An Involution for the Gauss Identity

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Abstract. We present an involution for a classical identity on the alternate sum of the Gauss coefficients in terms of the traditional Ferrers diagram. It turns out that the refinement of our involution with restrictions on the height of Ferrers diagram implies a generalization of the Gauss identity, which is a terminating form of the q-Kummer identity. Furthermore, we extend the Gauss identity to the p-th root of unity.

Keywords: involution, Ferrers diagram, Gauss identity, Gauss coefficients, q-Kummer identity.

1. Introduction

We use the standard notation on q-series. The *q-shifted factorial* $(a;q)_n$ is defined by

$$(a;q)_n = \begin{cases} 1, & n = 0 \\ (1-a)(1-aq)\cdots(1-aq^{n-1}) & n = 1, 2, \ldots. \end{cases}$$

The *q-binomial coefficient*, or the *Gauss coefficient*, is given by

$$\begin{bmatrix} n \\ k \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}.$$

Note that the parameter q is often omitted in the notation of Gauss coefficients when no confusion arises. The *Rogers-Szegö polynomial*

$$h_n(x|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^k$$

plays an important role in the theory of basic hypergeometric series [1]. In particular, for $x = -1$, we have the following evaluation which is due to Gauss [4]:

Theorem 1.1 (Gauss) *We have*

$$\sum_{r=0}^{m} (-1)^r \begin{bmatrix} m \\ r \end{bmatrix} = \begin{cases} 0, & \text{if } m \text{ is odd,} \\ (1-q)(1-q^3)\cdots(1-q^{m-1}), & \text{if } m \text{ is even.} \end{cases} \tag{1.1}$$

There have been several proofs of this identity. Rademacher proves it in order to determine the sign of quadratic Gauss sums [9]. It is also a by-product of results of Littlewood [7] on the evaluation of symmetric functions at roots of unity, and plethysm with power sums (cf. [6]). Kupershmidt finds a generalization of the Gauss identity [5]. Andrews gives a combinatorial proof in terms of generating functions for partitions with certain properties associated with successive ranks [2]. In this paper, we obtain a simple combinatorial proof of this identity based on Ferrers diagrams. Through a refinement of our involution by considering the heights of the Ferrers diagrams, we get a generalization of the Gauss identity, which is a terminating form of the q-Kummer identity. Moreover, we give an extension of the Gauss identity to the p-th root of unity.

2. An Involution for the Gauss Identity

Our combinatorial setting for the proof of the Gauss identity is based on the following equivalent form:

$$\sum_{r=0}^{m}(-1)^r \frac{q^r}{(q;q)_r} \cdot \frac{q^{m-r}}{(q;q)_{m-r}} = \begin{cases} 0, & \text{if } m \text{ is odd,} \\ \dfrac{q^m}{(q^2;q^2)_{m/2}}, & \text{if } m \text{ is even.} \end{cases} \tag{2.1}$$

We proceed to describe our involution for the above identity. First, let us recall the standard notation on partitions used in [8]. The set of nonnegative integer is denoted by $\mathbb{N}$. A partition λ is a sequence of nonnegative integers

$$(\lambda_1, \lambda_2, \ldots, \lambda_i, \ldots) \tag{2.2}$$

in decreasing order $\lambda_1 \geq \lambda_2 \geq \cdots$ and containing only a finite number of nonzero terms. If $\lambda_i = 0$ for all $i > n$, we also write λ as the finite form $(\lambda_1, \ldots, \lambda_n)$. Especially, the partition $(0, 0, \ldots)$ is denoted by 0. The nonzero λ_i in (2.2) are called the parts of λ. The number of parts and the sum of parts are called the length and the weight of λ, denoted by $\ell(\lambda)$ and $|\lambda|$, respectively. We also use the exponential notation $\lambda = 1^{m_1}2^{m_2}\cdots r^{m_r}\cdots$ for a partition with exactly m_i parts equal to i.

Let $\lambda = (\lambda_1, \ldots, \lambda_i, \ldots)$, $\mu = (\mu_1, \ldots, \mu_i, \ldots)$ be two partitions. We define the addition of λ and μ to be the partition $\lambda + \mu = (\lambda_1 + \mu_1, \ldots, \lambda_i + \mu_i, \ldots)$. The substraction of two partitions $\lambda - \mu$ is defined similarly, given that the resulting partition exists. The Ferrers diagram of a partition is a left-justified array of n squares with λ_i squares in the i-th row.

Let $\mathcal{P}_r$ be the set of partitions λ with maximal component r. Define $\mathcal{W}_r$ and $\mathcal{W}$ as follows:

$$\mathcal{W}_r := \{(\lambda, \mu; r) : \lambda \in \mathcal{P}_r, \mu \in \mathcal{P}_{m-r}\}, \qquad \mathcal{W} := \bigcup_{r=0}^{m} \mathcal{W}_r.$$

It's easy to see that

$$\sum_{\lambda \in \mathcal{P}_r} q^{|\lambda|} = \frac{q^r}{(q;q)_r},$$

and hence, the left hand side of (2.1) becomes

$$\begin{aligned}
&\sum_{r=0}^{m}(-1)^r\left[\left(\sum_{\lambda\in\mathcal{P}_r} q^{|\lambda|}\right)\left(\sum_{\mu\in\mathcal{P}_{m-r}} q^{|\mu|}\right)\right]\\
&\quad= \sum_{r=0}^{m}(-1)^r \sum_{(\lambda,\mu;r)\in\mathcal{W}_r} q^{|\lambda|+|\mu|}\\
&\quad= \sum_{(\lambda,\mu;r)\in\mathcal{W}} (-1)^r q^{|\lambda|+|\mu|}. \qquad (2.3)
\end{aligned}$$

For $(\lambda,\mu;r)\in\mathcal{W}$, let $s=\ell(\lambda)$ and $t=\ell(\mu)$. We define a map $\sigma\colon (\lambda,\mu;r)\longmapsto (\lambda',\mu';r')$ as follows:

Case 1: $s<t$

Let

$$\lambda'=\lambda+1^t,\ \mu'=\mu-1^t,\ \text{and } r'=r+1.$$

Clearly, $\lambda'\in\mathcal{P}_{r'}$ and $\mu'\in\mathcal{P}_{m-r'}$. Noting that $t>0$, r must be less than m, and hence $(\lambda',\mu';r')\in\mathcal{W}$ (The Ferrers diagrams are shown in Figure 1).

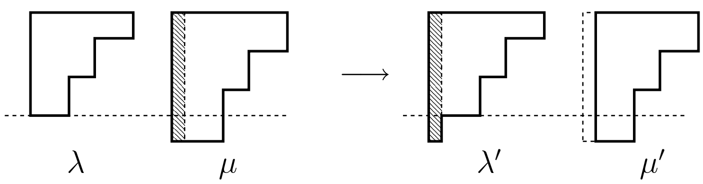

Figure 1: Case $\ell(\lambda)<\ell(\mu)$.

Case 2: $s\geq t$

2.1 There exists at least one odd number among $\lambda_{t+1},\ldots,\lambda_s$.

Suppose λ_p is odd, while $\lambda_{p+1},\ldots,\lambda_s$ are all even. Let

$$\lambda'=\lambda-1^p,\ \mu'=\mu+1^p,\ \text{and } r'=r-1.$$

Noting that $\lambda_p>0$, which implies $r=\lambda_1>0$, we have $(\lambda',\mu';r')\in\mathcal{W}$ (see Figure 2).

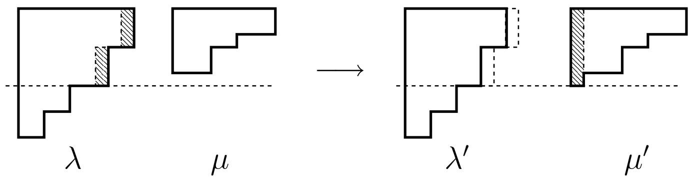

Figure 2: Case λ_p is odd.

2.2 $\lambda_{t+1}, \ldots, \lambda_s$ are all even, and $t > 0$.

If λ_t is odd, we define

$$\lambda' = \lambda - 1^t, \ \mu' = \mu + 1^t, \ \text{and } r' = r - 1.$$

Then, $(\lambda', \mu'; r') \in \mathcal{W}$. (see Figure 3).

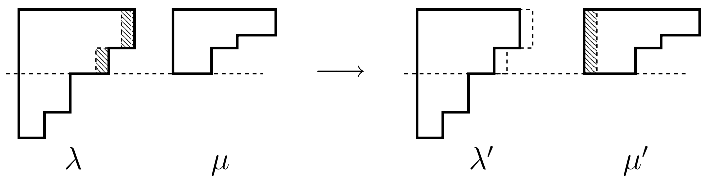

Figure 3: Case λ_t is odd.

If λ_t is even, similar to Case 1, we define

$$\lambda' = \lambda + 1^t, \ \mu' = \mu_1 - 1^t, \ \text{and } r' = r + 1.$$

Then, $(\lambda', \mu'; r') \in \mathcal{W}$.(see Figure 4).

2.3 $\lambda_1, \ldots, \lambda_s$ are all even, and $t = 0$.

Since $t = 0$, the partition μ is the zero partition, whose maximal part is regarded as 0. From the definition of $\mathcal{W}$, $m - r = 0$. In this situation, we define the image of $(\lambda, 0; m)$ to be itself.

We can check that σ^2 is the identity map on W. In fact,

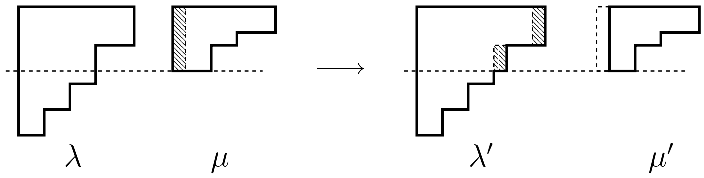

Figure 4: Case λ_t is even.

- In Case 1, we have $\ell(\lambda') = t \geq \ell(\mu')$ and $\lambda'_t = 1$. Hence $\sigma(\lambda', \mu'; r') = (\lambda' - 1^t, \mu' + 1^t; r' - 1) = (\lambda, \mu; r)$.

- In Case 2.1, there are two occasions. One is $\ell(\lambda') \geq p = \ell(\mu') > 0$ and λ'_i are all even for $i \geq p$. Hence, $\sigma(\lambda', \mu'; r') = (\lambda' + 1^p, \mu' - 1^p; r' + 1) = (\lambda, \mu; r)$. The other occasion is $\ell(\lambda') < p = \ell(\mu')$. We also have $\sigma(\lambda', \mu'; r') = (\lambda' + 1^p, \mu' - 1^p; r' + 1) = (\lambda, \mu; r)$.

- In Case 2.2, there are also two occasions. One is that $\ell(\lambda') = \ell(\lambda)$. In this situation, it's clear that $\sigma(\lambda', \mu'; r') = (\lambda, \mu; r)$. The other occasion is $\ell(\lambda') < \ell(\lambda)$. Then $s = t$ and $\lambda_s = 1$. Hence $\ell(\lambda') < \ell(\mu) = t$ and $\sigma(\lambda', \mu'; r') = (\lambda' + 1^t, \mu' - 1^t; r' + 1) = (\lambda, \mu; r)$

- In Case 2.3, σ is the identity map.

All together, σ^2 becomes the identity map on W.

It is clear that $|\lambda'| + |\mu'| = |\lambda| + |\mu|$. Furthermore, except for Case 2.3, we have $|r - r'| = 1$, which implies that $(-1)^{r'} q^{|\lambda'|+|\mu'|} + (-1)^r q^{|\lambda|+|\mu|} = 0$. Hence,

$$\sum_{(\lambda,\mu;r)\in\mathcal{W}} (-1)^r q^{|\lambda|+|\mu|} = \sum_{\substack{(\lambda,\mu;r) \text{ being} \\ \text{fixed point of } \sigma}} (-1)^r q^{|\lambda|+|\mu|} = \sum_{\substack{(\lambda,0;m)\in\mathcal{W}_m \\ \lambda_i \text{ all even}}} (-1)^m q^{|\lambda|}. \tag{2.4}$$

Since $(\lambda, 0; m) \in \mathcal{W}_m$ implies that $m = \lambda_1$, (2.4) becomes

$$\sum_{(\lambda,\mu;r)\in\mathcal{W}} (-1)^r q^{|\lambda|+|\mu|} = \begin{cases} 0, & \text{if } m \text{ is odd,} \\ \dfrac{q^m}{(q^2;q^2)_{m/2}}, & \text{if } m \text{ is even.} \end{cases}$$

∎

The above involution can be refined in terms of the height of the Ferrers diagram and one can derive the following identity which generalizes the Gauss identity.

Theorem 2.1 *We have*

$$\sum_{r=0}^{m}(-1)^r \begin{bmatrix} n+r \\ n \end{bmatrix} \begin{bmatrix} n+m-r \\ n \end{bmatrix} = \begin{cases} 0, & m \text{ is odd,} \\ \begin{bmatrix} n+m/2 \\ n \end{bmatrix}_{q^2}, & m \text{ is even.} \end{cases} \tag{2.5}$$

Proof. Note that the above involution σ preserves the bigger length between λ and μ. Therefore, this involution can be restricted to the partitions with length not greater than $n+1$ and the arguments are still valid. Let $\mathcal{P}_{n,r}$ be the set of partitions with maximal component r and length not greater than $n+1$. Since the generating function of partitions in $\mathcal{P}_{n,r}$ is (see [10, Proposition 1.3.19])

$$\sum_{\lambda \in \mathcal{P}_{n,r}} q^{|\lambda|} = q^r \begin{bmatrix} n+r \\ n \end{bmatrix},$$

we immediately have (2.5). ∎

Note that the identity (1.1) is the limiting case of (2.5) by taking $n \to \infty$. We now reformulate Theorem 2.1 into a symmetric form.

Theorem 2.2 *We have*

$$\sum_{r=0}^{m}(-1)^r \frac{(a;q)_r}{(q;q)_r}\frac{(a;q)_{m-r}}{(q;q)_{m-r}} = \begin{cases} 0, & m \text{ is odd,} \\ \dfrac{(a^2;q^2)_{m/2}}{(q^2;q^2)_{m/2}}, & m \text{ is even.} \end{cases} \tag{2.6}$$

Proof. We can rewrite the identity (2.5) as

$$\sum_{r=0}^{m}(-1)^r \frac{(q^n;q)_r}{(q;q)_r}\frac{(q^n;q)_{m-r}}{(q;q)_{m-r}} = \begin{cases} 0, & m \text{ is odd,} \\ \dfrac{(q^{2n};q^2)_{m/2}}{(q^2;q^2)_{m/2}}, & m \text{ is even.} \end{cases}$$

Setting $a = q^n$, we obtain the desired formulation. ∎

Note that the case of $a = 0$ in (2.6) gives the Gauss identity and the case $a = \infty$ also leads to the Gauss identity with parameter q replaced by q^{-1}. As pointed out by the referee, Theorem 2.2 is a specialization of the q-Kummer identity, which is usually stated as follows [3]:

$$\sum_{k=0}^{\infty} \frac{(a;q)_k(b;q)_k}{(q;q)_k(aq/b;q)_k}(-q/b)^k = \frac{(-q;q)_\infty(aq;q^2)_\infty(aq^2/b^2;q^2)_\infty}{(aq/b;q)_\infty(-q/b;q)_\infty}.$$

Exchanging a and b, we get

$$\sum_{k=0}^{\infty} \frac{(a;q)_k(b;q)_k}{(q;q)_k(bq/a;q)_k}(-q/a)^k = \frac{(-q;q)_\infty(bq;q^2)_\infty(bq^2/b^2;q^2)_\infty}{(bq/a;q)_\infty(-q/a;q)_\infty}.$$

Setting $b = q^{-m}$, we arrive at Theorem 2.2.

3. Generalization to the p-th root of unity

In this section, we consider a generalization of the Gauss identity to the p-th root of unity. This generalization reduces to the classical case when $p = 2$.

Theorem 3.1 *Let $\zeta = e^{\frac{2\pi i}{p}}$ be the p-th root of 1. Then*

$$\sum_{r_1+\cdots+r_p=m} \zeta^{r_1+2r_2+\cdots+pr_p} \begin{bmatrix} n+r_1 \\ n \end{bmatrix} \begin{bmatrix} n+r_2 \\ n \end{bmatrix} \cdots \begin{bmatrix} n+r_p \\ n \end{bmatrix}$$

$$= \begin{cases} 0, & \text{if } p \nmid m, \\ \begin{bmatrix} n+m/p \\ n \end{bmatrix}_{q^p}, & \text{if } p \mid m. \end{cases} \qquad (3.1)$$

Proof. Let $\mathcal{P}_{n,r}$ be the set of partitions with maximal component r and length not greater than $n+1$. Define the set W to be

$$W := \{(\lambda^1, \ldots, \lambda^p; r_1, \ldots, r_p) \colon r_1 + \cdots + r_p = m \text{ and } \lambda^k \in \mathcal{P}_{n,r_k}\},$$

and the weight of $x = (\lambda^1, \ldots, \lambda^p; r_1, \ldots, r_p) \in W$ to be

$$w(x) := \zeta^{r_1+2r_2+\cdots+pr_p} q^{|\lambda^1|+\cdots+|\lambda^p|}.$$

As stated in the proof of Theorem 2.1, the generating function of partitions in $\mathcal{P}_{n,r}$ is

$$\sum_{\lambda\in\mathcal{P}_{n,r}} q^{|\lambda|} = q^r \begin{bmatrix} n+r \\ n \end{bmatrix},$$

Therefore, (3.1) is equivalent to

$$\sum_{x\in W} w(x) = \begin{cases} 0, & \text{if } p \nmid m, \\ \displaystyle\sum_{\lambda\in\mathcal{P}_{n,m/p}} q^{p|\lambda|}, & \text{if } p \mid m. \end{cases}$$

Let $\overline{a}$ $(0 \le \overline{a} < p)$ denote the residue of a modulo p. Define

$$\begin{aligned} W_h := \{&(\lambda^1,\ldots,\lambda^p; r_1,\ldots,r_p) \in W: \\ &h \text{ is the smallest integer such that } \overline{\lambda^1_k} = \lambda^2_k = \cdots = \lambda^p_k = 0,\ \forall\, k > h\}, \end{aligned}$$

and

$$\begin{aligned} W_{h,s} := \{&(\lambda^1,\ldots,\lambda^p; r_1,\ldots,r_p) \in W_h: \\ &s \text{ is the smallest integer such that } \overline{\lambda^1_h} + \lambda^2_h + \cdots + \lambda^s_h \ge p - s + 1\}, \end{aligned}$$

where λ^i_j denote the j-th component of the partition λ^i. Noting that $\overline{\lambda^1_h} < p$, $W_{h,1} = \emptyset$ for any $h > 0$. Hence,

$$W = \biguplus_{h=0}^{n+1} W_h = W_0 \biguplus \left(\biguplus_{h=1}^{n+1} \biguplus_{s=2}^{p} W_{h,s} \right),$$

where $\biguplus$ denotes the disjoint union.

In the following, we focus on $W_{h,s}$ with $h > 0$ and $s \ge 2$.

For $x = (\lambda^1,\ldots,\lambda^p; r_1,\ldots,r_p) \in W_{h,s}$. Let

$$\begin{aligned} d_x &:= (p - s + 1) - (\overline{\lambda^1_h} + \lambda^2_h + \cdots + \lambda^{s-1}_h), \\ I_x &:= (\overline{\lambda^1_h}, \lambda^2_h, \ldots, \lambda^{s-1}_h, d_x). \end{aligned}$$

From the definition of $W_{h,s}$,

$$\overline{\lambda^1_h} + \lambda^2_h + \cdots + \lambda^{s-1}_h < p - (s-1) + 1 = p - s + 2,$$

which implies $I_x \in \mathbb{N}^s$. Moreover,

$$\lambda_h^s - d_x = (\overline{\lambda_h^1} + \lambda_h^2 + \cdots + \lambda_h^s) - (p - s + 1) \geq 0.$$

For each $I = (i_1, \ldots, i_s) \in \mathbb{N}^s$ with $|I| = i_1 + \cdots + i_s = p - s + 1$, define

$$W_{h,s,I} := \{x \in W_{h,s} \colon I_x = I\}.$$

Since I_x are uniquely determined by x, $W_{h,s}$ is the disjoint union of $W_{h,s,I}$:

$$W_{h,s} = \biguplus_{\substack{I \in \mathbb{N}^s \\ |I| = p-s+1}} W_{h,s,I}.$$

For $I = (i_1, \ldots, i_s), J = (j_1, \ldots, j_s) \in \mathbb{N}^s$ with $|I| = |J| = p - s + 1$, there is a bijection $\sigma_{I,J}$ from $W_{h,s,I}$ to $W_{h,s,J}$ defined by

$$\begin{array}{cccc} \sigma_{I,J}: & W_{h,s,I} & \to & W_{h,s,J} \\ & x & \mapsto & y, \end{array}$$

where

$$x = (\lambda^1, \ldots, \lambda^p; r_1, \ldots, r_p), \quad y = (\mu^1, \ldots, \mu^p; r'_1, \ldots, r'_p),$$

$$\mu^k = \begin{cases} \lambda^k - (i_k)^h + (j_k)^h, & \text{if } 1 \leq k \leq s, \\ \lambda^k, & s < k \leq p, \end{cases}$$

$$r'_k = \begin{cases} r_k - i_k + j_k, & \text{if } 1 \leq k \leq s, \\ r_k, & s < k \leq p. \end{cases}$$

We will show that $\sigma_{I,J}$ is well defined, i.e., $y \in W_{h,s,J}$. Since $I_x = I$, we have

$$\lambda_h^1 \geq \overline{\lambda_h^1} = i_1, \quad \lambda_h^k = i_k \text{ for } k = 2, \ldots, s-1, \quad \lambda_h^s \geq d_x = i_s.$$

Noting also that $\overline{\lambda_k^1} = \lambda_k^2 = \cdots = \lambda_k^p = 0$ for $k > h$, the difference partition $\lambda^k - (i_k)^h$ is well defined, and hence, $\mu^k = \lambda^k - (i_k)^h + (j_k)^h$ is well defined. Furthermore, the maximal components of μ^k is $r_k - i_k + j_k = r'_k$. Therefore, $y \in W$. It's easy to see that

$$(\overline{\mu_h^1}, \mu_h^2, \ldots, \mu_h^s) = (j_1, \ldots, j_{s-1}, (\lambda_h^s - d_s) + j_s). \tag{3.2}$$

Since $J \neq (0,\ldots,0)$, $(\mu_h^1,\mu_h^2,\ldots,\mu_h^s) \neq (0,\ldots,0)$, which implies that $y \in W_h$. Noting that $\mu_h^1 + \mu_h^2 + \cdots + \mu_h^t \leq |J| < p-t+1$ for $t < s$ and $\mu_h^1 + \mu_h^2 + \cdots + \mu_h^s = \lambda_h^1 + \lambda_h^2 + \cdots + \lambda_h^s \geq p-s+1$, we have $y \in W_{h,s}$. Furthermore, (3.2) and $|I_y| = |J| = p-s+1$ imply that $I_y = J$, that is, $y \in W_{h,s,J}$. Thus, we prove that $\sigma_{I,J}$ is a well defined map from $W_{h,s,I}$ to $W_{h,s,J}$.

It's easy to derive that $\sigma_{I,J} \circ \sigma_{J,I}$ and $\sigma_{J,I} \circ \sigma_{I,J}$ are the identity maps on $W_{h,s,I}$ and $W_{h,s,J}$, respectively. Therefore, $\sigma_{I,J}$ is a bijection from $W_{h,s,I}$ to $W_{h,s,J}$. Moreover,

$$
\begin{aligned}
w(\sigma_{I,J}(x)) &= w(y) \\
&= \zeta^{r'_1+2r'_2+\cdots+pr'_p} q^{|\mu^1|+\cdots+|\mu^p|} \\
&= \zeta^{r_1+2r_2+\cdots+pr_p} \zeta^{-(i_1+2i_2+\cdots+si_s)} \zeta^{j_1+2j_2+\cdots+sj_s} q^{|\lambda^1|+\cdots+|\lambda^p|} \\
&= \zeta^{-(i_1+2i_2+\cdots+si_s)} w(x) \zeta^{j_1+2j_2+\cdots+sj_s}
\end{aligned}
$$

Let $I_0 = (p-s+1,0,\ldots,0) \in \mathbb{N}^s$. Using the bijection $\sigma_{I_0,J}$, we have

$$
\begin{aligned}
\sum_{x \in W_{h,s}} w(x) &= \sum_{\substack{J \in \mathbb{N}^s \\ |J|=p-s+1}} \sum_{x \in W_{h,s,J}} w(x) \\
&= \sum_{\substack{J \in \mathbb{N}^s \\ |J|=p-s+1}} \sum_{x \in W_{h,s,I_0}} w(\sigma_{I_0,J}(x)) \\
&= \sum_{\substack{J \in \mathbb{N}^s \\ |J|=p-s+1}} \sum_{x \in W_{h,s,I_0}} \zeta^{-(p-s+1)} w(x) \zeta^{j_1+2j_2+\cdots+sj_s} \\
&= \sum_{x \in W_{h,s,I_0}} \zeta^{-(p-s+1)} w(x) \sum_{\substack{J \in \mathbb{N}^s \\ |J|=p-s+1}} \zeta^{j_1+2j_2+\cdots+sj_s}.
\end{aligned}
$$

Now,

$$\begin{aligned}
\sum_{J\in\mathbb{N}^s, |J|=p-s+1} t^{j_1+2j_2+\cdots+sj_s} &= \sum_{\substack{\lambda=1^{j_1}2^{j_2}\cdots s^{j_s}\\ j_1+\cdots+j_s=p-s+1}} t^{|\lambda|}\\
&= \sum_{\substack{\ell(\lambda)=p-s+1\\ \text{each part } \le s}} t^{|\lambda|}\\
&= \sum_{\substack{\ell(\lambda)\le s\\ \text{maximal part is } p-s+1}} t^{|\lambda|}\\
&= \sum_{\lambda\in\mathcal{P}_{s-1,p-s+1}} t^{|\lambda|}\\
&= t^{p-s+1}\frac{(t;t)_p}{(t;t)_{s-1}(t;t)_{p-s+1}}.
\end{aligned}$$

Noting that $1-\zeta^p = 0$ and $1-\zeta^k \neq 0$ for $k = 1, \ldots, p-1$,

$$\sum_{\substack{J\in\mathbb{N}^s\\ |J|=p-s+1}} \zeta^{j_1+2j_2+\cdots+sj_s} = 0, \quad \forall\, 2\le s\le p,$$

which implies that $\sum_{x\in W_{h,s}} w(x)$ vanishes for all $h>0$ and $2\le s\le p$. Hence, $\sum_{x\in W} w(x) = \sum_{x\in W_0} w(x)$. From the definition,

$$W_0 = \{(\lambda, 0, \ldots, 0; \lambda_1, 0, \ldots, 0) \in W \colon \lambda_1 = m \text{ and } p \mid \lambda_i\ \forall\, i\ge 1\}.$$

If $p\nmid m$, W_0 will be the empty set. Otherwise, there is a bijection between W_0 and $\mathcal{P}_{n,m/p}$ by dividing each component of λ by p. Hence,

$$\sum_{x\in W_0} w(x) = \begin{cases} 0, & \text{if } p\nmid m,\\ \displaystyle\sum_{\lambda\in\mathcal{P}_{n,m/p}} q^{p|\lambda|}, & \text{if } p\mid m. \end{cases}$$

∎

Setting $n\to\infty$, we get the generalized form of the Gauss identity.

Corollary 3.2 *Let $\zeta = e^{\frac{2\pi i}{p}}$ be the p-th root of 1. Then*

$$\sum_{r_1+\cdots+r_p=m} \zeta^{r_1+2r_2+\cdots+pr_p} \begin{bmatrix} m \\ r_1,\ldots,r_p \end{bmatrix} = \begin{cases} 0, & \text{if } p \nmid m, \\ \displaystyle\prod_{\substack{1\le k\le m \\ p\nmid k}} (1-q^k), & \text{if } p \mid m, \end{cases}$$

where $\left[\begin{smallmatrix} m \\ r_1,\ldots,r_p \end{smallmatrix}\right] = \frac{(q;q)_m}{(q;q)_{r_1}\cdots(q;q)_{r_p}}$ is the q-multinomial coefficient.

Analogous to Theorem 2.2, we have

Theorem 3.3 *Let $\zeta = e^{\frac{2\pi i}{p}}$ be the p-th root of 1. Then*

$$\sum_{r_1+\cdots+r_p=m} \zeta^{r_1+2r_2+\cdots+pr_p} \frac{(a;q)_{r_1}}{(q;q)_{r_1}} \cdots \frac{(a;q)_{r_p}}{(q;q)_{r_p}} = \begin{cases} 0, & \text{if } p \nmid m, \\ \dfrac{(a^p;q^p)_{m/p}}{(q^p;q^p)_{m/p}}, & \text{if } p \mid m. \end{cases}$$

To conclude this paper, we present an algebraic proof of Theorem 3.3 from the Cauchy identity:

$$\sum_{r=0}^{\infty} \frac{(a;q)_r}{(q;q)_r} t^r = \prod_{r=0}^{\infty} \frac{(1-atq^r)}{(1-tq^r)}$$

Algebraic Proof of Theorem 3.3. Let $\zeta = e^{\frac{2\pi i}{p}}$ be the p-th root of 1. Then

$$\begin{aligned} &\prod_{r=0}^{\infty} \frac{(1-a\zeta tq^r)}{(1-\zeta tq^r)} \cdot \prod_{r=0}^{\infty} \frac{(1-a\zeta^2 tq^r)}{(1-\zeta^2 tq^r)} \cdots \prod_{r=0}^{\infty} \frac{(1-a\zeta^p tq^r)}{(1-\zeta^p tq^r)} \\ &= \sum_{r=0}^{\infty} \frac{(a;q)_r}{(q;q)_r} (\zeta t)^r \cdot \sum_{r=0}^{\infty} \frac{(a;q)_r}{(q;q)_r} (\zeta^2 t)^r \cdots \sum_{r=0}^{\infty} \frac{(a;q)_r}{(q;q)_r} (\zeta^p t)^r \\ &= \sum_{m=0}^{\infty} t^m \sum_{r_1+\cdots+r_p=m} \frac{(a;q)_{r_1}}{(q;q)_{r_1}} \cdots \frac{(a;q)_{r_p}}{(q;q)_{r_p}} \zeta^{r_1+2r_2+\cdots+pr_p}. \end{aligned}$$

On the other hand, from the relation $1 - x^p = (1-x)(1-\zeta x)\cdots(1-\zeta^{p-1}x)$ and the Cauchy identity, it follows that

$$\begin{aligned}
&\prod_{r=0}^{\infty} \frac{(1-a\zeta tq^r)}{(1-\zeta tq^r)} \cdot \prod_{r=0}^{\infty} \frac{(1-a\zeta^2 tq^r)}{(1-\zeta^2 tq^r)} \cdots \prod_{r=0}^{\infty} \frac{(1-a\zeta^p tq^r)}{(1-\zeta^p tq^r)} \\
&\quad = \prod_{r=0}^{\infty} \frac{(1-atq^r)(1-\zeta atq^r)\cdots(1-\zeta^{p-1}atq^r)}{(1-tq^r)(1-\zeta tq^r)\cdots(1-\zeta^{p-1}tq^r)} \\
&\quad = \prod_{r=0}^{\infty} \frac{1-(atq^r)^p}{1-(tq^r)^p} \\
&\quad = \prod_{r=0}^{\infty} \frac{1-a^p t^p (q^p)^r}{1-t^p(q^p)^r} \\
&\quad = \sum_{r=0}^{\infty} \frac{(a^p;q^p)_r}{(q^p;q^p)_r} (t^p)^r.
\end{aligned}$$

Comparing the coefficients of t^m, we obtain Theorem 3.3. ∎

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