A Class of Kazhdan-Lusztig R-Polynomials and q-Fibonacci Numbers

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Abstract

Let S_n denote the symmetric group on $\{1, 2, \ldots, n\}$. For two permutations $u, v \in S_n$ such that $u \le v$ in the Bruhat order, let $R_{u,v}(q)$ and $\widetilde{R}_{u,v}(q)$ denote the Kazhdan-Lusztig R-polynomial and $\widetilde{R}$-polynomial, respectively. Let $v_n = 34 \cdots n\, 12$, and let σ be a permutation such that $\sigma \le v_n$. We obtain a formula for the $\widetilde{R}$-polynomials $\widetilde{R}_{\sigma,v_n}(q)$ in terms of the q-Fibonacci numbers depending on a parameter determined by the reduced expression of σ. When σ is the identity e, this reduces to a formula obtained by Pagliacci. In another direction, we obtain a formula for the $\widetilde{R}$-polynomial $\widetilde{R}_{e,\,v_{n,i}}(q)$, where $v_{n,i} = 34 \cdots i\, n\, (i+1) \cdots (n-1)\, 12$. In a more general context, we conjecture that for any two permutations $\sigma, \tau \in S_n$ such that $\sigma \le \tau \le v_n$, the $\widetilde{R}$-polynomial $\widetilde{R}_{\sigma,\tau}(q)$ can be expressed as a product of q-Fibonacci numbers multiplied by a power of q.

Keywords: Kazhdan-Lusztig R-polynomial, q-Fibonacci number, symmetric group

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1 Introduction

Let S_n denote the symmetric group on $\{1, 2, \ldots, n\}$. For two permutations $u, v \in S_n$ such that $u \le v$ in the Bruhat order, let $R_{u,v}(q)$ be the Kazhdan-Lusztig R-polynomial, and $\widetilde{R}_{u,v}(q)$ be the Kazhdan-Lusztig $\widetilde{R}$-polynomial. Let $v_n = 34 \cdots n\, 12$, and let σ be a permutation such that $\sigma \le v_n$. The main result of this paper is a formula for the $\widetilde{R}$-polynomials $\widetilde{R}_{\sigma,\,v_n}(q)$ in terms of the q-Fibonacci numbers depending on a parameter determined by the reduced expression of σ. When σ is the identity permutation e, a formula for the $\widetilde{R}$-polynomials has been given by Pagliacci [6, Theorem 4.1].

We also derive a formula for the $\widetilde{R}$-polynomials $\widetilde{R}_{e,\,v_{n,i}}(q)$, where $v_{n,i} = 34 \cdots i\, n\, (i+1) \cdots (n-1)\, 12$, which can be viewed as a generalization of Pagliacci's formula [6, Theorem 4.1] in another direction. We conclude this paper with a conjecture that for any two permutations $\sigma, \tau \in S_n$ such that $\sigma \le \tau \le v_n$, the $\widetilde{R}$-polynomial $\widetilde{R}_{\sigma,\,\tau}(q)$ can be expressed as a product of q-Fibonacci numbers and a power of q.

Let us give an overview of some notation and background. For each permutation π in S_n, it is known that π can be expressed as a product of simple transpositions $s_i = (i, i+1)$ subject to the following braid relations

$$s_i\, s_j = s_j\, s_i, \qquad \text{for } |i-j|>1;$$
$$s_i\, s_{i+1}\, s_i = s_{i+1}\, s_i\, s_{i+1}, \qquad \text{for } 1\leq i\leq n-2.$$

An expression ω of π is said to be reduced if the number of simple transpositions appearing in ω is minimum. The following word property is due to Tits, see Björner and Brenti [1, Theorem 3.3.1].

Theorem 1.1 (Word Property) *Let π be a permutation of S_n, and ω_1 and ω_2 be two reduced expressions of π. Then ω_1 and ω_2 can be obtained from each other by applying a sequence of braid relations.*

Let $\ell(\pi)$ denote the length of π, that is, the number of simple transpositions in a reduced expression of π. Write $D_R(\pi)$ for the set of right descents of π, namely,

$$D_R(\pi) = \{s_i\colon 1\leq i\leq n-1,\, \ell(\pi s_i)<\ell(\pi)\}.$$

The exchange condition gives a characterization for the (right) descents of a permutation in terms of reduced expressions, see Humphreys [4, Section 1.7].

Theorem 1.2 (Exchange Condition) *Let $\pi = s_{i_1}s_{i_2}\cdots s_{i_k}$ be a reduced expression of π. If $s_i\in D_R(\pi)$, then there exists an index i_j for which $\pi s_i = s_{i_1}\cdots \widehat{s}_{i_j}\cdots s_{i_k}$, where $\widehat{s}_{i_j}$ means that s_{i_j} is missing. In particular, π has a reduced expression ending with s_i if and only if $s_i\in D_R(\pi)$.*

The following subword property serves as a definition of the Bruhat order. For other equivalent definitions of the Bruhat order, see Björner and Brenti [1]. For a reduced expression $\omega = s_{i_1}s_{i_2}\cdots s_{i_k}$, we say that $s_{i_{j_1}}s_{i_{j_2}}\cdots s_{i_{j_m}}$ is a subword of ω if $1\leq j_1<j_2<\cdots<j_m\leq k$.

Theorem 1.3 (Subword Property) *Let u and v be two permutation in S_n. Then $u\leq v$ in the Bruhat order if and only if every reduced expression of v has a subword that is a reduced expression of u.*

The Bruhat order satisfies the following lifting property, see Björner and Brenti [1, Proposition 2.2.7].

Theorem 1.4 (Lifting Property) *Suppose that u and v are two permutations in S_n such that $u<v$. For any simple transposition s_i in $D_R(v)\backslash D_R(u)$, we have $u\leq vs_i$ and $us_i\leq v$.*

The Kazhdan-Lusztig R-polynomials, which were introduced by Kazhdan and Lusztig [5], can be recursively determined by the following properties, see also Humpreys [4, Section 7.5].

Theorem 1.5 *For any $u,v\in S_n$,*

(i) *$R_{u,v}(q)=0$, if $u\not\leq v$;*

(ii) $R_{u,v}(q) = 1$, *if* $u = v$;

(iii) *If* $u < v$ *and* $s \in D_R(v)$,

$$R_{u,v}(q) = \begin{cases} R_{us,vs}(q), & \text{if } s \in D_R(u); \\ qR_{us,vs}(q) + (q-1)R_{u,vs}(q), & \text{if } s \notin D_R(u). \end{cases}$$

While R-polynomials may contain negative coefficients, a variant of the R-polynomials introduced by Dyer [3], which has been called the $\widetilde{R}$-polynomials, has only nonnegative coefficients. For an alternative definition of the $\widetilde{R}$-polynomials for the symmetric group, see Brenti [2]. The following two theorems are due to Dyer [3], see also Brenti [2].

Theorem 1.6 *Let* $u, v \in S_n$ *with* $u \le v$. *Then, for* $s \in D_R(v)$,

$$\widetilde{R}_{u,v}(q) = \begin{cases} \widetilde{R}_{us,vs}(q), & \text{if } s \in D_R(u); \\ \widetilde{R}_{us,vs}(q) + q\widetilde{R}_{u,vs}(q), & \text{if } s \notin D_R(u). \end{cases} \tag{1.1}$$

Theorem 1.7 *Let* $u, v \in S_n$ *with* $u \le v$. *Then*

$$R_{u,v}(q) = q^{\frac{\ell(v)-\ell(u)}{2}} \widetilde{R}_{u,v}(q^{\frac{1}{2}} - q^{-\frac{1}{2}}).$$

Recall that $v_n = 34\cdots n\,12$ and $v_{n,i} = 34\cdots i\,n\,(i+1)\cdots(n-1)\,12$. We shall use the recurrence relations in Theorem 1.6 to deduce a formula for the $\widetilde{R}$-polynomials $\widetilde{R}_{\sigma,\,v_n}(q)$, from which we also find a formula for the $\widetilde{R}$-polynomial $\widetilde{R}_{e,\,v_{n,i}}(q)$.

2 Main result

The main result of this paper is an equation for $\widetilde{R}_{\sigma,\,v_n}(q)$, where $v_n = 34\cdots n\,12$ and $\sigma \le v_n$ in the Bruhat order. Combining this equation with a formula of Pagliacci [6], we obtain an expression of $\widetilde{R}_{\sigma,\,v_n}(q)$ in terms of q-Fibonacci numbers. To describe our result, we need the following reduced expression of v_n.

Proposition 2.1 *For* $n \ge 3$,

$$\Omega_n = s_2 s_1 s_3 s_2 \,\cdots\, s_{n-1} s_{n-2}$$

is a reduced expression of v_n.

Let σ be a permutation of S_n such that $\sigma \le v_n$. By the subword property in Theorem 1.3, σ can be expressed as a reduced subword of Ω_n. We introduce two statistics of a reduced subword of Ω_n.

Let $\omega = s_{i_1} s_{i_2} \cdots s_{i_k}$ be a reduced subword of Ω_n. Define

$$D(\omega) = \{1 \le t < k \colon i_t - i_{t+1} = 1\}.$$

We use $d(\omega)$ to denote the cardinality of $D(\omega)$, and let

$$h(\omega) = n - \ell(\omega) + d(\omega). \tag{2.1}$$

For example, for a reduced subword $\omega = s_2s_3s_4s_3s_6s_5s_7$ of Ω_9, we have $D(\omega) = \{3, 5\}$, and thus $d(\omega) = 2$ and $h(\omega) = 4$. Note that $h(\omega)$ depends on both ω and n. This causes no confusion since the index n is always clear from the context.

The main result in this paper is the following equation for the $\widetilde{R}$-polynomials $\widetilde{R}_{\sigma, v_n}(q)$.

Theorem 2.2 *For $n \geq 3$, let σ be a permutation in S_n such that $\sigma \leq v_n$, and let ω be any reduced expression of σ that is a subword of Ω_n. Then we have*

$$\widetilde{R}_{\sigma, v_n}(q) = q^{\ell(\omega)-2d(\omega)} \widetilde{R}_{e, v_{h(\omega)}}(q). \tag{2.2}$$

Let $F_n(q)$ be the q-Fibonacci numbers, that is, $F_0(q) = F_1(q) = 1$ and for $n \geq 2$,

$$F_n(q) = F_{n-1}(q) + qF_{n-2}(q).$$

Pagliacci [6, Theorem 4.1] has shown that

$$\widetilde{R}_{e, v_n}(q) = q^{2n-4} F_{n-2}(q^{-2}). \tag{2.3}$$

As a consequence of Theorem 2.2 and formula (2.3), we obtain an expression of $\widetilde{R}_{\sigma, v_n}(q)$ in terms of q-Fibonacci numbers.

Corollary 2.3 *For $n \geq 3$, let σ be a permutation in S_n such that $\sigma \leq v_n$, and let ω be any reduced expression of σ that is a subword of Ω_n. Then we have*

$$\widetilde{R}_{\sigma, v_n}(q) = q^{2n-\ell(\sigma)-4} F_{h(\omega)-2}(q^{-2}). \tag{2.4}$$

To give an inductive proof of Theorem 2.2, we need three lemmas. Assume that ω is a reduced subword of Ω_n. When $\omega s_{n-1} \leq \Omega_n$, the first two lemmas are concerned with the existence of a reduced expression ω' of ωs_{n-1} such that $d(\omega') = d(\omega)$. When $\omega s_{n-1} \not\leq \Omega_n$, the third lemma shows that $h(\omega) = 2$.

Lemma 2.4 *Let ω be a reduced subword of Ω_n. If $\omega s_{n-1} \leq \Omega_n$ and $s_{n-1} \in D_R(\omega)$, then there exists a reduced expression ω' of ωs_{n-1} such that ω' is a subword of Ω_n and $d(\omega') = d(\omega)$.*

Proof. We use induction on n. It is easy to check that the lemma holds for $n \leq 3$. Assume that $n > 3$ and the assertion holds for $n - 1$. We now consider the case for n. By definition, we have $\Omega_n = \Omega_{n-1}s_{n-1}s_{n-2}$. Since ω is a subword of Ω_n, we can write $\omega = \omega_1\omega_2$, where ω_1 is a subword of Ω_{n-1} and ω_2 is a subword of $s_{n-1}s_{n-2}$. Because $s_{n-1} \in D_R(\omega)$, we have the following two cases.

Case 1: $\omega = \omega_1 s_{n-1}$. Set $\omega' = \omega_1$. Clearly, ω' is a reduced expression of ωs_{n-1}. Moreover, it is easy to check that $D(\omega) = D(\omega')$, and thus $d(\omega') = d(\omega)$, that is, ω' is a desired reduced expression of ωs_{n-1}.

Case 2: $\omega = \omega_1 s_{n-1} s_{n-2}$. Since $s_{n-1} \in D_R(\omega)$, by Theorem 1.2, there exists a reduced expression of ω ending with s_{n-1}. Hence the word property in Theorem 1.1 ensures the existence of a reduced expression of ω_1 ending with s_{n-2}. This implies that s_{n-2} belongs to $D_R(\omega_1)$. By the induction hypothesis, there exists a reduced expression ω_1' of $\omega_1 s_{n-2}$ such that ω_1' is a subword of Ω_{n-1} and $d(\omega_1') = d(\omega_1)$.

Set

$$\omega' = \omega_1' s_{n-1} s_{n-2}.$$

We deduce that ω' is a desired reduced subword. Since

$$\omega' = \omega_1' s_{n-1} s_{n-2} = \omega_1 s_{n-2} s_{n-1} s_{n-2} = \omega_1 s_{n-1} s_{n-2} s_{n-1} = \omega s_{n-1},$$

we see that ω' is an expression of ωs_{n-1}. On the other hand, since ω' consists of $\ell(\omega_1') + 2$ simple transpositions and

$$\ell(\omega_1') + 2 = \ell(\omega_1) + 1 = \ell(\omega) - 1 = \ell(\omega s_{n-1}),$$

we conclude that ω' is a reduced expression of ωs_{n-1}. By the construction of ω', we have

$$d(\omega') = d(\omega_1') + 1 = d(\omega_1) + 1 = d(\omega).$$

This completes the proof. ∎

The next lemma deals with the case $s_{n-1} \notin D_R(\omega)$.

Lemma 2.5 *Let ω be a reduced subword of Ω_n. If $\omega s_{n-1} \le \Omega_n$ and $s_{n-1} \notin D_R(\omega)$, then there exists a reduced expression ω' of ωs_{n-1} such that ω' is a subword of Ω_n and $d(\omega') = d(\omega)$.*

Proof. We use induction on n. It is easily checked that the lemma holds for $n \le 3$. Assume that $n > 3$ and the assertion holds for $n-1$. We now consider the case for n. Let $\omega = \omega_1\omega_2$, where ω_1 is a subword of Ω_{n-1} and ω_2 is a subword of $s_{n-1}s_{n-2}$. Since $s_{n-1} \notin D_R(\omega)$, we have the following three cases.

Case 1: $\omega = \omega_1$. Set $\omega' = \omega_1 s_{n-1}$. It is easily seen that ω' is a desired reduced expression.

Case 2: $\omega = \omega_1 s_{n-2}$. We claim that $\omega = \omega_1 s_{n-2} \le \Omega_{n-1}$. Note that s_{n-1} does not appear in ω_1. Since $\omega s_{n-1} = \omega_1 s_{n-2} s_{n-1}$, by Theorem 1.1, there does not exist any reduced expression of ωs_{n-1} ending with s_{n-2}. This implies that s_{n-2} does not belong to $D_R(\omega s_{n-1})$. Thus, by the lifting property in Theorem 1.4, we deduce that

$$\omega s_{n-1} \le \Omega_n s_{n-2} = \Omega_{n-1} s_{n-1}.$$

This implies that $\omega = \omega_1 s_{n-2} \le \Omega_{n-1}$, as claimed.

Since $\omega = \omega_1 s_{n-2}$ is reduced, we see that $s_{n-2} \notin D_R(\omega_1)$. By the induction hypothesis, there exists a reduced expression ω_1' of $\omega_1 s_{n-2}$ such that ω_1' is a subword of Ω_{n-1} and $d(\omega_1') = d(\omega_1)$. Set $\omega' = \omega_1' s_{n-1}$. We find that ω' is a reduced expression of ωs_{n-1} such that $d(\omega') = d(\omega)$.

Case 3: $\omega = \omega_1 s_{n-1} s_{n-2}$. We claim that $s_{n-2} \notin D_R(\omega_1)$. Suppose to the contrary that $s_{n-2} \in D_R(\omega_1)$. By Theorem 1.2, there exists a reduced expression of ω_1 ending with s_{n-2}. Write $\omega_1 = \mu s_{n-2}$, where μ is a reduced expression. Then we get

$$\omega = \mu s_{n-2} s_{n-1} s_{n-2} = \mu s_{n-1} s_{n-2} s_{n-1},$$

contradicting the assumption that $s_{n-1} \notin D_R(\omega)$. So the claim is proved.

On the other hand, since

$$\omega_1 s_{n-2}s_{n-1}s_{n-2} = \omega_1 s_{n-1}s_{n-2}s_{n-1} = \omega s_{n-1} \leq \Omega_n,$$

we have $\omega_1 s_{n-2} \leq \Omega_{n-1}$. It follows from the induction hypothesis that there exists a reduced expression ω_1' of $\omega_1 s_{n-2}$ such that ω_1' is a subword of Ω_{n-1} and $d(\omega_1') = d(\omega_1)$.

Let

$$\omega' = \omega_1' s_{n-1}s_{n-2}.$$

Since

$$\omega' = \omega_1' s_{n-1}s_{n-2} = \omega_1 s_{n-2}s_{n-1}s_{n-2} = \omega_1 s_{n-1}s_{n-2}s_{n-1} = \omega s_{n-1},$$

we deduce that ω' is a reduced expression of ωs_{n-1}. By the construction of ω', we obtain that

$$d(\omega') = d(\omega_1') + 1 = d(\omega_1) + 1 = d(\omega),$$

as required. ∎

We now come to the third lemma.

Lemma 2.6 *Let ω be a reduced subword of Ω_n. If $\omega s_{n-1} \not\leq \Omega_n$, then we have $h(\omega) = 2$.*

Proof. We proceed by induction on n. It can be verified that the lemma holds for $n \leq 3$. Assume that $n > 3$ and the assertion holds for $n-1$. Consider the case for n. Write $\omega = \omega_1\omega_2$, where ω_1 is a subword of Ω_{n-1} and ω_2 is a subword of $s_{n-1}s_{n-2}$. Since $\omega s_{n-1} \not\leq \Omega_n$, we see that s_{n-1} is not a right descent of ω. We have the following two cases.

Case 1: $\omega = \omega_1 s_{n-2}$. Since $\omega s_{n-1} \not\leq \Omega_n$, we have $\omega = \omega_1 s_{n-2} \not\leq \Omega_{n-1}$. Thus, by the induction hypothesis, we get $h(\omega_1) = 2$. Noticing that $\ell(\omega_1) = \ell(\omega) - 1$ and $d(\omega_1) = d(\omega)$, we obtain that

$$h(\omega) = n - \ell(\omega) + d(\omega) = n - 1 - \ell(\omega_1) + d(\omega_1) = h(\omega_1) = 2,$$

as required.

Case 2: $\omega = \omega_1 s_{n-1}s_{n-2}$. We claim that $\omega_1 s_{n-2} \not\leq \Omega_{n-1}$. Suppose to the contrary that $\omega_1 s_{n-2} \leq \Omega_{n-1}$. Note that

$$\omega s_{n-1} = \omega_1 s_{n-1}s_{n-2}s_{n-1} = \omega_1 s_{n-2}s_{n-1}s_{n-2}.$$

This yields $\omega s_{n-1} \leq \Omega_n$, contradicting the assumption that $\omega s_{n-1} \not\leq \Omega_n$. So the claim is verified.

By the induction hypothesis, we have $h(\omega_1) = 2$. Since $\ell(\omega_1) = \ell(\omega) - 2$ and $d(\omega_1) = d(\omega) - 1$, we find that

$$h(\omega) = n - \ell(\omega) + d(\omega) = n - 1 - \ell(\omega_1) + d(\omega_1) = h(\omega_1) = 2,$$

as requied. ∎

Next we give a proof of Theorem 2.2 based on the above lemmas.

Proof of Theorem 2.2. Let

$$T_n(q) = \widetilde{R}_{e,\, v_n}(q).$$

and

$$g(\omega) = \ell(\omega) - 2d(\omega). \tag{2.5}$$

Then equation (2.2) can be rewritten as

$$\widetilde{R}_{\sigma, v_n}(q) = q^{g(\omega)} T_{h(\omega)}(q). \tag{2.6}$$

We proceed to prove (2.6) by induction on n. It can be checked that (2.6) holds for $n \leq 3$. Assume that $n > 3$ and (2.6) holds for $n-1$. For the case for n, let $\omega = \omega_1\omega_2$, where ω_1 is a subword of Ω_{n-1} and ω_2 is a subword of $s_{n-1}s_{n-2}$. There are four cases.

Case 1: $\omega = \omega_1 s_{n-2}$. It follows from (1.1) that

$$\begin{aligned}
\widetilde{R}_{\sigma, v_n}(q) &= \widetilde{R}_{\omega_1 s_{n-2}, \Omega_{n-1}s_{n-1}s_{n-2}}(q) \\
&= \widetilde{R}_{\omega_1, \Omega_{n-1}s_{n-1}}(q) \\
&= \widetilde{R}_{\omega_1 s_{n-1}, \Omega_{n-1}}(q) + q\widetilde{R}_{\omega_1, \Omega_{n-1}}(q).
\end{aligned} \tag{2.7}$$

Since $\omega_1 s_{n-1} \not\leq \Omega_{n-1}$, we see that the first term in (2.7) vanishes. Thus, (2.7) becomes

$$\widetilde{R}_{\sigma, v_n}(q) = q\widetilde{R}_{\omega_1, \Omega_{n-1}}(q). \tag{2.8}$$

By the induction hypothesis, we have

$$\widetilde{R}_{\omega_1, \Omega_{n-1}}(q) = q^{g(\omega_1)} T_{h(\omega_1)}. \tag{2.9}$$

Since $d(\omega_1) = d(\omega)$, we get

$$g(\omega_1) = \ell(\omega_1) - 2d(\omega_1) = \ell(\omega) - 1 - 2d(\omega) = g(\omega) - 1 \tag{2.10}$$

and

$$h(\omega_1) = n - 1 - \ell(\omega_1) + d(\omega_1) = n - \ell(\omega) + d(\omega) = h(\omega). \tag{2.11}$$

Plugging (2.10) and (2.11) into (2.9), we obtain

$$\widetilde{R}_{\omega_1, \Omega_{n-1}}(q) = q^{g(\omega_1)} T_{h(\omega_1)} = q^{g(\omega)-1} T_{h(\omega)}(q),$$

which leads to

$$\widetilde{R}_{\sigma, v_n}(q) = q\widetilde{R}_{\omega_1, \Omega_{n-1}}(q) = q^{g(\omega)} T_{h(\omega)}(q).$$

Case 2: $\omega = \omega_1 s_{n-1}$. By Theorem 1.1, there is no reduced expression of σ that ends with s_{n-2}. This implies that s_{n-2} is not a right descent of σ, so that

$$\begin{aligned}
\widetilde{R}_{\sigma, v_n}(q) &= \widetilde{R}_{\omega_1 s_{n-1}s_{n-2}, \Omega_{n-1}s_{n-1}}(q) + q\widetilde{R}_{\omega_1 s_{n-1}, \Omega_{n-1}s_{n-1}}(q) \\
&= \widetilde{R}_{\omega_1 s_{n-1}s_{n-2}, \Omega_{n-1}s_{n-1}}(q) + q\widetilde{R}_{\omega_1, \Omega_{n-1}}(q).
\end{aligned} \tag{2.12}$$

We claim that the first term in (2.12) vanishes, or equivalently, $\omega_1 s_{n-1}s_{n-2} \not\leq \Omega_{n-1}s_{n-1}$. Suppose to the contrary that $\omega_1 s_{n-1}s_{n-2} \leq \Omega_{n-1}s_{n-1}$. By Theorem 1.3, there exists a subword μ of $\Omega_{n-1}s_{n-1}$ that is a reduced expression of $\omega_1 s_{n-1}s_{n-2}$. Since s_{n-1} must appear in μ, we may write μ in the following form

$$\mu = s_{i_1}s_{i_2}\cdots s_{i_k}s_{n-1},$$

where $s_{i_1}s_{i_2}\cdots s_{i_k}$ is a reduced subword of Ω_{n-1}. By the word property in Theorem 1.1, $\omega_1 s_{n-1}s_{n-2}$ can be obtained from μ by applying the braid relations. However, this is impossible since any simple transposition s_{n-2} appearing in μ cannot be moved to the last position by applying the braid relations. So the claim is proved, and hence (2.12) becomes

$$\widetilde{R}_{\sigma,\, v_n}(q) = q\widetilde{R}_{\omega_1,\, \Omega_{n-1}}(q).$$

It is easily seen that

$$g(\omega_1) = g(\omega) - 1 \quad \text{and} \quad h(\omega_1) = h(\omega).$$

By the induction hypothesis, we deduce that

$$\widetilde{R}_{\sigma,\, v_n}(q) = q\widetilde{R}_{\omega_1,\, \Omega_{n-1}}(q) = q^{g(\omega_1)+1} T_{h(\omega_1)}(q) = q^{g(\omega)} T_{h(\omega)}(q).$$

Case 3: $\omega = \omega_1 s_{n-1} s_{n-2}$. It is clear from (1.1) that

$$\widetilde{R}_{\sigma,\, v_n}(q) = \widetilde{R}_{\omega_1 s_{n-1},\, \Omega_{n-1} s_{n-1}}(q) = \widetilde{R}_{\omega_1,\, \Omega_{n-1}}(q). \tag{2.13}$$

Noting that $d(\omega) = d(\omega_1) + 1$, we obtain

$$g(\omega_1) = \ell(\omega_1) - 2d(\omega_1) = \ell(\omega_1) + 2 - 2d(\omega) = \ell(\omega) - 2d(\omega) = g(\omega)$$

and

$$h(\omega_1) = n - 1 - \ell(\omega_1) + d(\omega_1) = n - 2 - \ell(\omega_1) + d(\omega) = n - \ell(\omega) + d(\omega) = h(\omega).$$

Thus, by (2.13) and the induction hypothesis, we find that

$$\widetilde{R}_{\sigma,\, v_n}(q) = \widetilde{R}_{\omega_1,\, \Omega_{n-1}}(q) = q^{g(\omega_1)} T_{h(\omega_1)}(q) = q^{g(\omega)} T_{h(\omega)}(q).$$

Case 4: $\omega = \omega_1$. Here are two subcases.

Subcase 1: $s_{n-2} \in D_R(\omega_1)$. By (1.1), we deduce that

$$\begin{aligned}\widetilde{R}_{\sigma,\, v_n}(q) &= \widetilde{R}_{\omega_1 s_{n-2},\, \Omega_{n-1} s_{n-1}}(q) \\ &= q\widetilde{R}_{\omega_1 s_{n-2},\, \Omega_{n-1}}(q).\end{aligned} \tag{2.14}$$

By Lemma 2.4, there exists a reduced expression ω_1' of $\omega_1 s_{n-2}$ such that ω_1' is a subword of Ω_{n-1} and $d(\omega_1') = d(\omega_1)$. Consequently,

$$g(\omega_1') = \ell(\omega_1') - 2d(\omega_1') = \ell(\omega_1) - 1 - 2d(\omega_1) = g(\omega_1) - 1$$

and

$$h(\omega_1') = n - 1 - \ell(\omega_1') + d(\omega_1') = n - \ell(\omega_1) + d(\omega_1) = h(\omega_1) + 1.$$

By the induction hypothesis, we obtain that

$$\begin{aligned}\widetilde{R}_{\omega_1 s_{n-2},\, \Omega_{n-1}}(q) &= \widetilde{R}_{\omega_1',\, \Omega_{n-1}}(q) \\ &= q^{g(\omega_1')} T_{h(\omega_1')}(q) \\ &= q^{g(\omega_1)-1} T_{h(\omega_1)+1}(q).\end{aligned} \tag{2.15}$$

But $h(\omega) = h(\omega_1) + 1$, substituting (2.15) into (2.14) gives

$$\begin{aligned}\widetilde{R}_{\sigma,\, v_n}(q) &= q\widetilde{R}_{\omega_1 s_{n-2},\, \Omega_{n-1}}(q) \\ &= q^{g(\omega)} T_{h(\omega)}(q).\end{aligned}$$

Subcase 2: $s_{n-2} \notin D_R(\omega_1)$. By (1.1), we see that

$$\begin{aligned}\widetilde{R}_{\sigma,\, v_n}(q) &= \widetilde{R}_{\omega_1 s_{n-2},\, \Omega_{n-1}s_{n-1}}(q) + q\widetilde{R}_{\omega_1,\, \Omega_{n-1}s_{n-1}}(q) \\ &= q\widetilde{R}_{\omega_1 s_{n-2},\, \Omega_{n-1}}(q) + q^2\widetilde{R}_{\omega_1,\, \Omega_{n-1}}(q) \end{aligned} \tag{2.16}$$

By the induction hypothesis, the second term in (2.16) equals

$$q^2\widetilde{R}_{\omega_1,\, \Omega_{n-1}}(q) = q^{g(\omega_1)+2}T_{h(\omega_1)}(q) = q^{g(\omega)+2}T_{h(\omega)-1}(q). \tag{2.17}$$

It remains to compute the first term in (2.16). To this end, we have the following two cases.

Subcase 2a: $\omega_1 s_{n-2} \leq \Omega_{n-1}$. By Lemma 2.5, there exists a reduced expression ω_1' of $\omega_1 s_{n-2}$ such that ω_1' is a subword of Ω_{n-1} and $d(\omega_1) = d(\omega_1')$. Hence

$$g(\omega_1') = \ell(\omega_1') - 2d(\omega_1') = \ell(\omega_1) + 1 - 2d(\omega_1) = g(\omega_1) + 1 = g(\omega) + 1$$

and

$$h(\omega_1') = n - 1 - \ell(\omega_1') + d(\omega_1') = n - 2 - \ell(\omega_1) + d(\omega_1) = h(\omega_1) - 1 = h(\omega) - 2.$$

By the induction hypothesis, we obtain that

$$\widetilde{R}_{\omega_1 s_{n-2},\, \Omega_{n-1}}(q) = q^{g(\omega_1')}T_{h(\omega_1')}(q) = q^{g(\omega)+1}T_{h(\omega)-2}(q). \tag{2.18}$$

Putting (2.17) and (2.18) into (2.16), we deduce that

$$\begin{aligned}\widetilde{R}_{\sigma,v_n}(q) &= q\widetilde{R}_{\omega_1 s_{n-2},\, \Omega_{n-1}}(q) + q^2\widetilde{R}_{\omega_1,\, \Omega_{n-1}}(q) \\ &= q^{g(\omega)+2}T_{h(\omega)-2}(q) + q^{g(\omega)+2}T_{h(\omega)-1}(q) \\ &= q^{g(\omega)}(q^2T_{h(\omega)-2}(q) + q^2T_{h(\omega)-1}(q)). \end{aligned} \tag{2.19}$$

In view of the following relation due to Pagliacci [6]

$$T_n(q) = q^2T_{n-2}(q) + q^2T_{n-1}(q),$$

(2.19) can be rewritten as

$$\widetilde{R}_{\sigma,\, v_n}(q) = q^{g(\omega)}T_{h(\omega)}(q).$$

Subcase 2b: $\omega_1 s_{n-2} \not\leq \Omega_{n-1}$. In this case, we have

$$\widetilde{R}_{\omega_1 s_{n-2},\, \Omega_{n-1}}(q) = 0. \tag{2.20}$$

By Lemma 2.6, we find that $h(\omega_1) = 2$. Thus (2.17) reduces to

$$q^2\widetilde{R}_{\omega_1,\, \Omega_{n-1}}(q) = q^{g(\omega)+2}T_2(q) = q^{g(\omega)+2}. \tag{2.21}$$

Putting (2.20) and (2.21) into (2.16), we get

$$\widetilde{R}_{\sigma,\, v_n}(q) = q^{g(\omega)+2}. \tag{2.22}$$

Since $T_3(q) = q^2$ and $h(\omega) = h(\omega_1) + 1 = 3$, it follows from (2.22) that

$$\widetilde{R}_{\sigma,\, v_n}(q) = q^{g(\omega)}T_3(q) = q^{g(\omega)}T_{h(\omega)}(q),$$

and hence the proof is complete. ∎

For $2 \le i \le n-1$, let

$$v_{n,i} = \begin{cases} n34\cdots(n-1)\,12, & \text{if } i=2; \\ 34\cdots i\,n\,(i+1)\cdots(n-1)\,12, & \text{if } 3 \le i \le n-1. \end{cases}$$

We obtain the following formula for $\widetilde{R}_{e,\,v_{n,i}}(q)$, which reduces to formula (2.3) due to Pagliacci in the case $i = n-1$.

Theorem 2.7 *Let $n \ge 3$ and $2 \le i \le n-1$. Then we have*

$$\widetilde{R}_{e,\,v_{n,i}}(q) = \sum_{k=0}^{n-i-1} q^{3n-i-2k-5} \binom{n-i-1}{k} F_{n-k-2}(q^{-2}). \tag{2.23}$$

Proof. Recall that

$$T_n(q) = \widetilde{R}_{e,\,v_n}(q) = q^{2n-4} F_{n-2}(q^{-2}),$$

see (2.3). Hence (2.23) can be rewritten as

$$\widetilde{R}_{e,\,v_{n,i}}(q) = \sum_{k=0}^{n-i-1} q^{n-i-1} \binom{n-i-1}{k} T_{n-k}(q). \tag{2.24}$$

Note that $\Omega_n s_{n-3} s_{n-4} \cdots s_{i-1}$ is a reduced expression of the permutation $v_{n,i}$. By the defining relation (1.1) of $\widetilde{R}$-polynomials, we obtain that

$$\begin{aligned}
\widetilde{R}_{e,\,v_{n,i}}(q) =& \widetilde{R}_{e,\,\Omega_n s_{n-3} s_{n-4}\cdots s_{i-1}}(q) \\
=& \widetilde{R}_{s_{i-1},\,\Omega_n s_{n-3} s_{n-4}\cdots s_i}(q) + q\widetilde{R}_{e,\,\Omega_n s_{n-3} s_{n-4}\cdots s_i}(q) \\
=& \left(\widetilde{R}_{s_{i-1}s_i,\,\Omega_n s_{n-3} s_{n-4}\cdots s_{i+1}}(q) + q\widetilde{R}_{s_{i-1},\,\Omega_n s_{n-3} s_{n-4}\cdots s_{i+1}}(q)\right) \\
&+ q\left(\widetilde{R}_{s_i,\,\Omega_n s_{n-3} s_{n-4}\cdots s_{i+1}}(q) + q\widetilde{R}_{e,\,\Omega_n s_{n-3} s_{n-4}\cdots s_{i+1}}(q)\right) \\
=& \cdots \\
=& \sum_{i-1 \le i_1 < \cdots < i_k \le n-3} q^{n-i-1-k} \widetilde{R}_{s_{i_1}\cdots s_{i_k},\,\Omega_n}(q).
\end{aligned} \tag{2.25}$$

Observe that $s_{i_1} \cdots s_{i_k}$ is a reduced subword of Ω_n with $d(s_{i_1} \cdots s_{i_k}) = 0$. By Corollary 2.3, we find that

$$\widetilde{R}_{s_{i_1}\cdots s_{i_k},\,\Omega_n}(q) = q^k T_{n-k}(q). \tag{2.26}$$

Substituting (2.26) into (2.25), we get

$$\begin{aligned}
\widetilde{R}_{e,\,v_{n,i}}(q) &= \sum_{i-1 \le i_1 < \cdots < i_k \le n-3} q^{n-i-1-k} \widetilde{R}_{s_{i_1}\cdots s_{i_k},\,\Omega_n}(q) \\
&= \sum_{i-1 \le i_1 < \cdots < i_k \le n-3} q^{n-i-1}\, T_{n-k}(q) \\
&= \sum_{k=0}^{n-i-1} q^{n-i-1} \binom{n-i-1}{k} T_{n-k}(q),
\end{aligned}$$

as required. ∎

We conclude this paper with the following conjecture, which has been verified for $n \le 9$.

Conjecture 2.8 *For $n \geq 2$ and $e \leq \sigma_1 \leq \sigma_2 \leq \Omega_n$, we have*

$$\widetilde{R}_{\sigma_1,\sigma_2}(q) = q^{g(\sigma_1,\sigma_2)} \prod_{i=1}^{k} F_{h_i(\sigma_1,\sigma_2)}(q^{-2}), \tag{2.27}$$

where k, $g(\sigma_1, \sigma_2)$ and $h_i(\sigma_1, \sigma_2)$ are integers depending on σ_1 and σ_2.

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