The Homogeneous q-Difference Operator

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Abstract. We introduce a q-differential operator D_{xy} on functions in two variables which turns out to be suitable for dealing with the homogeneous form of the q-binomial theorem as studied by Andrews, Goldman and Rota, Roman, Ihrig and Ismail, et al. The homogeneous versions of the q-binomial theorem and the Cauchy identity are often useful for their specializations of the two parameters. Using this operator, we derive an equivalent form of the Goldman-Rota binomial identity and show that it is a homogeneous generalization of the q-Vandermonde identity. Moreover, the inverse identity of Goldman and Rota also follows from our unified identity. We also obtain the q-Leibniz formula for this operator. In the last section, we introduce the homogeneous Rogers-Szegö polynomials and derive their generating function by using the homogeneous q-shift operator.

Keywords: q-binomial theorem, Cauchy polynomials, q-Vandermonde identity, homogeneous q-difference operator, q-Leibniz formula, homogeneous Rogers-Szegö polynomials

1. Introduction

We adopt the common conventions and notations on q-series. So we always assume that $|q| < 1$ and use the following notation of the q-shifted factorial:

$$(x;q)_0 = 1;\ \ (x;q)_n = \prod_{j=0}^{n-1}(1-q^j x), n = 1, 2, ..., \infty.$$

The basic hypergeometric series ${}_r\phi_s$ is defined as follows [6]:

$${}_r\phi_s(x_1, x_2, \cdots, x_r; y_1, y_2, \cdots, y_s; q, t) = {}_r\phi_s\left[\begin{array}{c} x_1, x_2, \cdots, x_r \\ y_1, y_2, \cdots, y_s \end{array}; q, t\right]$$
$$= \sum_{n=0}^{\infty} \frac{(x_1;q)_n(x_2;q)_n\cdots(x_r;q)_n}{(y_1;q)_n(y_2;q)_n\cdots(y_s;q)_n}\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r} t^n,$$

where $q \neq 0$ when $r > s+1$.

The q-binomial coefficient is given by:

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q;q)_n}{(q;q)_{n-k}(q;q)_k}.$$

The following is the homogeneous form of the q-shifted factorial:

$$P_n(x,y) = (y/x;q)_n \, x^n = (x-y)(x-qy)\cdots(x-q^{n-1}y).$$

We also have the following basic relations:

$$\begin{bmatrix} n \\ k \end{bmatrix}(-1)^k q^{\binom{k}{2}} = \frac{(q^{-n};q)_k q^{nk}}{(q;q)_k},$$

$$P_n(x,y) = (-1)^n q^{\binom{n}{2}} P_n(y, q^{1-n}x),$$

$$P_{n-k}(x, q^{1-n}y) = (-1)^{n-k} q^{\binom{k}{2}-\binom{n}{2}} P_{n-k}(y, q^k x).$$

The polynomials $P_n(x,y)$ are important in the q-umbral calculus as studied by Andrews [1, 2], Goldman-Rota [5], Goulden-Jackson [7], Ihrig and Ismail [8], Roman [13], Johnson[11], et al. In the q-umbral calculus, the polynomial sequence $P_n(x,y)$ is a homogeneous Eulerian family. By vector space arguments, Goldman and Rota [5] have shown the following q-binomial identity, which we call the Goldman-Rota q-binomial theorem. This identity may be known earlier, but we do not have accurate information on the reference:

$$P_n(x,y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} P_k(x,z) P_{n-k}(z,y). \tag{1.1}$$

Let V_n be an n-dimensional vector space over the finite field of q elements, and X, Y Z be vector spaces over $GF(q)$ such that $|X| = x$, $|Y| = y$ and $|Z| = z$ where $|X|$ denotes the number of vectors in X. Assuming that $Z \subset Y \subset X$ and $\dim V_n < \dim Z$, Goldman and Rota [5] show that the above identity counts in two ways the set of all one-to-one linear transformations $f\colon V_n \to X$ such that $f^{-1}(Z) = 0$. Setting $y = 0$ and $z = 1$ in (1.1), one obtains the following identity due to Cauchy:

$$x^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (x-1)(x-q)\cdots(x-q^{k-1}). \tag{1.2}$$

Note that the polynomials $P_n(x,1) = (x-1)(x-q)\cdots(x-q^{n-1})$ are sometimes called the Gauss polynomials. A direct combinatorial argument for the above identity of Cauchy is also given by Goldman and Rota [5]. For further background on the above q-binomial theorem and its specializations, the reader is referred to the introduction written by Kung [12]. Moreover, by Möbius inversion, Goldman and Rota obtain an identity which leads to a partition identity, generalizing Durfee's identity.

$$P_n(x,y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (-1)^k q^{\binom{k}{2}} P_k(y,1) P_{n-k}(x,q^k). \tag{1.3}$$

It was not obvious how to show the equivalence of the above two q-binomial theorems (1.1) and (1.3). Here we give a derivation:

$$\begin{aligned} P_n(x,y) &= (-1)^n q^{\binom{n}{2}} P_n(y, q^{1-n}x) \\ &= (-1)^n q^{\binom{n}{2}} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} P_k(y,1) P_{n-k}(1, q^{1-n}x) \\ &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (-1)^k q^{\binom{k}{2}} P_k(y,1) P_{n-k}(x,q^k) \end{aligned}$$

Goulden and Jackson [7] give a similar derivation of (1.3) from (1.1). Moreover, they give an interpretation of the polynomials $Q_n(x,y) = P_n(x,-y)$ in terms of q-counting of certain permutations (bimodal permutations). The following exchange property of $Q_n(x,y)$ is given by Goulden and Jackson [7]

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} Q_k(x,y) Q_{n-k}(w,z) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} Q_k(w,y) Q_{n-k}(x,z).$$

Note that there is a notation for $Q_n(x,y)$ in the literature following F. H. Jackson [9] as mentioned by Johnson [11]:

$$(x+y)^{[n]} = (x+y)(x+qy)\cdots(x+q^{n-1}y).$$

Because the polynomials $P_n(x,y)$ occur so often in q-series that they may deserve a name. We propose to call them the *Cauchy polynomials* for the reason that they are the coefficients in the expansion of the homogenous version of the Cauchy identity (or the q-binomial theorem):

$$\sum_{n=0}^{\infty} \frac{P_n(x,y)}{(q;q)_n} t^n = \frac{(yt;q)_\infty}{(xt;q)_\infty}. \tag{1.4}$$

Setting $y=0$, the Cauchy identity becomes Euler's identity:

$$\frac{1}{(xt;q)_\infty} = \sum_{n=0}^{\infty} \frac{x^n t^n}{(q;q)_n}. \tag{1.5}$$

It seems to be neglected that the q-binomial theorem of Goldman and Rota, and the above exchange property of $Q_n(x, y)$ both are immediate from the above homogeneous form of the Cauchy identity.

The main result of this paper is to introduce the operator D_{xy} on functions in two variables x and y. This operator turns out to be suitable for dealing with the Cauchy polynomials $P_n(x, y)$. We derive a binomial identity which unifies the two identities of Rota and Goldman, as well as the q-Vandermonde identity. Moreover, our identity can be shown to be equivalent to the Goldman-Rota binomial identity, and the it can be regarded as a homogeneous generalization of the q-Vandermonde identity.

Based on the q-Leibniz formula for the classical q-difference operator, we obtain the q-Leibniz formula for the homogeneous q-difference operator. It turns out the Cauchy polynomials also appear in the homogeneous q-Leibniz formula. In the last section, we introduce the homogeneous Rogers-Szegö polynomials and the q-shift operator. The generating function of the homogeneous Rogers-Szegö polynomials is derived.

2. The Homogeneous q-difference Operator

Recall that the classical q-difference operator, or the q-derivative, acting on functions on variable x, D_q is defined by:

$$D_q f(x) = \frac{f(x) - f(qx)}{x}.$$

Note that when the function f is in the context of hypergeometric functions, the variable x is often used as a parameter, but throughout this paper D_q is always acting on x. The operator D_q is also the Euler-Jackson difference operator [10]. It may also be expressed in terms of the q-shift operator on the variable x:

$$\eta_x f(x) = f(qx).$$

Thus, we may write

$$D_q = \frac{1 - \eta_x}{x}.$$

Notice that the inverse of η_x is denoted by $\theta_x = \eta_x^{-1}$.

Andrews [1, 2] employs the q-difference operator to study the Cauchy polynomials for the case $y = 1$, and observes the following relation:

$$D_q P_n(x, 1) = (1 - q^n) P_{n-1}(x, 1).$$

The objective of this paper to introduce a new operator which is suitable for the study of the Cauchy polynomials:

$$D_{xy}f(x,y)=\frac{f(x,q^{-1}y)-f(qx,y)}{x-q^{-1}y}, \tag{2.1}$$

where x and y are variables. We now give the frist theorem of this paper, which is straightforward to verify.

Theorem 2.1 *We have*

$$D_{xy}\{P_n(x,y)\}=(1-q^n)P_{n-1}(x,y). \tag{2.2}$$

Obviously, for any constant c, one has $D_{xy}c=0$. Moreover, one may have the following property of the q-difference operator.

Proposition 2.2 *If $f(x,y)$ and $g(x,y)$ are homogeneous polynomials of the same degree n, and $H(x,y)=\dfrac{f(x,y)}{g(x,y)}$, then we have*

$$D_{xy}H(x,y)=0.$$

From (2.2), we obtain the following property:

Proposition 2.3 *We have*

$$D_{xy}\left\{\frac{(yt;q)_\infty}{(xt;q)_\infty}\right\} = t\frac{(yt;q)_\infty}{(xt;q)_\infty}, \tag{2.3}$$

$$D_{xy}^k\left\{\frac{(yt;q)_\infty}{(xt;q)_\infty}\right\} = t^k\frac{(yt;q)_\infty}{(xt;q)_\infty}. \tag{2.4}$$

We use θ_y for the operator acting on the variable y. Clearly,

$$\theta_y\eta_x=\eta_x\theta_y. \tag{2.5}$$

We define $P_n(\theta_y,\eta_x)$ as the following operator:

$$P_n(\theta_y,\eta_x)=(\theta_y-\eta_x)(\theta_y-q\eta_x)\cdots(\theta_y-q^{n-1}\eta_x). \tag{2.6}$$

The following theorem gives the expansion of the power of D_{xy} in terms of operations on x and y individually.

Theorem 2.4 *We have*

$$D_{xy}f(x,y)=\frac{(\theta_y-\eta_x)\{f(x,y)\}}{x-q^{-1}y}, \tag{2.7}$$

$$D_{xy}^n f(x,y)=\frac{P_n(\theta_y,q^{1-n}\eta_x)\{f(x,y)\}}{P_n(x,q^{-n}y)}. \tag{2.8}$$

Proof.

$$\begin{aligned}
&D_{xy}^{n+1}\{f(x,y)\}(x-q^{-1}y)\\
&=\frac{\theta_y P_n(\theta_y,q^{1-n}\eta_x)\{f(x,y)\}}{P_n(x,q^{-n-1}y)}-\frac{\eta_x P_n(\theta_y,q^{1-n}\eta_x)\{f(x,y)\}}{P_n(qx,q^{-n}y)}\\
&=\frac{(\theta_y-q^{-n}\eta_x)P_n(\theta_y,q^{1-n}\eta_x)\{f(x,y)\}}{P_n(x,q^{-n-1}y)}\\
&=\frac{P_{n+1}(\theta_y,q^{-n}\eta_x)\{f(x,y)\}}{P_n(x,q^{-n-1}y)}.
\end{aligned}$$

∎

From (2.5) and (2.6), we have

Lemma 2.5 *We have*

$$P_n(\theta_y,\eta_x)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}(-1)^k q^{\binom{k}{2}}\eta_x^k\theta_y^{n-k}. \tag{2.9}$$

Theorem 2.4 can rewritten as:

Theorem 2.6 *The operator D_{xy}^n has the following expansion:*

$$\begin{aligned}
&D_{xy}^n\{f(x,y)\}\\
&=\frac{1}{\prod_{k=1}^{n}\theta_y^k\{x-y\}}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}(-1)^k q^{\binom{k}{2}}q^{(1-n)k}\eta_x^k\theta_y^{n-k}\{f(x,y)\}\\
&=\frac{1}{P_n(x,q^{-n}y)}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}(-1)^k q^{\binom{k}{2}}q^{(1-n)k}f(q^kx,q^{k-n}y).
\end{aligned}$$

From (2.4) and Theorem 2.6, we have

$$
\begin{aligned}
& D_{xy}^{n}\left\{\frac{(yt;q)_\infty}{(xt;q)_\infty}\right\} \\
= \quad & \frac{1}{P_n(x,q^{-n}y)} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (-1)^k q^{\binom{k}{2}} q^{(1-n)k} \frac{(q^{k-n}yt;q)_\infty}{(q^k xt;q)_\infty} \\
= \quad & \frac{(yt;q)_\infty}{(xt;q)_\infty} \frac{1}{P_n(x,q^{-n}y)} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (-1)^k q^{\binom{k}{2}} q^{(1-n)k} (xt;q)_k (q^{k-n}yt;q)_{n-k}.
\end{aligned}
$$

We now arrive at the following identity:

$$
t^n P_n(x,q^{-n}y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (-1)^k q^{\binom{k}{2}} q^{(1-n)k} (xt;q)_k (q^{k-n}yt;q)_{n-k}. \tag{2.10}
$$

Note that the above identity is an equivalent form of the Goldman-Rota q-binomial identity. However, this form has the advantage of specializing to the inverse Goldman-Rota identity (1.3) and it can be viewed as a homogeneous version of the q-Vandermonde identity:

$$
{}_2\phi_1(q^{-n},x;y;q,q) = \frac{(y/x;q)_n}{(y;q)_n} x^n, \tag{2.11}
$$

For given n, we may specialize the values of the parameters in (2.10) to obtain some classical results.

- Setting $t \to 1/z$, $q^{-1}y \to y$, and exchanging x and y, we obtain Goldman-Rota q-binomial identity(1.1). Thus, we may say that the formula (2.10) is equivalent to the Goldman-Rota q-binomial theorem.
- Setting $t \to 1$ and $q^{-n}y \to y$, we obtain the q-Vandermonde identity (2.11). Indeed, setting $1/t \to z$ and $q^{-n}y \to y$ one may rewrite (2.10) in the following form:

$$
P_n(x,y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{(1-n)k} P_k(q^{k-1}x,z)\, P_{n-k}(z,q^k y).
$$

- Setting $t \to q^{1-n}$ and $q^{-n}y \to y$, we get the inverse Goldman-Rota identity (1.3). In (1.3), setting $1/y$ by y and $1/x$ by x then setting $n \to \infty$, we obtain the following identity [6]:

$$
{}_1\phi_1(y;x;q,x/y) = \frac{(x/y;q)_\infty}{(x;q)_\infty}.
$$

3. The homogeneous q-Leibniz formula

In this section, we give the homogeneous q-Leibniz formula for the operator D_{xy}. In order to present a non-inductive proof, we will use the q-Leibniz formula for the classical q-difference operator D_q [13, 14]

$$D_q^n\{f(x)g(x)\} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{k(k-n)} D_q^k\{f(x)\} D_q^{n-k}\{g(q^k x)\}.$$

Theorem 3.7 *For $n \geq 0$, we have*

$$\begin{aligned} & D_{xy}^n\{f(x,y)g(x,y)\} \\ = & \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{P_{n-k}(q^{-1}y, x)}{P_{n-k}(q^{-1}y, q^k x)} D_{xy}^k\{g(q^{n-k}x, y)\} D_{xy}^{n-k}\{f(x, q^{-k}y)\}. \end{aligned}$$

Proof. Let $y = xzq$, then we have $F(x,z) = f(x,y)$, and $G(x,z) = g(x,y)$. It follows that

$$D_{xy} = \frac{1}{1-z} D_q \theta_z \tag{3.12}$$

and

$$D_q \theta_z = \theta_z D_q. \tag{3.13}$$

Therefore,

$$D_{xy}^k = \frac{1}{(q^{1-k}z;q)_k} D_q^k \theta_z^k. \tag{3.14}$$

Thus, we have

$$\begin{aligned} & D_{xy}^n\{f(x,y)g(x,y)\} \\ & = \frac{1}{(q^{1-n}z;q)_n} D_q^n \theta_z^n \{F(x,z)G(x,z)\} \\ & = \frac{1}{(q^{1-n}z;q)_n} \theta_z^n D_q^n \{F(x,z)G(x,z)\} \\ & = \frac{1}{(q^{1-n}z;q)_n} \theta_z^n \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{k(k-n)} D_q^k\{F(x,z)\} D_q^{n-k}\{G(q^k x, z)\} \\ & = \frac{1}{(q^{1-n}z;q)_n} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{k(k-n)} D_q^k \theta_z^k\{F(x, q^{k-n}z)\} D_q^{n-k} \theta_z^{n-k}\{G(q^k x, q^{-k}z)\} \\ & = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{P_k(q^{-1}y, x)}{P_k(q^{-1}y, q^{n-k}x)} D_{xy}^k\{f(x, q^{k-n}y)\} D_{xy}^{n-k}\{g(q^k x, y)\} \\ & = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{P_{n-k}(q^{-1}y, x)}{P_{n-k}(q^{-1}y, q^k x)} D_{xy}^k\{g(q^{n-k}x, y)\} D_{xy}^{n-k}\{f(x, q^{-k}y)\}. \end{aligned}$$

∎

Clearly, setting $z = 0$, namely, $y = 0$, we have:

$$D_{xy}^k = D_q^k.$$

Corollary 3.8 *We have*

$$D_{xy}^n\{f(x,y)g(x)\} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{(-x)^k q^{\binom{k}{2}}}{P_k(q^{-1}y, q^{n-k}x)} D_q^k\{g(q^{n-k}x)\} D_{xy}^{n-k}\{f(x, q^{-k}y)\}.$$

4. The homogeneous q-shift operator

Based on the homogeneous q-difference operator, one can build up the homogeneous q-shift operator as the q-exponential of the homogeneous q-difference operator:

$$\mathbb{E}(D_{xy}) = \sum_{k=0}^{\infty} \frac{D_{xy}^k}{(q;q)_k}. \tag{4.15}$$

The following proposition for the homogeneous q-shift operator immediately follows from Proposition 2.3:

Proposition 4.9 *We have*

$$\mathbb{E}(D_{xy}) \left\{ \frac{(yt;q)_\infty}{(xt;q)_\infty} \right\} = \frac{(yt;q)_\infty}{(t;q)_\infty (xt;q)_\infty}.$$

The q-shift operator is suitable for the study of the homogeneous Rogers-Szegö polynomials which are defined by

$$h_n(x,y|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} P_k(x,y).$$

Note that setting $y = 0$ the polynomials $h_n(x,y)$ reduces to the classical Rogers-Szegö polynomials $h_n(x|q)$. Recall that $h_n(x|q)$ can be expressed in terms of the q-shift operator $T(D_q)x^n$, where

$$T(D_q) = \sum_{n=0}^{\infty} \frac{D_q^n}{(q;q)_n}.$$

The operator $T(D_q)$ called the augmentation operator in [4], which can be used to derive the generating function of $h_n(x|q)$:

$$\sum_{n=0}^{\infty} \frac{h_n(x|q)t^n}{(q;q)_n} = \frac{1}{(t;q)_\infty (xt;q)_\infty} \tag{4.16}$$

From (2.2), we obtain the following formula:

$$E(D_{xy})\{P_n(x,y)\} = h_n(x,y|q). \tag{4.17}$$

Next we present the generating function for the homogeneous Roger-Szegö polynomials.

Theorem 4.10 *We have*

$$\sum_{n=0}^{\infty} \frac{h_n(x,y|q)t^n}{(q;q)_n} = \frac{(yt;q)_\infty}{(t;q)_\infty (xt;q)_\infty}.$$

Proof. By Proposition 4.9, we have

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{h_n(x,y|q)t^n}{(q;q)_n} &= E(D_{xy}) \left\{ \frac{P_n(x,y)t^n}{(q;q)_n} \right\} \\
&= E(D_{xy}) \left\{ \frac{(yt;q)_\infty}{(xt;q)_\infty} \right\} \\
&= \frac{(yt;q)_\infty}{(t;q)_\infty (xt;q)_\infty}.
\end{aligned}$$

This completes the proof. ∎

Setting $y = 1$ in the above theorem, by Euler's identity (1.5) we are led to the evaluation $h_n(x,1|q) = x^n$, which is the Cauchy identity (1.2).

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