Semi-Finite Forms of Bilateral Basic Hypergeometric Series

William Y. C. Chen[1] and Amy M. Fu[2]

Center for Combinatorics, LPMC
Nankai University, Tianjin 300071, P.R. China

Email: [1]chen@nankai.edu.cn, [2]fu@nankai.edu.cn

Abstract. We show that several classical bilateral summation and transformation formulas have semi-finite forms. We obtain these semi-finite forms from unilateral summation and transformation formulas. Our method can be applied to derive Ramanujan's ${}_1\psi_1$ summation, Bailey's ${}_2\psi_2$ transformations, and Bailey's ${}_6\psi_6$ summation.

Corresponding Author: William Y. C. Chen, Email: chen@nankai.edu.cn

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1. Introduction

We follow the terminology for basic hypergeometric series in [6]. Assuming $|q| < 1$, let

$$(a;q)_\infty = (1-a)(1-aq)(1-aq^2)\cdots .$$

For any integer n, the q-shifted factorial $(a;q)_n$ is given by

$$(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}.$$

For $n \geq 0$, we have the following relation which is crucial for this paper:

$$(a;q)_{-n} = \frac{1}{(aq^{-n};q)_n} = \frac{(-q/a)^n q^{\binom{n}{2}}}{(q/a;q)_n}. \tag{1.1}$$

For convenience, we employ the following usual notation:

$$(a_1, a_2, \ldots, a_m; q)_n = (a_1;q)_n (a_2;q)_n \ldots (a_m;q)_n.$$

The (unilateral) basic hypergeometric series ${}_{r+1}\phi_r$ is defined by

$${}_{r+1}\phi_r \left[\begin{matrix} a_1, a_2, \cdots, a_{r+1} \\ b_1, b_2, \cdots, b_r \end{matrix} ; q, z \right] = \sum_{k=0}^{\infty} A(k), \tag{1.2}$$

where

$$A(k) = \frac{(a_1, a_2, \cdots, a_{r+1}; q)_k}{(b_1, b_2, \cdots, b_r, q; q)_k} z^k.$$

The bilateral basic hypergeometric series ${}_s\psi_s$ is defined as follows,

$$ {}_s\psi_s \left[\begin{matrix} a_1, a_2, \cdots, a_s \\ b_1, b_2, \cdots, b_s \end{matrix} ; q, z \right] = \sum_{k=-\infty}^{\infty} B(k), \tag{1.3} $$

where

$$ B(k) = \frac{(a_1, a_2, \cdots, a_s; q)_k}{(b_1, b_2, \cdots, b_s; q)_k} z^k. $$

In this paper, we propose the following method of deriving bilateral summation and transformation formulas using *semi-finite forms*. For a bilateral series ${}_s\psi_s$ as given in (1.3), we construct a summand $G(k, m)$ which implies a unilateral series ${}_{r+s+1}\phi_{r+s}$, where r is a nonnegative integer, such that

$$ \lim_{m\to\infty} G(k, m) = B(k) $$

for all k, and the summation

$$ \sum_{k=-m}^{\infty} G(k, m) \tag{1.4} $$

can be easily accomplished as a Laurent extension of the summation

$$ \sum_{k=0}^{\infty} G(k - m, m) = G(-m, m) \sum_{k=0}^{\infty} A(k), \tag{1.5} $$

where $G(k, m)$ can be written as

$$ G(k - m, m) = G(-m, m)A(k) $$

for some $A(k)$. The bilateral series (1.3) is then obtained from (1.4) as $m \to \infty$, subject to suitable convergence conditions. We apply this procedure to derive bilateral series identities from suitable unilateral ones. The above summation (1.4) is called the *semi-finite form* of the bilateral summation (1.3). A method similar to ours was recently used by Schlosser [9], and Jouhet and Schlosser [8], who derived summations for bilateral series from *finite forms*. We also note that another method, which uses a similar factorization as above, for deriving bilateral series identities from unilateral ones was used by Ismail [7], and Askey and Ismail [2]. Rather than taking limits, they apply analytic continuation as the main ingredient.

In this paper, we present semi-finite forms of several classical bilateral summation and transformation formulas such as Ramanujan's ${}_1\psi_1$ formula, Bailey's ${}_2\psi_2$ transformations, and Bailey's ${}_6\psi_6$ summation.

2. From ${}_2\phi_1$ to ${}_1\psi_1$

Using the well known Gauss summation formula

$$ {}_2\phi_1 \left[\begin{matrix} a, b \\ c \end{matrix} ; q, c/ab \right] = \frac{(c/a, c/b; q)_\infty}{(c, c/ab; q)_\infty}, \tag{2.1} $$

where $|c/ab| < 1$, we get a semi-finite form of Ramanujan's summation of the general ${}_1\psi_1$,

$$ {}_1\psi_1\left[\begin{array}{c} a \\ b \end{array}; q, z\right] = \sum_{k=-\infty}^{\infty} \frac{(a;q)_k}{(b;q)_k} z^k = \frac{(q;q)_\infty (b/a;q)_\infty (az;q)_\infty (q/az;q)_\infty}{(b;q)_\infty (q/a;q)_\infty (z;q)_\infty (b/az;q)_\infty}, \tag{2.2} $$

where $|b/a| < |z| < 1$.

Proposition 2.1 *For $|z| < 1$, the following identity holds:*

$$ \sum_{k=-m}^{\infty} \frac{(a;q)_k (bq^m/az;q)_k}{(q^{1+m};q)_k (b;q)_k} z^k = \frac{(q;q)_m (q/az;q)_m}{(q/a;q)_m (b/az;q)_m} \frac{(b/a;q)_\infty (az;q)_\infty}{(b;q)_\infty (z;q)_\infty}. \tag{2.3} $$

Proof. The left hand side of (2.3) can be rewritten as

$$ \begin{aligned} &\sum_{k=0}^{\infty} \frac{(a;q)_{k-m} (bq^m/az;q)_{k-m}}{(q^{1+m};q)_{k-m} (b;q)_{k-m}} z^{k-m} \\ &= \quad z^{-m} \frac{(a;q)_{-m} (bq^m/az;q)_{-m}}{(q^{1+m};q)_{-m} (b;q)_{-m}} \sum_{k=0}^{\infty} \frac{(aq^{-m};q)_k (b/az;q)_k}{(q;q)_k (bq^{-m};q)_k} z^k \\ &\overset{(2.1)}{=} \quad z^{-m} \frac{(a;q)_{-m} (bq^m/az;q)_{-m}}{(q^{1+m};q)_{-m} (b;q)_{-m}} \frac{(b/a;q)_\infty (azq^{-m};q)_\infty}{(bq^{-m};q)_\infty (z;q)_\infty} \\ &\overset{(1.1)}{=} \quad z^{-m} \frac{(q;q)_m (azq^{-m};q)_m}{(aq^{-m};q)_m (b/az;q)_m} \frac{(az;q)_\infty (b/a;q)_\infty}{(b;q)_\infty (z;q)_\infty}, \end{aligned} $$

which equals the right hand side of (2.3). ∎

Taking the limit $m \to \infty$ in Proposition 2.1 while assuming $|b/az| < 1$, we immediately obtain (2.2).

We remark that our method is different from the method of M. Jackson's elementary proof of (2.2) (see the exposition of Schlosser [9]) in the sense that Jackson's proof does not give a semi-finite form although the Gauss summation is also the basic ingredient. We should also note that a finite form of Ramanujan's ${}_1\psi_1$ summation has been given by Schlosser [10] using the terminating q-Pfaff-Saalschütz summation.

3. From ${}_3\phi_2$ to ${}_2\psi_2$

In this section, we use two ${}_3\phi_2$ summation and transformation formulas to give the semi-finite forms of ${}_2\psi_2$ formulas due to Bailey. We begin with the following ${}_2\psi_2$ transformation formula [6, Ex. 5.20(i)] valid for $|z|, |cd/abz|, |d/a|, |c/b| < 1$:

$$ {}_2\psi_2\left[\begin{array}{c} a, b \\ c, d \end{array}; q, z\right] = \frac{(az, d/a, c/b, dq/abz;q)_\infty}{(z, d, q/b, cd/abz;q)_\infty} {}_2\psi_2\left[\begin{array}{c} a, abz/d \\ az, c \end{array}; q, \frac{d}{a}\right]. \tag{3.1} $$

Using a q-analogue of the Kummer-Thomae-Whipple formula [6, Eq. (3.2.7)]:

$$ {}_3\phi_2\left[\begin{matrix} a,b,c \\ d,e \end{matrix};q,\frac{de}{abc}\right] = \frac{(e/a,de/bc;q)_\infty}{(e,de/abc;q)_\infty} {}_3\phi_2\left[\begin{matrix} a,d/b,d/c \\ d,de/bc \end{matrix};q,\frac{e}{a}\right], \tag{3.2} $$

where $|de/abc|<1$ and $|e/a|<1$, we get a semi-finite form of (3.1).

Proposition 3.1 *For $|z|<1$ and $|d/a|<1$, we have*

$$ \begin{aligned} \sum_{k=-m}^{\infty} \frac{(a,b;q)_k (cdq^m/abz;q)_k}{(c,d;q)_k (q^{1+m};q)_k} z^k &= \frac{(az,d/a;q)_\infty}{(z,d;q)_\infty} \frac{(c/b,dq/abz;q)_m}{(q/b,cd/abz;q)_m} \\ &\quad \cdot \sum_{k=-m}^{\infty} \frac{(a,cq^m/b,abz/d;q)_k}{(c,q^{1+m},az;q)_k} (d/a)^k. \end{aligned} \tag{3.3} $$

Proof. The left hand side of (3.3) equals

$$ \begin{aligned} & z^{-m} \frac{(a,b,cdq^m/abz;q)_{-m}}{(c,d,q^{1+m};q)_{-m}} \sum_{k=0}^{\infty} \frac{(aq^{-m},bq^{-m},cd/abz;q)_k}{(cq^{-m},dq^{-m},q;q)_k} z^k \\ \overset{(3.2)}{=} \; & z^{-m} \frac{(a,b,cdq^m/abz;q)_{-m}}{(c,d,q^{1+m};q)_{-m}} \frac{(d/a,azq^{-m};q)_\infty}{(dq^{-m},z;q)_\infty} \\ & \cdot \sum_{k=0}^{\infty} \frac{(aq^{-m},c/b,abzq^{-m}/d;q)_k}{(q,cq^{-m},azq^{-m};q)_k} \left(\frac{d}{a}\right)^k \\ \overset{(1.1)}{=} \; & \frac{(d/a,az;q)_\infty}{(d,z;q)_\infty} \frac{(c/b,abzq^{-m}/d;q)_m}{(bq^{-m},cd/abz;q)_m} \left(\frac{d}{az}\right)^m \\ & \cdot \sum_{k=0}^{\infty} \frac{(a,cq^m/b,abz/d;q)_{k-m}}{(c,q^{1+m},az;q)_{k-m}} (d/a)^{k-m}, \end{aligned} $$

which can be rewritten in the form of the right hand side of (3.3). ∎

The next ${}_2\psi_2$ transformation formula we consider is the following [6, Ex. 5.20(ii)]:

$$ {}_2\psi_2\left[\begin{matrix} a,b \\ c,d \end{matrix};q,z\right] = \frac{(az,bz,cq/abz,dq/abz;q)_\infty}{(q/a,q/b,c,d;q)_\infty} {}_2\psi_2\left[\begin{matrix} abz/c,abz/d \\ az,bz \end{matrix};q,\frac{cd}{abz}\right]. \tag{3.4} $$

Using a summation of Hall [6, Eq. (3.2.10)]:

$$ {}_3\phi_2\left[\begin{matrix} a,b,c \\ d,e \end{matrix};q,\frac{de}{abc}\right] = \frac{(b,de/ab,de/bc;q)_\infty}{(d,e,de/abc;q)_\infty} {}_3\phi_2\left[\begin{matrix} d/b,e/b,de/abc \\ de/ab,de/bc \end{matrix};q,b\right], \tag{3.5} $$

where $|de/abc|<1$ and $|b|<1$, we obtain the following semi-finite form of (3.4).

Proposition 3.2 *For $|z|<1$ and $|cd/abz|<1$, we have*

$$ \begin{aligned} \sum_{k=-m}^{\infty} \frac{(a,b;q)_k (cdq^m/abz;q)_k}{(c,d;q)_k (q^{1+m};q)_k} z^k &= \frac{(az,bz,cd/abz;q)_\infty}{(c,d,z;q)_\infty} \frac{(cq/abz,dq/abz,z;q)_m}{(q/a,q/b,cd/abz;q)_m} \\ &\quad \cdot \sum_{k=-m}^{\infty} \frac{(abz/c,abz/d,zq^m;q)_k}{(az,bz,q^{1+m};q)_k} (cd/abz)^k. \end{aligned} $$

4. From nonterminating ${}_8\phi_7$ to ${}_6\psi_6$

In this section, we give a semi-finite form of Bailey's ${}_6\psi_6$ summation formula by using Bailey's 3-term transformation formula for a nonterminating very-well-poised ${}_8\phi_7$ series [6, Eq. (2.11.1)]:

$$
\begin{aligned}
&{}_8\phi_7\left[\begin{matrix} a, qa^{\frac12}, -qa^{\frac12}, b, c, d, e, f \\ a^{\frac12}, -a^{\frac12}, aq/b, aq/c, aq/d, aq/e, aq/f \end{matrix}; q, \frac{a^2q^2}{bcdef}\right] \\
&\quad = \frac{(aq, aq/de, aq/df, aq/ef, eq/c, fq/c, b/a, bef/a; q)_\infty}{(aq/d, aq/e, aq/f, aq/def, q/c, efq/c, be/a, bf/a; q)_\infty} \\
&\qquad \cdot {}_8\phi_7\left[\begin{matrix} ef/c, q(ef/c)^{\frac12}, -q(ef/c)^{\frac12}, aq/bc, aq/cd, ef/a, e, f \\ (ef/c)^{\frac12}, -(ef/c)^{\frac12}, bef/a, def/a, aq/c, fq/c, eq/c \end{matrix}; q, \frac{bd}{a}\right] \\
&\qquad + \frac{b}{a}\frac{(aq, bq/a, bq/c, bq/d, bq/e, bq/f, d, e, f; q)_\infty}{(aq/b, aq/c, aq/d, aq/e, aq/f, bd/a, be/a, bf/a, def/a; q)_\infty} \\
&\qquad \cdot \frac{(aq/bc, bdef/a^2, a^2q/bdef; q)_\infty}{(aq/def, q/c, b^2q/a; q)_\infty} \\
&\qquad \cdot {}_8\phi_7\left[\begin{matrix} b^2/a, qba^{-\frac12}, -qba^{-\frac12}, b, bc/a, bd/a, be/a, bf/a \\ ba^{-\frac12}, -ba^{-\frac12}, bq/a, bq/c, bq/d, bq/e, bq/f \end{matrix}; q, \frac{a^2q^2}{bcdef}\right], \qquad (4.1)
\end{aligned}
$$

where $|bd/a| < 1$ and $|a^2q^2/bcdef| < 1$.

Proposition 4.1 *When $|bd/a| < 1$ and $|a^2q^2/bcdef| < 1$, we have*

$$
\begin{aligned}
&\sum_{k=-m}^{\infty} \frac{(q^{m-k+1}/a, fq^m; q)_k}{(q^{m-k}f/a, q^{1+m}; q)_k} \frac{(qa^{\frac12}, -qa^{\frac12}, b, c, d, e; q)_k}{(a^{\frac12}, -a^{\frac12}, aq/b, aq/c, aq/d, aq/e; q)_k} \left(\frac{qa^2}{bcde}\right)^k \\
&= \frac{1 - efq^{2m}/c}{1 - efq^m/c} \frac{(q/a, df/a, ef/a, aq/bc, aq/cd, efq^m/a; q)_m}{(f/a, q/b, q/c, q/d, def/a, fq^{1+m}/c; q)_m} \\
&\quad \times \frac{(aq, aq/de, aq/df, aq/ef, eq^{1+m}/c, fq^{1+m}/c, b/a, befq^m/a; q)_\infty}{(aq/d, aq/e, aq/f, aq/def, q^{1+m}/c, efq^{1+m}/c, be/a, bfq^m/a; q)_\infty} \\
&\quad \times \sum_{k=-m}^{\infty} \frac{(efq^m/c, q^{1+m}(ef/c)^{\frac12}, -q^{1+m}(ef/c)^{\frac12}, aq^{1+m}/bc; q)_k}{(q^{1+m}, q^m(ef/c)^{\frac12}, -q^m(ef/c)^{\frac12}, befq^m/a; q)_k} \\
&\qquad \cdot \frac{(aq^{1+m}/cd, efq^{2m}/a, e, fq^m; q)_k}{(defq^m/a, aq/c, fq^{1+2m}/c, eq^{1+m}/c; q)_k} \left(\frac{bd}{a}\right)^k \\
&\quad + \frac{b}{a}\frac{1 - b^2q^{2m}/a}{1 - b^2q^m/a} \left(\frac{a^2q}{bcde}\right)^m \frac{(q/a, bc/a; q)_m}{(f/a; q)_m} \frac{(aq, bq^{1+2m}/a, bq^{1+m}/c; q)_\infty}{(aq/b, aq/c, aq/d; q)_\infty} \\
&\quad \times \frac{(bq^{1+m}/d, bq^{1+m}/e, bq/f, d, e, fq^m, aq/bc, bdef/a^2, a^2q/bdef; q)_\infty}{(aq/e, aq/f, bdq^m/a, beq^m/a, bfq^{2m}/a, def/a, aq/def, q/c, b^2q^{1+m}/a; q)_\infty} \\
&\quad \times \sum_{k=-m}^{\infty} \frac{(b^2q^m/a, q^{1+m}ba^{-\frac12}, -q^{1+m}ba^{-\frac12}; q)_k}{(q^{1+m}, q^m ba^{-\frac12}, -q^m ba^{-\frac12}; q)_k} \\
&\qquad \cdot \frac{(b, bcq^m/a, bdq^m/a, beq^m/a, bfq^{2m}/a; q)_k}{(bq^{1+2m}/a, bq^{1+m}/c, bq^{1+m}/d, bq^{1+m}/e, bq/f; q)_k} \left(\frac{a^2q^2}{bcdef}\right)^k. \qquad (4.2)
\end{aligned}
$$

Proof. The left hand side of (4.2) equals

$$
\begin{aligned}
&\sum_{k=-m}^{\infty} \frac{(aq^{-m}, fq^m; q)_k}{(aq^{1-m}/f, q^{1+m}; q)_k} \frac{(qa^{\frac{1}{2}}, -qa^{\frac{1}{2}}, b, c, d, e; q)_k}{(a^{\frac{1}{2}}, -a^{\frac{1}{2}}, aq/b, aq/c, aq/d, aq/e; q)_k} \left(\frac{q^2a^2}{bcdef}\right)^k \\
&\overset{(1.5)}{=} \frac{(aq^{-m}, fq^m, qa^{\frac{1}{2}}, -qa^{\frac{1}{2}}, b, c, d, e; q)_{-m}}{(aq^{1-m}/f, q^{1+m}, a^{\frac{1}{2}}, -a^{\frac{1}{2}}, aq/b, aq/c, aq/d, aq/e; q)_{-m}} \left(\frac{q^2a^2}{bcdef}\right)^{-m} \\
&\quad \times \sum_{k=0}^{\infty} \frac{(aq^{-2m}, q^{1-m}a^{\frac{1}{2}}, -q^{1-m}a^{\frac{1}{2}}; q)_k}{(q, q^{-m}a^{\frac{1}{2}}, -q^{-m}a^{\frac{1}{2}}; q)_k} \\
&\qquad \cdot \frac{(bq^{-m}, cq^{-m}, dq^{-m}, eq^{-m}, f; q)_k}{(aq^{1-m}/b, aq^{1-m}/c, aq^{1-m}/d, aq^{1-m}/e, aq^{1-2m}/f; q)_k} \left(\frac{q^2a^2}{bcdef}\right)^k \\
&\overset{(4.1)}{=} \frac{(aq^{-m}, fq^m, qa^{\frac{1}{2}}, -qa^{\frac{1}{2}}, b, c, d, e; q)_{-m}}{(aq^{1-m}/f, q^{1+m}, a^{\frac{1}{2}}, -a^{\frac{1}{2}}, aq/b, aq/c, aq/d, aq/e; q)_{-m}} \left(\frac{q^2a^2}{bcdef}\right)^{-m} \\
&\quad \times \frac{(aq^{1-2m}, aq/de, aq^{1-m}/df, aq^{1-m}/ef, eq/c, fq^{1+m}/c, bq^m/a, bef/a; q)_\infty}{(aq^{1-m}/d, aq^{1-m}/e, aq^{1-2m}/f, aq/def, q^{1+m}/c, efq/c, be/a, bfq^m/a; q)_\infty} \\
&\quad \times {}_8\phi_7 \left[\begin{matrix} ef/c, q(ef/c)^{\frac{1}{2}}, -q(ef/c)^{\frac{1}{2}}, aq/bc, aq/cd, efq^m/a, eq^{-m}, f \\ (ef/c)^{\frac{1}{2}}, -(ef/c)^{\frac{1}{2}}, bef/a, def/a, aq^{1-m}/c, fq^{1+m}/c, eq/c \end{matrix} ; q, \frac{bd}{a} \right] \\
&\quad + \frac{(aq^{-m}, fq^m, qa^{\frac{1}{2}}, -qa^{\frac{1}{2}}, b, c, d, e; q)_{-m}}{(aq^{1-m}/f, q^{1+m}, a^{\frac{1}{2}}, -a^{\frac{1}{2}}, aq/b, aq/c, aq/d, aq/e; q)_{-m}} \left(\frac{q^2a^2}{bcdef}\right)^{-m} \\
&\quad \times \frac{bq^m}{a} \frac{(aq^{1-2m}, bq^{1+m}/a, bq/c, bq/d, bq/e, bq^{1-m}/f, dq^{-m}; q)_\infty}{(aq^{1-m}/b, aq^{1-m}/c, aq^{1-m}/d, aq^{1-m}/e, aq^{1-2m}/f, bd/a, be/a; q)_\infty} \\
&\quad \times \frac{(eq^{-m}, f, aq/bc, bdefq^m/a^2, a^2q^{1-m}/bdef; q)_\infty}{(bfq^m/a, def/a, aq/def, q^{1+m}/c, b^2q/a; q)_\infty} \\
&\quad \times {}_8\phi_7 \left[\begin{matrix} b^2/a, qba^{-\frac{1}{2}}, -qba^{-\frac{1}{2}}, bq^{-m}, bc/a, bd/a, be/a, bfq^m/a \\ ba^{-\frac{1}{2}}, -ba^{-\frac{1}{2}}, bq^{1+m}/a, bq/c, bq/d, bq/e, bq^{1-m}/f \end{matrix} ; q, \frac{a^2q^2}{bcdef} \right],
\end{aligned}
$$

which equals to the right hand side of (4.2). ∎

The above proposition can be viewed as a semi-finite form of Bailey's ${}_6\psi_6$ summation formula. By taking $f = b$ and $m \to \infty$ in (4.2) while assuming $|a^2q/bcde| < 1$, we get

$$
\begin{aligned}
&{}_6\psi_6\left[\begin{matrix} qa^{\frac{1}{2}}, -qa^{\frac{1}{2}}, b, c, d, e \\ a^{\frac{1}{2}}, -a^{\frac{1}{2}}, aq/b, aq/c, aq/d, aq/d, aq/e \end{matrix} ; q, \frac{a^2q}{bcde}\right] \\
&= \frac{(q/a, bd/a, aq/bc, aq/cd, aq, aq/de, aq/bd, aq/be; q)_\infty}{(q/b, q/c, q/d, aq/b, aq/d, aq/e, aq/bde, bde/a; q)_\infty} \\
&\quad \times \sum_{k=-\infty}^{\infty} \frac{(e;q)_k}{(aq/c;q)_k} (bd/a)^k \\
&\overset{(2.2)}{=} \frac{(q/a, bd/a, aq/bc, aq/cd, aq, aq/de, aq/bd, aq/be; q)_\infty}{(q/b, q/c, q/d, aq/b, aq/d, aq/e, aq/bde, bde/a; q)_\infty} \\
&\quad \times \frac{(q, aq/ce, bde/a, aq/bde; q)_\infty}{(aq/c, q/e, bd/a, a^2q/bcde; q)_\infty} \\
&= \frac{(aq, aq/bc, aq/bd, aq/be, aq/ce, aq/cd, aq/de, q, q/a; q)_\infty}{(aq/b, aq/c, aq/d, aq/e, q/b, q/c, q/d, q/e, qa^2/bcde; q)_\infty}.
\end{aligned}
$$

Many proofs of above identity have been found, see, for example, Slater and Lakin [11], Andrews [1], Askey and Ismail [2], Askey [3], Chen and Liu [5], Schlosser [9] and Jouhet and Schlosser [8]. Our proof shows that the semi-finite form of the ${}_6\psi_6$ summation is in essence a shifted version of the ${}_8\phi_7$ summation. This proof utilizes Ramanujan's ${}_1\psi_1$ summation (2.2). It would be interesting to find a proof using a semi-finite (or even finite) form which yields Bailey's ${}_6\psi_6$ summation in a direct limit, without the need to invoke another summation formula as above.

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