

The Homogeneous q -Difference Operator

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Abstract. We introduce a q -differential operator D_{xy} on functions in two variables which turns out to be suitable for dealing with the homogeneous form of the q -binomial theorem as studied by Andrews, Goldman and Rota, Roman, Ihrig and Ismail, et al. The homogeneous versions of the q -binomial theorem and the Cauchy identity are often useful for their specializations of the two parameters. Using this operator, we derive an equivalent form of the Goldman-Rota binomial identity and show that it is a homogeneous generalization of the q -Vandermonde identity. Moreover, the inverse identity of Goldman and Rota also follows from our unified identity. We also obtain the q -Leibniz formula for this operator. In the last section, we introduce the homogeneous Rogers-Szegö polynomials and derive their generating function by using the homogeneous q -shift operator.

Keywords: q -binomial theorem, Cauchy polynomials, q -Vandermonde identity, homogeneous q -difference operator, q -Leibniz formula, homogeneous Rogers-Szegö polynomials

1. Introduction

We adopt the common conventions and notations on q -series. So we always assume that $|q| < 1$ and use the following notation of the q -shifted factorial:

$$(x; q)_0 = 1; (x; q)_n = \prod_{j=0}^{n-1} (1 - q^j x), n = 1, 2, \dots, \infty.$$

The basic hypergeometric series ${}_r\phi_s$ is defined as follows [6]:

$$\begin{aligned} {}_r\phi_s(x_1, x_2, \dots, x_r; y_1, y_2, \dots, y_s; q, t) &= {}_r\phi_s \left[\begin{matrix} x_1, x_2, \dots, x_r \\ y_1, y_2, \dots, y_s \end{matrix} ; q, t \right] \\ &= \sum_{n=0}^{\infty} \frac{(x_1; q)_n (x_2; q)_n \cdots (x_r; q)_n}{(y_1; q)_n (y_2; q)_n \cdots (y_s; q)_n} \left[(-1)^n q^{\binom{n}{2}} \right]^{1+s-r} t^n, \end{aligned}$$

where $q \neq 0$ when $r > s + 1$.

The q -binomial coefficient is given by:

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q; q)_n}{(q; q)_{n-k} (q; q)_k}.$$

The following is the homogeneous form of the q -shifted factorial:

$$P_n(x, y) = (y/x; q)_n x^n = (x - y)(x - qy) \cdots (x - q^{n-1}y).$$

We also have the following basic relations:

$$\begin{aligned} \begin{bmatrix} n \\ k \end{bmatrix} (-1)^k q^{\binom{k}{2}} &= \frac{(q^{-n}; q)_k q^{nk}}{(q; q)_k}, \\ P_n(x, y) &= (-1)^n q^{\binom{n}{2}} P_n(y, q^{1-n}x), \\ P_{n-k}(x, q^{1-n}y) &= (-1)^{n-k} q^{\binom{k}{2} - \binom{n}{2}} P_{n-k}(y, q^k x). \end{aligned}$$

The polynomials $P_n(x, y)$ are important in the q -umbral calculus as studied by Andrews [1, 2], Goldman-Rota [5], Goulden-Jackson [7], Ihrig and Ismail [8], Roman [13], Johnson[11], et al. In the q -umbral calculus, the polynomial sequence $P_n(x, y)$ is a homogeneous Eulerian family. By vector space arguments, Goldman and Rota [5] have shown the following q -binomial identity, which we call the Goldman-Rota q -binomial theorem. This identity may be known earlier, but we do not have accurate information on the reference:

$$P_n(x, y) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} P_k(x, z) P_{n-k}(z, y). \quad (1.1)$$

Let V_n be an n -dimensional vector space over the finite field of q elements, and X, Y, Z be vector spaces over $GF(q)$ such that $|X| = x$, $|Y| = y$ and $|Z| = z$ where $|X|$ denotes the number of vectors in X . Assuming that $Z \subset Y \subset X$ and $\dim V_n < \dim Z$, Goldman and Rota [5] show that the above identity counts in two ways the set of all one-to-one

linear transformations $f: V_n \rightarrow X$ such that $f^{-1}(Z) = 0$. Setting $y = 0$ and $z = 1$ in (1.1), one obtains the following identity due to Cauchy:

$$x^n = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} (x-1)(x-q) \cdots (x-q^{k-1}). \quad (1.2)$$

Note that the polynomials $P_n(x, 1) = (x-1)(x-q) \cdots (x-q^{n-1})$ are sometimes called the Gauss polynomials. A direct combinatorial argument for the above identity of Cauchy is also given by Goldman and Rota [5]. For further background on the above q -binomial theorem and its specializations, the reader is referred to the introduction written by Kung [12]. Moreover, by Möbius inversion, Goldman and Rota obtain an identity which leads to a partition identity, generalizing Durfee's identity.

$$P_n(x, y) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} (-1)^k q^{\binom{k}{2}} P_k(y, 1) P_{n-k}(x, q^k). \quad (1.3)$$

It was not obvious how to show the equivalence of the above two q -binomial theorems (1.1) and (1.3). Here we give a derivation:

$$\begin{aligned} P_n(x, y) &= (-1)^n q^{\binom{n}{2}} P_n(y, q^{1-n}x) \\ &= (-1)^n q^{\binom{n}{2}} \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} P_k(y, 1) P_{n-k}(1, q^{1-n}x) \\ &= \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} (-1)^k q^{\binom{k}{2}} P_k(y, 1) P_{n-k}(x, q^k) \end{aligned}$$

Goulden and Jackson [7] give a similar derivation of (1.3) from (1.1). Moreover, they give an interpretation of the polynomials $Q_n(x, y) = P_n(x, -y)$ in terms of q -counting of certain permutations (bimodal permutations). The following exchange property of $Q_n(x, y)$ is given by Goulden and Jackson [7]

$$\sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} Q_k(x, y) Q_{n-k}(w, z) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} Q_k(w, y) Q_{n-k}(x, z).$$

Note that there is a notation for $Q_n(x, y)$ in the literature following F. H. Jackson [9] as mentioned by Johnson [11]:

$$(x+y)^{[n]} = (x+y)(x+qy) \cdots (x+q^{n-1}y).$$

Because the polynomials $P_n(x, y)$ occur so often in q -series that they may deserve a name. We propose to call them the *Cauchy polynomials* for the reason that they are

the coefficients in the expansion of the homogenous version of the Cauchy identity (or the q -binomial theorem):

$$\sum_{n=0}^{\infty} \frac{P_n(x, y)}{(q; q)_n} t^n = \frac{(yt; q)_{\infty}}{(xt; q)_{\infty}}. \quad (1.4)$$

Setting $y = 0$, the Cauchy identity becomes Euler's identity:

$$\frac{1}{(xt; q)_{\infty}} = \sum_{n=0}^{\infty} \frac{x^n t^n}{(q; q)_n}. \quad (1.5)$$

It seems to be neglected that the q -binomial theorem of Goldman and Rota, and the above exchange property of $Q_n(x, y)$ both are immediate from the above homogeneous form of the Cauchy identity.

The main result of this paper is to introduce the operator D_{xy} on functions in two variables x and y . This operator turns out to be suitable for dealing with the Cauchy polynomials $P_n(x, y)$. We derive a binomial identity which unifies the two identities of Rota and Goldman, as well as the q -Vandermonde identity. Moreover, our identity can be shown to be equivalent to the Goldman-Rota binomial identity, and it can be regarded as a homogeneous generalization of the q -Vandermonde identity.

Based on the q -Leibniz formula for the classical q -difference operator, we obtain the q -Leibniz formula for the homogeneous q -difference operator. It turns out the Cauchy polynomials also appear in the homogeneous q -Leibniz formula. In the last section, we introduce the homogeneous Rogers-Szegö polynomials and the q -shift operator. The generating function of the homogeneous Rogers-Szegö polynomials is derived.

2. The Homogeneous q -difference Operator

Recall that the classical q -difference operator, or the q -derivative, acting on functions on variable x , D_q is defined by:

$$D_q f(x) = \frac{f(x) - f(qx)}{x}.$$

Note that when the function f is in the context of hypergeometric functions, the variable x is often used as a parameter, but throughout this paper D_q is always acting

on x . The operator D_q is also the Euler-Jackson difference operator [10]. It may also be expressed in terms of the q -shift operator on the variable x :

$$\eta_x f(x) = f(qx).$$

Thus, we may write

$$D_q = \frac{1 - \eta_x}{x}.$$

Notice that the inverse of η_x is denoted by $\theta_x = \eta_x^{-1}$.

Andrews [1, 2] employs the q -difference operator to study the Cauchy polynomials for the case $y = 1$, and observes the following relation:

$$D_q P_n(x, 1) = (1 - q^n) P_{n-1}(x, 1).$$

The objective of this paper to introduce a new operator which is suitable for the study of the Cauchy polynomials:

$$D_{xy} f(x, y) = \frac{f(x, q^{-1}y) - f(qx, y)}{x - q^{-1}y}, \quad (2.1)$$

where x and y are variables. We now give the first theorem of this paper, which is straightforward to verify.

Theorem 2.1 *We have*

$$D_{xy} \{P_n(x, y)\} = (1 - q^n) P_{n-1}(x, y). \quad (2.2)$$

Obviously, for any constant c , one has $D_{xy} c = 0$. Moreover, one may have the following property of the q -difference operator.

Proposition 2.2 *If $f(x, y)$ and $g(x, y)$ are homogeneous polynomials of the same degree n , and $H(x, y) = \frac{f(x, y)}{g(x, y)}$, then we have*

$$D_{xy} H(x, y) = 0.$$

From (2.2), we obtain the following property:

Proposition 2.3 *We have*

$$D_{xy} \left\{ \frac{(yt; q)_\infty}{(xt; q)_\infty} \right\} = t \frac{(yt; q)_\infty}{(xt; q)_\infty}, \quad (2.3)$$

$$D_{xy}^k \left\{ \frac{(yt; q)_\infty}{(xt; q)_\infty} \right\} = t^k \frac{(yt; q)_\infty}{(xt; q)_\infty}. \quad (2.4)$$

We use θ_y for the operator acting on the variable y . Clearly,

$$\theta_y \eta_x = \eta_x \theta_y. \quad (2.5)$$

We define $P_n(\theta_y, \eta_x)$ as the following operator:

$$P_n(\theta_y, \eta_x) = (\theta_y - \eta_x)(\theta_y - q\eta_x) \cdots (\theta_y - q^{n-1}\eta_x). \quad (2.6)$$

The following theorem gives the expansion of the power of D_{xy} in terms of operations on x and y individually.

Theorem 2.4 *We have*

$$D_{xy} f(x, y) = \frac{(\theta_y - \eta_x) \{f(x, y)\}}{x - q^{-1}y}, \quad (2.7)$$

$$D_{xy}^n f(x, y) = \frac{P_n(\theta_y, q^{1-n}\eta_x) \{f(x, y)\}}{P_n(x, q^{-n}y)}. \quad (2.8)$$

Proof.

$$\begin{aligned} & D_{xy}^{n+1} \{f(x, y)\} (x - q^{-1}y) \\ &= \frac{\theta_y P_n(\theta_y, q^{1-n}\eta_x) \{f(x, y)\}}{P_n(x, q^{-n-1}y)} - \frac{\eta_x P_n(\theta_y, q^{1-n}\eta_x) \{f(x, y)\}}{P_n(qx, q^{-n}y)} \\ &= \frac{(\theta_y - q^{-n}\eta_x) P_n(\theta_y, q^{1-n}\eta_x) \{f(x, y)\}}{P_n(x, q^{-n-1}y)} \\ &= \frac{P_{n+1}(\theta_y, q^{-n}\eta_x) \{f(x, y)\}}{P_n(x, q^{-n-1}y)}. \end{aligned}$$

■

From (2.5) and (2.6), we have

Lemma 2.5 *We have*

$$P_n(\theta_y, \eta_x) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} (-1)^k q^{\binom{k}{2}} \eta_x^k \theta_y^{n-k}. \quad (2.9)$$

Theorem 2.4 can be rewritten as:

Theorem 2.6 *The operator D_{xy}^n has the following expansion:*

$$\begin{aligned} & D_{xy}^n \{f(x, y)\} \\ &= \frac{1}{\prod_{k=1}^n \theta_y^k \{x - y\}} \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} (-1)^k q^{\binom{k}{2}} q^{(1-n)k} \eta_x^k \theta_y^{n-k} \{f(x, y)\} \\ &= \frac{1}{P_n(x, q^{-n}y)} \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} (-1)^k q^{\binom{k}{2}} q^{(1-n)k} f(q^k x, q^{k-n}y). \end{aligned}$$

From (2.4) and Theorem 2.6, we have

$$\begin{aligned} & D_{xy}^n \left\{ \frac{(yt; q)_\infty}{(xt; q)_\infty} \right\} \\ &= \frac{1}{P_n(x, q^{-n}y)} \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} (-1)^k q^{\binom{k}{2}} q^{(1-n)k} \frac{(q^{k-n}yt; q)_\infty}{(q^k xt; q)_\infty} \\ &= \frac{(yt; q)_\infty}{(xt; q)_\infty} \frac{1}{P_n(x, q^{-n}y)} \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} (-1)^k q^{\binom{k}{2}} q^{(1-n)k} (xt; q)_k (q^{k-n}yt; q)_{n-k}. \end{aligned}$$

We now arrive at the following identity:

$$t^n P_n(x, q^{-n}y) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} (-1)^k q^{\binom{k}{2}} q^{(1-n)k} (xt; q)_k (q^{k-n}yt; q)_{n-k}. \quad (2.10)$$

Note that the above identity is an equivalent form of the Goldman-Rota q -binomial identity. However, this form has the advantage of specializing to the inverse Goldman-Rota identity (1.3) and it can be viewed as a homogeneous version of the q -Vandermonde identity:

$${}_2\phi_1(q^{-n}, x; y; q, q) = \frac{(y/x; q)_n}{(y; q)_n} x^n, \quad (2.11)$$

For given n , we may specialize the values of the parameters in (2.10) to obtain some classical results.

- Setting $t \rightarrow 1/z$, $q^{-1}y \rightarrow y$, and exchanging x and y , we obtain Goldman-Rota q -binomial identity(1.1). Thus, we may say that the formula (2.10) is equivalent to the Goldman-Rota q -binomial theorem.
- Setting $t \rightarrow 1$ and $q^{-n}y \rightarrow y$, we obtain the q -Vandermonde identity (2.11). Indeed, setting $1/t \rightarrow z$ and $q^{-n}y \rightarrow y$ one may rewrite (2.10) in the following form:

$$P_n(x, y) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} q^{(1-n)k} P_k(q^{k-1}x, z) P_{n-k}(z, q^k y).$$

- Setting $t \rightarrow q^{1-n}$ and $q^{-n}y \rightarrow y$, we get the inverse Goldman-Rota identity (1.3). In (1.3), setting $1/y$ by y and $1/x$ by x then setting $n \rightarrow \infty$, we obtain the following identity [6]:

$${}_1\phi_1(y; x; q, x/y) = \frac{(x/y; q)_\infty}{(x; q)_\infty}.$$

3. The homogeneous q -Leibniz formula

In this section, we give the homogeneous q -Leibniz formula for the operator D_{xy} . In order to present a non-inductive proof, we will use the q -Leibniz formula for the classical q -difference operator D_q [13, 14]

$$D_q^n \{f(x)g(x)\} = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} q^{k(k-n)} D_q^k \{f(x)\} D_q^{n-k} \{g(q^k x)\}.$$

Theorem 3.7 *For $n \geq 0$, we have*

$$\begin{aligned} & D_{xy}^n \{f(x, y)g(x, y)\} \\ &= \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} \frac{P_{n-k}(q^{-1}y, x)}{P_{n-k}(q^{-1}y, q^k x)} D_{xy}^k \{g(q^{n-k}x, y)\} D_{xy}^{n-k} \{f(x, q^{-k}y)\}. \end{aligned}$$

Proof. Let $y = xzq$, then we have $F(x, z) = f(x, y)$, and $G(x, z) = g(x, y)$. It follows that

$$D_{xy} = \frac{1}{1-z} D_q \theta_z \quad (3.12)$$

and

$$D_q \theta_z = \theta_z D_q. \quad (3.13)$$

Therefore,

$$D_{xy}^k = \frac{1}{(q^{1-k}z; q)_k} D_q^k \theta_z^k. \quad (3.14)$$

Thus, we have

$$\begin{aligned} & D_{xy}^n \{f(x, y)g(x, y)\} \\ &= \frac{1}{(q^{1-n}z; q)_n} D_q^n \theta_z^n \{F(x, z)G(x, z)\} \\ &= \frac{1}{(q^{1-n}z; q)_n} \theta_z^n D_q^n \{F(x, z)G(x, z)\} \\ &= \frac{1}{(q^{1-n}z; q)_n} \theta_z^n \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} q^{k(k-n)} D_q^k \{F(x, z)\} D_q^{n-k} \{G(q^k x, z)\} \\ &= \frac{1}{(q^{1-n}z; q)_n} \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} q^{k(k-n)} D_q^k \theta_z^k \{F(x, q^{k-n}z)\} D_q^{n-k} \theta_z^{n-k} \{G(q^k x, q^{-k}z)\} \\ &= \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} \frac{P_k(q^{-1}y, x)}{P_k(q^{-1}y, q^{n-k}x)} D_{xy}^k \{f(x, q^{k-n}y)\} D_{xy}^{n-k} \{g(q^k x, y)\} \\ &= \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} \frac{P_{n-k}(q^{-1}y, x)}{P_{n-k}(q^{-1}y, q^k x)} D_{xy}^k \{g(q^{n-k}x, y)\} D_{xy}^{n-k} \{f(x, q^{-k}y)\}. \end{aligned}$$

■

Clearly, setting $z = 0$, namely, $y = 0$, we have:

$$D_{xy}^k = D_q^k.$$

Corollary 3.8 *We have*

$$D_{xy}^n \{f(x, y)g(x)\} = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} \frac{(-x)^k q^{\binom{k}{2}}}{P_k(q^{-1}y, q^{n-k}x)} D_q^k \{g(q^{n-k}x)\} D_{xy}^{n-k} \{f(x, q^{-k}y)\}.$$

4. The homogeneous q -shift operator

Based on the homogeneous q -difference operator, one can build up the homogeneous q -shift operator as the q -exponential of the homogeneous q -difference operator:

$$\mathbb{E}(D_{xy}) = \sum_{k=0}^{\infty} \frac{D_{xy}^k}{(q; q)_k}. \quad (4.15)$$

The following proposition for the homogeneous q -shift operator immediately follows from Proposition 2.3:

Proposition 4.9 *We have*

$$\mathbb{E}(D_{xy}) \left\{ \frac{(yt; q)_{\infty}}{(xt; q)_{\infty}} \right\} = \frac{(yt; q)_{\infty}}{(t; q)_{\infty}(xt; q)_{\infty}}.$$

The q -shift operator is suitable for the study of the homogeneous Rogers-Szegö polynomials which are defined by

$$h_n(x, y|q) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} P_k(x, y).$$

Note that setting $y = 0$ the polynomials $h_n(x, y)$ reduces to the classical Rogers-Szegö polynomials $h_n(x|q)$. Recall that $h_n(x|q)$ can be expressed in terms of the q -shift operator $T(D_q)x^n$, where

$$T(D_q) = \sum_{n=0}^{\infty} \frac{D_q^n}{(q; q)_n}.$$

The operator $T(D_q)$ called the augmentation operator in [4], which can be used to derive the generating function of $h_n(x|q)$:

$$\sum_{n=0}^{\infty} \frac{h_n(x|q)t^n}{(q; q)_n} = \frac{1}{(t; q)_{\infty}(xt; q)_{\infty}} \quad (4.16)$$

From (2.2), we obtain the following formula:

$$E(D_{xy})\{P_n(x, y)\} = h_n(x, y|q). \quad (4.17)$$

Next we present the generating function for the homogeneous Roger-Szegö polynomials.

Theorem 4.10 *We have*

$$\sum_{n=0}^{\infty} \frac{h_n(x, y|q)t^n}{(q; q)_n} = \frac{(yt; q)_{\infty}}{(t; q)_{\infty}(xt; q)_{\infty}}.$$

Proof. By Proposition 4.9, we have

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{h_n(x, y|q)t^n}{(q; q)_n} &= E(D_{xy}) \left\{ \frac{P_n(x, y)t^n}{(q; q)_n} \right\} \\ &= E(D_{xy}) \left\{ \frac{(yt; q)_{\infty}}{(xt; q)_{\infty}} \right\} \\ &= \frac{(yt; q)_{\infty}}{(t; q)_{\infty}(xt; q)_{\infty}}. \end{aligned}$$

This completes the proof. ■

Setting $y = 1$ in the above theorem, by Euler's identity (1.5) we are led to the evaluation $h_n(x, 1|q) = x^n$, which is the Cauchy identity (1.2).

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