4,\ldots)$, we obtain a partition $\lambda = (\lambda_1,\,\lambda_2,\ldots)$ with odd parts. Figure 3 is an illustration of Pak's iterated Dyson's map.

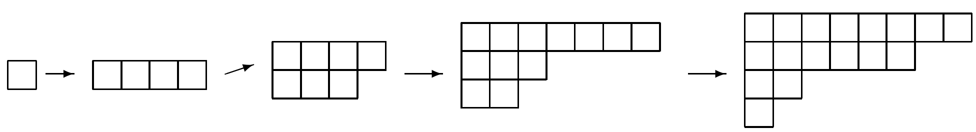

Figure 3: $\lambda = (5,\,5,\,3,\,3,\,1)$ and $\mu = (8,\,6,\,2,\,1)$.

When applying Sylvester's bijection, we see that each partition μ of n into distinct parts with maximal part μ_1 corresponds to a partition λ of n into odd parts with the maximal part λ_1 and the length $l(\lambda)$ such that $2\mu_1 + 1 = \lambda_1 + 2l(\lambda)$ or equivalently, $\mu_1 = \frac{\lambda_1 - 1}{2} + l(\lambda)$. Thus we obtain the following weighted forms of Euler's theorem:

Lemma 2.1 *The sum of $2\mu_1 + 1$ over all the partitions μ of n into distinct parts equals to the sum of $\lambda_1 + 2l(\lambda)$ over all the partitions λ of n into odd parts, namely,*

$$\sum_{\mu \in D} (2\mu_1 + 1) q^{|\mu|} = \sum_{\lambda \in O} (\lambda_1 + 2l(\lambda)) q^{|\lambda|}. \tag{2.7}$$

Lemma 2.2 *The sum of μ_1 over all the partitions μ of n into distinct parts equals to the sum of $\frac{\lambda_1 - 1}{2} + l(\lambda)$ over all the partitions λ of n into odd parts, namely,*

$$\sum_{\mu \in D} \mu_1 q^{|\mu|} = \sum_{\lambda \in O} \left(\frac{\lambda_1 - 1}{2} + l(\lambda) \right) q^{|\lambda|}, \tag{2.8}$$

We remark that these two weighted forms (2.7) and (2.8) of Euler's theorem can be deduced from a refinement of Euler's theorem due to Fine [15, p. 46, (23.91)].

From Pak's iterated Dyson's map, we see that a partition λ of n into odd parts with maximal part λ_1 corresponds to a partition μ of n into distinct parts with rank $r(\mu)$ such that

$$r(\mu)+\frac{1+(-1)^{r(\mu)}}{2}=\lambda_1.$$

Thus we obtain the following weighted form of Euler's theorem:

Lemma 2.3 *The sum of $\mu_1 - l(\mu) + \frac{1+(-1)^{r(\mu)}}{2}$ over all partitions μ of n into distinct parts equals the sum of λ_1 over all partitions λ of n into odd parts, namely,*

$$\sum_{\mu\in D}\left(\mu_1 - l(\mu)+\frac{1+(-1)^{r(\mu)}}{2}\right)q^{|\mu|}=\sum_{\lambda\in O}\lambda_1 q^{|\lambda|}. \tag{2.9}$$

It should be noted that this weighted form (2.9) of Euler's theorem can be deduced from Fine's another refinement of Euler's theorem in [14].

Now we consider the set of partitions μ of n into distinct parts with multiplicities $l(\mu)+\mu_1+\frac{1-(-1)^{r(\mu)}}{2}$. The number of such partitions of n with the multiplicities taken into account equals the number of elements in the set of partitions of n into distinct parts with multiplicities $2\mu_1+1$ minus the number of elements in the set of partitions of n into distinct parts with multiplicities $\mu_1 - l(\mu)+\frac{1+(-1)^{r(\mu)}}{2}$. In view of Lemmas 2.1 and 2.3, we are led to the weighted form (1.5) of Euler's theorem.

Next we consider the set of partitions μ of n into distinct parts with multiplicities $l(\mu)+\frac{1-(-1)^{r(\mu)}}{2}$. The number of such partitions with multiplicities equals the number of elements in the set of partitions of n into distinct parts with multiplicities μ_1+1 minus the number of elements in the set of partitions of n into distinct parts with multiplicities $\mu_1 - l(\mu)+\frac{1+(-1)^{r(\mu)}}{2}$, according to Lemmas 2.2, 2.3 and Euler's theorem, we obtain the weighted form (1.6) of Euler's theorem.

3 Rooted Partitions

Inspired by the suggestion of Andrews [4], we are guided to consider weighted counting of partitions in order to give combinatorial interpretations of the first sums on the right had sides of Ramanujan's identities (1.1) and (1.2). To this end, we introduce the notion of rooted partitions which can be regarded as a weighted version of ordinary partitions. In some sense, rooted partitions are related to "overpartitions" (see Corteel and Lovejoy [11]) and "partitions with designated summand" of Andrews-Lewis-Lovejoy [6].

A rooted partition of n can be formally defined as a pair of partitions (λ,μ), where $|\lambda|+|\mu|=n$ and μ is a nonempty partition with equal parts. The union of the parts of λ and μ are regarded as the parts of the rooted partition (λ,μ).

For example, there are twelve rooted partitions of 4:

$(\emptyset,\ (4))\ ((1),\ (3))\ ((3),\ (1))\ ((2),\ (2))\ (\emptyset,\ (2,2))\ ((1,1),\ (2))\ ((2,1),(1))\ ((2),(1,1))$

$((1,1,1),\ (1))\ ((1,1),\ (1,1))\ ((1),\ (1,1,1))\ (\emptyset,\ (1,1,1,1)).$

There are three rooted partitions of 4 with distinct parts: $(\emptyset,\ (4))\ ((1),\ (3))\ ((3),\ (1))$.

There are six rooted partitions of 4 with odd parts:

$$((1),\ (3))\ ((3),\ (1))\ ((1,1,1),\ (1))\ ((1,1),\ (1,1))\ ((1),\ (1,1,1))\ (\emptyset,\ (1,1,1,1)).$$

A rooted partition (λ,μ) is said to be a rooted partition with almost distinct parts if λ has distinct parts. As a convention, we shall assume that (λ,μ) is a rooted partition with almost distinct parts if $\lambda=\emptyset$. There are nine rooted partitions of 4 with almost distinct parts:

$$(\emptyset,\ (4))\ ((1),\ (3))\ ((3),(1))\ ((2),\ (2))\ (\emptyset,\ (2,2))\ ((2,1),\ (1))\ ((2),\ (1,1))$$

$$((1),\ (1,1,1))\ (\emptyset,\ (1,1,1,1)).$$

It is easy to see that the generating function for rooted partitions with distinct parts equals

$$\sum_{d=1}^{\infty} q^d \prod_{n\neq d}^{\infty}(1+q^n). \tag{3.10}$$

On the other hand, the generating function for rooted partitions with odd parts equals

$$\frac{1}{(q;q^2)_\infty}\sum_{d=0}^{\infty}\frac{q^{2d+1}}{1-q^{2d+1}}. \tag{3.11}$$

The generating function for rooted partitions with almost distinct parts equals

$$(-q;q)_\infty\sum_{d=1}^{\infty}\frac{q^d}{1-q^d}. \tag{3.12}$$

We now define the *root size of a rooted partition* $(\lambda,\ \mu)$ as the number of parts of μ. Then the generating function for rooted partitions into almost distinct parts with even root size equals

$$(-q;q)_\infty\sum_{d=1}^{\infty}\frac{q^{2d}}{1-q^{2d}}. \tag{3.13}$$

We have the following theorem on rooted partitions:

Theorem 3.1 *The number of rooted partitions of n into almost distinct parts with even root size plus the number of rooted partitions of n with distinct parts equals the number of rooted partitions of n with odd parts.*

We first give a generating function proof of the above theorem.

Proof. The sum of the two numbers have the following generating function

$$
\begin{aligned}
&(-q;q)_\infty \sum_{d=1}^{\infty} \frac{q^{2d}}{1-q^{2d}} + \sum_{d=1}^{\infty} q^d \prod_{n\neq d}^{\infty} (1+q^n) \\
&= (-q;q)_\infty \left(\sum_{d=1}^{\infty} \frac{q^{2d}}{1-q^{2d}} + \sum_{d=1}^{\infty} \frac{q^d - q^{2d}}{(1-q^d)(1+q^d)} \right) \\
&= (-q;q)_\infty \sum_{d=1}^{\infty} \frac{q^d + q^{2d} - q^{2d}}{1-q^{2d}} \\
&= (-q;q)_\infty \left(\sum_{d=1}^{\infty} \frac{q^d}{1-q^d} - \sum_{d=1}^{\infty} \frac{q^{2d}}{1-q^{2d}} \right) \\
&= \frac{1}{(q;q^2)_\infty} \sum_{d=0}^{\infty} \frac{q^{2d+1}}{1-q^{2d+1}}.
\end{aligned}
$$

This implies the desired statement for rooted partitions. ∎

We now present a combinatorial proof of the above theorem in terms of an involution and a bijection. We need the following fact:

Lemma 3.2 *The number of rooted partitions of n into almost distinct parts with odd root size equals the number of rooted partitions of n into almost distinct parts with even root size plus the number of rooted partitions of n with distinct parts.*

Proof. We construct an involution τ on the set of rooted partitions of n with almost distinct parts except those strictly with distinct parts. More precisely, the involution τ is on the set of rooted partitions (λ, μ) of n such that λ has distinct parts and the number of occurrences of the part of μ in both λ and μ is at least two.

Case 1: For a rooted partition (λ, μ) with almost distinct parts but not with distinct parts, if λ contains the part of μ, then move this part from λ to μ.

Case 2: For a rooted partition (λ, μ) with almost distinct parts but not with distinct parts, if λ does not contain the part of μ, then move this part from μ to λ.

It is easy to check that the above mapping is an involution. Moreover, τ changes the parity of the root size. ∎

For example, there are nine rooted partitions of 4 with almost distinct parts:

$(\emptyset, (4))$ $((1), (3))$ $((3), (1))$ $((2), (2))$ $(\emptyset, (2, 2))$ $((2, 1), (1))$ $((2), (1, 1))$

$((1), (1, 1, 1))$ $(\emptyset, (1, 1, 1, 1))$.

Applying the above involution, we get the following correspondence:

$$((2),\,(2)) \leftrightarrows (\emptyset,\,(2,2))\ ((2,1),\,(1)) \leftrightarrows ((2),\,(1,1))\ ((1),\,(1,1,1)) \leftrightarrows (\emptyset,\,(1,1,1,1)).$$

The above involution does not apply to rooted partitions with distinct parts:

$$(\emptyset,\,(4))\ ((1),\,(3))\ ((3),\,(1))$$

.

The following correspondence can be regarded as a rooted partition analogue of Euler's theorem. Here we need to define the conjugate of the partition. For a partition $\lambda = (\lambda_1, \ldots, \lambda_r)$, the conjugate partition $\lambda' = (\lambda'_1, \lambda'_2, \ldots, \lambda'_t)$ of λ by setting λ'_i to be the number of parts of λ that are greater than or equal to i. Clearly, we have $l(\lambda) = \lambda'_1$ and $\lambda_1 = l(\lambda')$. We have the following lemma:

Lemma 3.3 *The number of rooted partitions of n into almost distinct parts with odd root size equals to the number of rooted partitions of n with odd parts.*

Proof. We employ Sylvester's bijection to construct a map from the set of rooted partitions of n into almost distinct parts with odd root size to the set of rooted partitions of n with odd parts.

The map σ: For a rooted partition $(\lambda,\,\mu)$ into almost distinct parts with odd root size, we apply the inverse map of Sylvester's bijection φ^{-1} to λ to generate a partition α with odd parts. Let β be the conjugate of μ which is a partition with equal odd parts. Therefore $(\alpha,\,\beta)$ is a rooted partition with odd parts.

The inverse map σ^{-1}: For a rooted partition $(\alpha,\,\beta)$ with odd parts, we apply Sylvester's bijection φ to α to generate a partition λ with distinct parts. Let μ be conjugate of β, which is a partition into equal parts with odd length. Thus $(\lambda,\,\mu)$ is a rooted partition into almost distinct parts with odd root size.

From Sylvester's bijection, one sees that σ is also a bijection. ∎

For example, there are six rooted partitions of 4 into almost distinct parts with odd root size:

$$(\emptyset,\,(4))\ ((1),\,(3))\ ((3),\,(1))\ ((2),\,(2))\ ((2,1),\,(1))\ ((1),(1,1,1)),$$

and there are six rooted partitions of 4 with odd parts:

$$((1),\,(3))\ ((3),\,(1))\ ((1,1,1),\,(1))\ ((1,1),\,(1,1))\ ((1),\,(1,1,1))\ (\emptyset,\,(1,1,1,1)).$$

Using the above bijection, we have the following correspondence:

$$(\emptyset,\,(4)) \leftrightarrows (\emptyset,\,(1,1,1,1)) \quad ((1),\,(3)) \leftrightarrows ((1),\,(1,1,1)) \quad ((3),\,(1)) \leftrightarrows ((1,1,1),\,(1))$$
$$((2),\,(2)) \leftrightarrows ((1,1),\,(1,1)) \quad ((2,1),\,(1)) \leftrightarrows ((3),\,(1)) \quad ((1),\,(1,1,1)) \leftrightarrows ((1),\,(3)).$$

From the above Lemmas 3.2 and 3.3, we obtain Theorem 3.1 which serves as a combinatorial setting for Ramanujan's identity (1.2). For Ramanujan's identity (1.1), we need the following partition identity which also follows from the above two lemmas:

Theorem 3.4 *The number of rooted partitions of n with almost distinct parts plus the number of rooted partitions of n with distinct parts is twice the number of rooted partitions of n with odd parts.*

We now make a connection between rooted partitions with distinct parts and odd parts and ordinary partitions with distinct parts and odd parts. Chapman [10] has shown that the series (3.10)

$$\sum_{d=1}^{\infty} q^d \prod_{n\neq d}(1+q^n)$$

is the generating function for ordinary partitions with distinct parts with multiplicities being their lengths. Note that the above series is also the generating function for rooted partitions with distinct parts. This generating function identity implies that there should be a combinatorial correspondence between rooted partitions and ordinary partitions with distinct parts.

In fact, a simple correspondence goes as follows: From a partition α with distinct parts, we can get $l(\alpha)$ distinct rooted partitions (λ, μ) with distinct parts by designating any part of α as the part of μ and keeping the remaining parts of α as parts of λ. This map is clearly reversible.

For instance, there are two partitions of 4 with distinct parts: (4) (3, 1). The sum of their lengths is three, whereas there are three rooted partitions of 4 with distinct parts: $(\emptyset, (4))$ $((1), (3))$ $((3), (1))$.

Thus we have the following theorem on the relationship between rooted partitions with distinct parts and partitions with distinct parts.

Theorem 3.5 *The number of rooted partitions of n with distinct parts equals the sum of lengths over partitions of n with distinct parts.*

Chapman [10] has shown that (3.11) is also the generating function for the sum of the lengths of partitions with odd parts:

$$\begin{aligned}
\frac{1}{(q;q^2)_\infty}\sum_{d=0}^{\infty}\frac{q^{2d+1}}{1-q^{2d+1}} &= \sum_{d=0}^{\infty}\frac{1}{(q;q^2)_d(q^{2d+3};q^2)_\infty}\cdot\frac{q^{2d+1}}{(1-q^{2d+1})^2}\\
&= \sum_{d=0}^{\infty}\sum_{m=1}^{\infty}\frac{mq^{(2d+1)m}}{(q;q^2)_d(q^{2d+3};q^2)_\infty}\\
&= \sum_{d=0}^{\infty}\sum_{\lambda\in O} n_\lambda(2d+1)q^{|\lambda|}\\
&= \sum_{\lambda\in O} l(\lambda)q^{|\lambda|}.
\end{aligned}$$

where $n_\lambda(d)$ is the number of parts equal to d in λ. Using the formulation of rooted partitions with odd parts and the above generating function, we establish the following relation between rooted partitions and ordinary partitions, and we give a combinatorial proof of this fact. Theorem 3.5 and the following Theorem 3.6 will be necessary to transform the formulations of Ramanujan's identities with rooted partitions to combinatorial settings with ordinary partitions.

Theorem 3.6 *The number of rooted partitions of n with odd parts equals the sum of lengths over partitions of n with odd parts.*

Proof. In fact, for a partition β of n with odd parts, we may get $l(\beta)$ distinct rooted partitions $(\lambda,\ \mu)$ of n with odd parts by designating any part of β as the part of μ and keep the remaining parts of β as parts of λ. Assume that d is a part that appears m times ($m \geq 2$) in β. Then we may choose μ as the partition with d repeated i times, where $i = 1, 2, \ldots, m$. ∎

For example, there are two partitions of 4 with odd parts namely (3, 1) (1, 1, 1, 1), the sum of lengths is six. For rooted partitions of 4, we see that there are also six rooted partitions with odd parts:

$$((1),\ (3))\ ((3),\ (1))\ ((1,1,1),\ (1))\ ((1,1),\ (1,1))\ ((1),\ (1,1,1))\ (\emptyset,\ (1,1,1,1)).$$

4 Ramanujan's Identities

In this section, we will reformulate Ramanujan's identities (1.1) and (1.2) as the two weighted forms (1.5) and (1.6) of Euler's theorem. The left hand sides of (1.1) and (1.2) have partition interpretations as given by Andrews [4] and Chapman [10]. The first summations on the right hand sides of (1.1) and (1.2) can be interpreted combinatorially in term of ordinary partitions with multiplicities as given by Theorem 3.1 and

3.4. The second summations on the right hand sides of (1.1) and (1.2) have partition interpretations in terms of the rank.

Combining Theorems 3.5 and 3.6 on the relations between rooted partitions and ordinary partitions, we may transform Theorem 3.4 on rooted partitions to a statement on ordinary partitions in Lemma 1.1.

We proceed to demonstrate that with the aid of the above theorem, Ramanujan's identity (1.1) can be restated as the weighted form (1.5) of Euler's theorem. At first, one can check that the left side of Ramanujan's identity (1.1) equals the generating function for the sum of the largest parts over partitions with distinct parts according to Lemma 1 of [10]:

$$\sum_{n=0}^{\infty}[(-q;q)_\infty-(-q;q)_n]=\sum_{\mu\in D}\mu_1\, q^{|\mu|}. \tag{4.14}$$

It is easy to see that the second summation on the right hand of (1.1), that is,

$$1+\sum_{n=1}^{\infty}\frac{q^{\binom{n+1}{2}}}{(-q;q)_n},$$

equals the generating function for partitions into distinct parts with even rank minus the generating function for partitions into distinct parts with odd rank. Therefore, we have

$$-\frac{1}{2}(-q;q)_\infty+\frac{1}{2}\left[1+\sum_{n=1}^{\infty}\frac{q^{\binom{n+1}{2}}}{(-q;q)_n}\right]=-\sum_{\substack{\mu\in D\\ r(\mu)\ \text{odd}}}q^{|\mu|}. \tag{4.15}$$

From the above interpretations and Lemma 1.1, one sees the right side of Ramanujan's identity (1.1) equals the generating function for the sum of twice the lengths over partitions with odd parts and minus the generating function for the sum of lengths over partitions with distinct parts minus the generating function for partitions into distinct parts with odd rank:

$$\begin{aligned}(-q;q)_\infty&\left[-\frac{1}{2}+\sum_{d=1}^{\infty}\frac{q^d}{1-q^d}\right]+\frac{1}{2}\left[1+\sum_{n=1}^{\infty}\frac{q^{\binom{n+1}{2}}}{(-q;q)_n}\right]\\ &=\sum_{\lambda\in O}2l(\lambda)q^{|\lambda|}-\sum_{\mu\in D}l(\mu)q^{|\mu|}-\sum_{\substack{\mu\in D\\ r(\mu)\ \text{odd}}}q^{|\mu|}.\end{aligned} \tag{4.16}$$

We now reach the conclusion that Ramanujan's identity (1.1) can be restated as the weighted form (1.5) of Euler's theorem:

$$\sum_{\mu\in D}\left(\mu_1+l(\mu)+\frac{1-(-1)^{r(\mu)}}{2}\right)q^{|\mu|}=\sum_{\lambda\in O}2l(\lambda)q^{|\lambda|}.$$

Thus, we have obtained a combinatorial proof of (1.1) based on a weighted form of Euler's theorem.

Similarly, combining Theorems 3.5 and 3.6 on the relations between rooted partitions and ordinary partitions, we may transform Theorem 3.1 on rooted partitions to the assertion for ordinary partitions in Lemma 1.2.

As pointed out by Andrews [4], it is not difficult to see that the left hand side of Ramanujan's identity (1.2) equals the generating function of the sum of half of its largest part and minus one over partitions into odd parts:

$$\sum_{n=0}^{\infty}\left[\frac{1}{(q;q^2)_\infty}-\frac{1}{(q;q^2)_n}\right]=\sum_{\lambda\in O}\frac{\lambda_1-1}{2}\,q^{|\lambda|}. \tag{4.17}$$

By using the above relation (4.15) and Lemma 1.2, one sees that the right hand side of Ramanujan's identity (1.2) equals the generating function for the sum of lengths over partitions into odd parts minus the generating function for the sum of lengths over partitions into distinct parts minus the generating function for partitions into distinct parts with odd rank:

$$\begin{aligned}(-q;q)_\infty&\left[-\frac{1}{2}+\sum_{d=1}^{\infty}\frac{q^{2d}}{1-q^{2d}}\right]+\frac{1}{2}\left[1+\sum_{n=1}^{\infty}\frac{q^{\binom{n+1}{2}}}{(-q;q)_n}\right]\\ &=\sum_{\lambda\in O}l(\lambda)q^{|\lambda|}-\sum_{\mu\in D}l(\mu)q^{|\mu|}-\sum_{\substack{\mu\in D\\ r(\mu)\text{ odd}}}q^{|\mu|}.\end{aligned} \tag{4.18}$$

So we conclude that Ramanujan's identity (1.2) can be recast as the weighted form (1.6) of Euler's theorem:

$$\sum_{\mu\in D}\left(l(\mu)+\frac{1-(-1)^{r(\mu)}}{2}\right)q^{|\mu|}=\sum_{\lambda\in O}\left(l(\lambda)-\frac{\lambda_1-1}{2}\right)q^{|\lambda|}.$$

Acknowledgments. We are grateful to the referee for helpful suggestions, and we would like to thank George E. Andrews and Igor Pak for valuable comments. This work was supported by the 973 Project on Mathematical Mechanization, the National Science Foundation, the Ministry of Education, and the Ministry of Science and Technology of China.

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