Stanley's Lemma and Multiple Theta Function Identities

William Y. C. Chen[1] and Lisa H. Sun[2]

[1]Center for Applied Mathematics
Tianjin University, Tianjin 300072, P. R. China

[2]Center for Combinatorics, LPMC-TJKLC
Nankai University, Tianjin 300071, P.R. China

E-mail: [1]chenyc@tju.edu.cn, [2]sunhui@nankai.edu.cn

Abstract. We present a vector space approach to proving identities on multiple theta functions by verifying a finite number of simpler relations which are often verifiable by using Jacobi's triple product identity. Consider multiple theta function identities of the form $\sum_{k=1}^{m} c_k\theta_k(a_1, a_2, \ldots, a_r) = 0$, where $r, m \geq 2$, $\theta_k = \prod_{i=1}^{n_k} f_{k,i}(a_1, a_2, \ldots, a_r)$, $1 < n_k \leq r$ and each $f_{k,i}$ is of form $(p_{k,i}, q^{\beta_{k,i}}/p_{k,i}; q^{\beta_{k,i}})_\infty$ with $\beta_{k,i}$ being a positive integer and $p_{k,i}$ being a monomial in $a_1, a_2, \ldots, a_r$ and q. For such an identity, $\theta_1, \theta_2, \ldots, \theta_m$ satisfy the same set of linearly independent contiguous relations. Let W be the set of exponent vectors of $(a_1, a_2, \ldots, a_r)$ in the contiguous relations. We consider the case when the exponent vectors of $(a_1, a_2, \ldots, a_r)$ in $p_{k,1}, p_{k,2}, \ldots, p_{k,n_k}$ are linearly independent for any k. Let V_C denote the vector space spanned by multiple theta functions which satisfy the contiguous relations associated with the vectors in W. Using Stanley's lemma on the fundamental parallelepiped, we find an upper bound of the dimension of V_C. This implies that a multiple theta function identity may be reduced to a finite number of simpler relations. Many classical multiple theta function identities fall into this framework, such as Bailey's generalization of the quintuple product identity, the extended Riemann identity and Riemann's addition formula.

Keywords: theta function, multiple theta function, contiguous relation, Jacobi's triple product identity, addition formula

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1. Introduction

In this paper, we provide a vector space approach to proving multiple theta function identities by applying Stanley's lemma on the fundamental parallelepiped.

Let us begin with notation and terminology. Let $|q| < 1$. Recall that the q-shifted factorial of infinite order is defined by

$$(a; q)_\infty = \prod_{k=0}^{\infty}(1 - aq^k).$$

Write

$$(a_1, a_2, \ldots, a_m; q)_\infty = (a_1; q)_\infty (a_2; q)_\infty \cdots (a_m; q)_\infty.$$

The theta function $\theta(z)$ is defined by

$$\theta(z) = [z; q]_\infty = (z, q/z; q)_\infty, \tag{1.1}$$

see, for example, Krattenthaler [9]. A multiple theta function is defined by

$$[z_1, z_2, \cdots, z_n; q]_\infty = [z_1; q]_\infty [z_2; q]_\infty \cdots [z_n; q]_\infty. \tag{1.2}$$

A basic theta function identity is Jacobi's triple product identity

$$(q, z, q/z; q)_\infty = \sum_{n=-\infty}^{\infty} (-1)^n q^{\binom{n}{2}} z^n \tag{1.3}$$

with $z \neq 0$.

For the one variable case, theta function identities can be derived from the theta function solution space of the associated contiguous relation. It is known that theta functions satisfying the contiguous relation

$$f(zq^\alpha) = \frac{1}{z^n q^m} f(z) \tag{1.4}$$

form a vector space over $\mathbb{C}$ of dimension n, where r, n are positive integers and m is a nonnegative integer. This is a classical result in algebraic geometry, see, for example, [12, p. 212, Theorem 1]. Suppose that $f(z)$ can be written as

$$f(z) = \sum_{k \in \mathbb{Z}} a_k z^k. \tag{1.5}$$

Equating coefficients of z^k on both sides of the contiguous relation (1.4), we see that the coefficients a_k satisfy a recurrence relation of order n. Thus we can prove an identity on theta functions in one variable by showing that both sides satisfy the same contiguous relation in the form of (1.4) with the same initial conditions on the coefficients. This approach applies to many classical theta function identities, such as the quintuple product identity and the septuple product identity, see [1, 2, 5, 7, 16].

As an example, consider the quintuple product identity

$$(q, -z, -q/z; q)_\infty (qz^2, q/z^2; q^2)_\infty = (q^3, qz^3, q^2/z^3; q^3)_\infty + z(q^3, q^2z^3, q/z^3; q^3)_\infty, \tag{1.6}$$

see [8, 11, 15]. Denote the three theta functions in (1.6) by L, R_1 and R_2, respectively. It is easily checked that L, R_1 and R_2 satisfy the contiguous relation

$$f(zq) = -\frac{1}{z^3 q} f(z). \tag{1.7}$$

Suppose that $f(z)$ has an expansion with coefficients a_k as in (1.5). It is clearly from (1.7) that the coefficients a_k satisfy a recurrence relation of order three. Therefore, to prove (1.6), it suffices to show that

$$[z^{-1}]\, L = [z^{-1}]\, (R_1 + R_2),$$

$$[z^0]\,L = [z^0]\,(R_1 + R_2),$$

$$[z^1]\,L = [z^1]\,(R_1 + R_2),$$

where $[z^i]\,g(z)$ is the common notation for the coefficient of z^i in the power series expansion of $g(z)$.

We shall consider multiple theta functions of the following form

$$\theta(a_1, a_2, \ldots, a_r) = \prod_{i=1}^{m} f_i(a_1, a_2, \ldots, a_r), \tag{1.8}$$

where $1 < m \le r$ and

$$f_i(a_1, a_2, \ldots, a_r) = [(-1)^{\delta_i} a_1^{\gamma_{i,1}} a_2^{\gamma_{i,2}} \cdots a_r^{\gamma_{i,r}} q^{z_i}; q^{t_i}]_\infty \tag{1.9}$$

with $\delta_i \in \{0, 1\}$, $\gamma_{i,j} \in \mathbb{Z}$, $z_i \in \mathbb{Q}$ and $t_i \in \mathbb{Q}^+$. Moreover, we assume that for $1 \le i \le m$, the exponent vectors $\gamma_i = (\gamma_{i,1}, \gamma_{i,2}, \ldots, \gamma_{i,r})$ in f_i are linearly independent. For convenience, we denote the monomial $a_1^{\gamma_{i,1}} a_2^{\gamma_{i,2}} \cdots a_r^{\gamma_{i,r}}$ by a^{γ_i}, where a stands for $(a_1, a_2, \ldots, a_r)$. Remark that more generally, f_i may contains a factor a^κ with $\kappa \in \mathbb{Z}^r$. For brevity, we only deal with the cases when f_i is in the form of (1.9) while all the results presented in this paper still hold for the more general cases.

For a multiple theta function $\theta(a_1, a_2, \ldots, a_r)$, a contiguous relation is meant to be a relation of the following form

$$\frac{\theta(a_1 q^{\alpha_{i,1}}, a_2 q^{\alpha_{i,2}}, \ldots, a_r q^{\alpha_{i,r}})}{\theta(a_1, a_2, \ldots, a_r)} = \frac{(-1)^{\rho_i}}{a_1^{w_{i,1}} a_2^{w_{i,2}} \cdots a_r^{w_{i,r}} q^{s_i}}, \tag{1.10}$$

where $\alpha_{i,j}, w_{i,j}$ $(1 \le j \le r) \in \mathbb{Z}$, $s_i \in \mathbb{Q}$ and $\rho_i \in \{0, 1\}$. Notice that a contiguous relation (1.10) can be uniquely associated with a vector $w_i = (w_{i,1}, w_{i,2}, \ldots, w_{i,r})$. Furthermore, we say that two contiguous relations in the form of (1.10) are linearly independent if the corresponding exponent vectors w_i are linearly independent.

If we restrict our attention to multiple theta functions in the form of (1.8), then the solutions of a system of linearly independent contiguous relations form a finite dimensional vector space. The proof of this fact relies on Stanley's lemma on the fundamental parallelepiped. This vector space property enables us to verify a multiple theta function identity by reducing it to a finite number of simpler relations.

More precisely, consider the following multiple theta function identity

$$\sum_{k=1}^{m} c_k \theta_k(a_1, a_2, \ldots, a_r) = 0, \tag{1.11}$$

where $m \ge 2$ and $\theta_k(a_1, a_2, \ldots, a_r)$ are given in the form of (1.8). To prove identity (1.11), we first show that for $1 \le k \le m$, θ_k satisfy the same system C of linearly independent contiguous relations. Let $W = \{v_1, v_2, \ldots, v_d\}$ be the set of the exponent vectors of $(a_1, a_2, \ldots, a_r)$ in these contiguous relations in C. Let V_C denote the vector space spanned by the multiple theta functions as given in (1.8) that satisfy the contiguous relations in C. By Stanley's lemma on the fundamental parallelepiped, we show that $\dim V_C \le |\Pi_W|$, where

$$\Pi_W = \{\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_d v_d \mid 0 \le \lambda_i < 1,\ 1 \le i \le d\} \cap \mathbb{Z}^r. \tag{1.12}$$

Hence identity (1.11) can be justified by comparing the coefficients of the terms with exponent vectors in Π_W.

For example, let us consider the following identity,

$$(a, -b, q/a, -q/b\,; q)_\infty + (-a, b, -q/a, q/b\,; q)_\infty = \frac{2(ab, q^2/ab, aq/b, bq/a; q^2)_\infty}{(q; q^2)_\infty^2}. \tag{1.13}$$

Berndt [4, P. 45] proved (1.13) based on Ramanujan's general theta function $f(a, b)$. Chu [6] gave a proof of (1.13) by using q-difference equations.

Denote the three multiple theta functions in (1.13) by L_1, L_2 and R, respectively. It can be shown that L_1, L_2 and R satisfy the following two linearly independent contiguous relations

$$\frac{f(aq, bq)}{f(a, b)} = -\frac{1}{ab},$$
$$\frac{f(a, bq^2)}{f(a, b)} = \frac{1}{b^2 q}. \tag{1.14}$$

Thus $W = \{(1, 1), (0, 2)\}$. By Stanley's lemma, we see that the dimension of V_C is bounded by two since $\Pi_W = \{(0, 0), (0, 1)\}$. Hence identity (1.13) can be reduced to the following identities

$$[a^0 b^0]\,(L_1 + L_2) = [a^0 b^0]\,R,$$
$$[a^0 b^1]\,(L_1 + L_2) = [a^0 b^1]\,R,$$

which can be certified by Jacobi's triple product identity (1.3).

This paper is organized as follows. In Section 2, we define two vector spaces associated with a multiple theta function of the form (1.8). In Section 3, we employ Stanley's lemma on the fundamental parallelepiped to derive an upper bound on the dimension of the solution space of a system of linearly independent contiguous relations. In Sections 4 and 5, we apply the vector space approach to several well-known multiple theta function identities and addition formulas such as Bailey's generalization of the quintuple product identity, the extended Riemann identity and Riemann's addition formula.

2. Two vector spaces

In this section, we define two vector spaces over $\mathbb{Q}$ associated with a multiple theta function and we show that they are the same vector space. This property will be used in the next section to determine the number of linearly independent contiguous relations needed to verify a multiple theta function identity.

Recall that $\theta(a_1, a_2, \ldots, a_r)$ is a product of m factors

$$f_i(a_1, a_2, \ldots, a_r) = [(-1)^{\delta_i} a_1^{\gamma_{i,1}} a_2^{\gamma_{i,2}} \cdots a_r^{\gamma_{i,r}} q^{z_i}; q^{t_i}]_\infty.$$

Let U_θ denote the vector space spanned by exponent vectors of $(a_1, a_2, \ldots, a_r)$ in f_i, that is,

$$U_\theta = \big\langle \{\gamma_i = (\gamma_{i,1}, \gamma_{i,2}, \ldots, \gamma_{i,r}) \mid 1 \leq i \leq m\} \big\rangle.$$

Let V_θ denote the vector space generated by the exponent vectors $w_i = (w_{i,1}, w_{i,2}, \ldots, w_{i,r})$ of contiguous relations (1.10).

We have the following assertion.

Theorem 2.1. *Let $\theta(a_1, a_2, \ldots, a_r)$ be a multiple theta function as given in* (1.8) *and U_θ, V_θ defined as above. Then we have*

$$U_\theta = V_\theta.$$

The following properties of contiguous relations of theta functions will be needed in the proof of the above theorem.

Lemma 2.2. *Let*

$$f(a) = (a^v, q^t/a^v; q^t)_\infty$$

be a theta function, where v is an integer and t is a positive rational number. Assume that $f(a)$ has a contiguous relation of the following form

$$f(aq^\alpha) = \frac{1}{(-1)^\rho a^w q^s} f(a), \tag{2.1}$$

where $\alpha, w \in \mathbb{Z}$, $\rho \in \{0, 1\}$ and $s \in \mathbb{Q}$. Then we have $v|w$ and αv is an integer multiple of t.

Proof. By Jacobi's triple product identity (1.3), $f(a)$ can be written as

$$f(a) = \frac{1}{(q^t; q^t)_\infty} \sum_{n=-\infty}^{\infty} (-1)^n q^{t\binom{n}{2}} a^{vn}. \tag{2.2}$$

Substituting a with aq^α in (2.2) we get an expansion of $f(aq^\alpha)$. Note that the powers of a in the expansions of both sides are all multiples of v, so we deduce that $v|w$. Plugging the expansions of $f(a)$ and $f(aq^\alpha)$ into the contiguous relation (2.1) and equating coefficients of a^{vn}, we obtain that

$$(-1)^n q^{t\binom{n}{2} + \alpha vn} = (-1)^{\rho + n + \frac{w}{v}} q^{t\binom{n+\frac{w}{v}}{2} - s}.$$

It follows that $\rho + \frac{w}{v}$ is even and the powers of q on both sides are equal for all $n \in \mathbb{Z}$, namely,

$$t\binom{n}{2} + \alpha vn = t\binom{n + \frac{w}{v}}{2} - s.$$

Hence

$$\left(t\frac{w}{v} - \alpha v\right)n + t\binom{\frac{w}{v}}{2} - s = 0$$

for all $n \in \mathbb{Z}$, so that $t\frac{w}{v} = \alpha v$. Thus we have αv is an integer multiple of t. ∎

Recall that U_θ is the vector space spanned by $\gamma_1, \gamma_2, \ldots, \gamma_m$. The following lemma shows that any vector $v \in U_\theta$ gives rise to a contiguous relation of $\theta(a_1, a_2, \ldots, a_r)$.

Lemma 2.3. *Let $v = (v_1, v_2, \ldots, v_r)$ be a vector in U_θ such that*

$$v = k_1\gamma_1 + k_2\gamma_2 + \cdots + k_m\gamma_m,$$

where k_i's are not all zeros, and γ_i is associated with the factor f_i in (1.9). Then there exists a nonzero vector $(x_1, x_2, \ldots, x_r)$ and a constant $\varepsilon \neq 0$ such that the following contiguous relation holds

$$\frac{\theta(a_1q^{x_1}, a_2q^{x_2}, \ldots, a_rq^{x_r})}{\theta(a_1, a_2, \ldots, a_r)} = \frac{(-1)^{\sum_{i=1}^m (\delta_i+1)\varepsilon k_i}}{q^{\sum_{i=1}^m \varepsilon z_i k_i + \binom{\varepsilon k_i}{2} t_i} a^{\varepsilon v}}, \tag{2.3}$$

where $a = (a_1, a_2, \ldots, a_r)$ and $a^{\varepsilon v} = a_1^{\varepsilon v_1} a_2^{\varepsilon v_2} \cdots a_r^{\varepsilon v_r}$.

Proof. To prove the existence of a vector $(x_1, x_2, \ldots, x_r)$ and a nonzero constant ε satisfying (2.3), we show that the vector $(x_1, x_2, \ldots, x_r)$ and the nonzero constant ε can be chosen as a solution of the following system of linear equations

$$\begin{cases} \frac{x\gamma_i}{k_i t_i} - \varepsilon = 0, & \text{if } k_i \neq 0, \\ x\gamma_j = 0, & \text{if } k_j = 0, \end{cases} \tag{2.4}$$

where $1 \leq i, j \leq m$, $x = (x_1, x_2, \ldots, x_r)$, and ε is also considered as a variable. So there are m homogeneous linear equations in $r+1$ variables. In other words, the coefficient matrix A of the system of linear equations (2.4) has m rows and $r+1$ columns, which can be written as

$$A = \begin{pmatrix} \gamma_{1,1} & \gamma_{1,2} & \cdots & \gamma_{1,r} & \zeta_1 \\ \gamma_{2,1} & \gamma_{2,2} & \cdots & \gamma_{2,r} & \zeta_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \gamma_{m,1} & \gamma_{m,2} & \cdots & \gamma_{m,r} & \zeta_m \end{pmatrix}_{m\times(r+1)},$$

where $\zeta_i = -k_i t_i$, if $k_i \neq 0$ and $\zeta_i = 0$, otherwise. Since $1 < m \leq r$ and $\gamma_1, \gamma_2, \ldots, \gamma_m$ are linearly independent, the rank of the matrix A is m. Thus the solution space P of (2.4) in variables $x_1, x_2, \ldots, x_r, \varepsilon$ has dimension $r + 1 - m$.

Consider another system of linear equations

$$\begin{cases} x\gamma_1 = 0, \\ x\gamma_2 = 0, \\ \quad\vdots \\ x\gamma_m = 0. \end{cases} \tag{2.5}$$

Let Q denote the vector space of solutions $(x_1, x_2, \ldots, x_r)$ of the equations in (2.5). Let Q' be the vector space obtained from Q by substituting every vector $(x_1, x_2, \ldots, x_r) \in Q$ with $(x_1, x_2, \ldots, x_r, 0)$. Since $\gamma_1, \gamma_2, \ldots, \gamma_m$ are linearly independent, we see that $\dim Q' = \dim Q = r - m$. It is also clear that any solution $(x_1, x_2, \ldots, x_r)$ of (2.5) gives rise to a solution $(x_1, x_2, \ldots, x_r, 0)$ of (2.4). In other words, Q' is a subspace of P. Moreover, we have $\dim Q' < \dim P$. Hence there exists a solution $(x_1, x_2, \ldots, x_r, \varepsilon) \in P \setminus Q'$ such that $\varepsilon \neq 0$. Now we have

$$\frac{\theta(a_1q^{x_1}, a_2q^{x_2}, \ldots, a_rq^{x_r})}{\theta(a_1, a_2, \ldots, a_r)} = \prod_{i=1}^m \frac{f_i(a_1q^{x_1}, a_2q^{x_2}, \ldots, a_rq^{x_r})}{f_i(a_1, a_2, \ldots, a_r)}$$

$$= \prod_{i=1}^{m} \frac{[(-1)^{\delta_i} a_1^{\gamma_{i,1}} a_2^{\gamma_{i,2}} \cdots a_r^{\gamma_{i,r}} q^{x\gamma_i + z_i}; q^{t_i}]_\infty}{[(-1)^{\delta_i} a_1^{\gamma_{i,1}} a_2^{\gamma_{i,2}} \cdots a_r^{\gamma_{i,r}} q^{z_i}; q^{t_i}]_\infty}$$

$$= \prod_{\substack{i=1 \\ k_i \neq 0}}^{m} \frac{[(-1)^{\delta_i} a_1^{\gamma_{i,1}} a_2^{\gamma_{i,2}} \cdots a_r^{\gamma_{i,r}} q^{\varepsilon k_i t_i + z_i}; q^{t_i}]_\infty}{[(-1)^{\delta_i} a_1^{\gamma_{i,1}} a_2^{\gamma_{i,2}} \cdots a_r^{\gamma_{i,r}} q^{z_i}; q^{t_i}]_\infty}$$

$$= \prod_{\substack{i=1 \\ k_i \neq 0}}^{m} \frac{(-1)^{(\delta_i+1)\varepsilon k_i}}{q^{\varepsilon z_i k_i + \binom{\varepsilon k_i}{2} t_i} (a_1^{\gamma_{i,1}} a_2^{\gamma_{i,2}} \cdots a_r^{\gamma_{i,r}})^{\varepsilon k_i}}$$

$$= \frac{(-1)^{\sum_{i=1}^{m} (\delta_i+1)\varepsilon k_i}}{q^{\sum_{i=1}^{m} \varepsilon z_i k_i + \binom{\varepsilon k_i}{2} t_i} a^{\varepsilon(k_1\gamma_1 + k_2\gamma_2 + \cdots + k_m\gamma_m)}}$$

$$= \frac{(-1)^{\sum_{i=1}^{m} (\delta_i+1)\varepsilon k_i}}{q^{\sum_{i=1}^{m} \varepsilon z_i k_i + \binom{\varepsilon k_i}{2} t_i} a^{\varepsilon v}},$$

as desired. ∎

It should be noted that our approach does not require an explicit expression of a solution of the system of linear equations in (2.4). In particular, for $r = 1$, (2.4) reduces to the following equation

$$\frac{x\gamma}{kt} = \varepsilon.$$

Since $\gamma, k, t \in \mathbb{Q}$, it is obvious that the above equation has a solution x for any given $\varepsilon \neq 0$.

We are now ready to prove Theorem 2.1.

Proof of Theorem 2.1. Recall that

$$\theta(a_1, a_2, \ldots, a_r) = \prod_{i=1}^{m} f_i(a_1, a_2, \ldots, a_r),$$

with $1 < m \leq r$ and the factor $f_i = [(-1)^{\delta_i} a_1^{\gamma_{i,1}} a_2^{\gamma_{i,2}} \cdots a_r^{\gamma_{i,r}} q^{z_i}; q^{t_i}]_\infty$ as given by (1.9). For $1 \leq i \leq m$, let $\gamma_i = (\gamma_{i,1}, \gamma_{i,2}, \ldots, \gamma_{i,r})$. We first prove $V_\theta \subseteq U_\theta$. By definition, U_θ is spanned by the vectors $\gamma_1, \gamma_2, \ldots, \gamma_m$ and V_θ is the vector space consisting of vectors $w_i = (w_{i,1}, w_{i,2}, \ldots, w_{i,r})$ such that

$$\frac{\theta(a_1 q^{\alpha_{i,1}}, a_2 q^{\alpha_{i,2}}, \ldots, a_r q^{\alpha_{i,r}})}{\theta(a_1, a_2, \ldots, a_r)} = \frac{(-1)^{\rho_i}}{a_1^{w_{i,1}} a_2^{w_{i,2}} \cdots a_r^{w_{i,r}} q^{s_i}}, \tag{2.6}$$

where $\alpha_{i,j}, w_{i,j} \in \mathbb{Z}$, $s_i \in \mathbb{Q}$ and $\rho_i \in \{0, 1\}$.

We proceed to show that any vector in V_θ can be expressed as a linear combination of the vectors $\gamma_1, \gamma_2, \ldots, \gamma_m$. From (2.6), we find that

$$\frac{\theta(a_1 q^{\alpha_{i,1}}, a_2 q^{\alpha_{i,2}}, \ldots, a_r q^{\alpha_{i,r}})}{\theta(a_1, a_2, \ldots, a_r)} = \prod_{i=1}^{m} \frac{f_i(a_1 q^{\alpha_{i,1}}, a_2 q^{\alpha_{i,2}}, \ldots, a_r q^{\alpha_{i,r}})}{f_i(a_1, a_2, \ldots, a_r)}$$

$$= \prod_{i=1}^{m} \frac{[(-1)^{\delta_i} (a_1 q^{\alpha_{i,1}})^{\gamma_{i,1}} \cdots (a_r q^{\alpha_{i,r}})^{\gamma_{i,r}} q^{z_i}; q^{t_i}]_\infty}{[(-1)^{\delta_i} a_1^{\gamma_{i,1}} \cdots a_r^{\gamma_{i,r}} q^{z_i}; q^{t_i}]_\infty}. \tag{2.7}$$

Let

$$\mu_i = \alpha_{i,1}\gamma_{i,1} + \alpha_{i,2}\gamma_{i,2} + \cdots + \alpha_{i,r}\gamma_{i,r},$$

so that (2.7) can be rewritten as

$$\frac{\theta(a_1q^{\alpha_{i,1}}, a_2q^{\alpha_{i,2}}, \ldots, a_rq^{\alpha_{i,r}})}{\theta(a_1, a_2, \ldots, a_r)} = \prod_{i=1}^{m} \frac{[(-1)^{\delta_i} a_1^{\gamma_{i,1}} a_2^{\gamma_{i,2}} \cdots a_r^{\gamma_{i,r}} q^{z_i+\mu_i}; q^{t_i}]_\infty}{[(-1)^{\delta_i} a_1^{\gamma_{i,1}} a_2^{\gamma_{i,2}} \cdots a_r^{\gamma_{i,r}} q^{z_i}; q^{t_i}]_\infty}. \tag{2.8}$$

In view of Lemma 2.2, we see that μ_i can be written as $\mu_i = \upsilon_i t_i$ with υ_i being an integer. It follows that

$$\begin{aligned}\frac{\theta(a_1q^{\alpha_{i,1}}, a_2q^{\alpha_{i,2}}, \ldots, a_rq^{\alpha_{i,r}})}{\theta(a_1, a_2, \ldots, a_r)} &= \prod_{i=1}^{m} \frac{\prod_{k=0}^{\upsilon_i-1}\left(1 - \frac{1}{(-1)^{\delta_i} a_1^{\gamma_{i,1}} a_2^{\gamma_{i,2}} \cdots a_r^{\gamma_{i,r}} q^{z_i+kt_i}}\right)}{\prod_{k=0}^{\upsilon_i-1}\left(1 - (-1)^{\delta_i} a_1^{\gamma_{i,1}} a_2^{\gamma_{i,2}} \cdots a_r^{\gamma_{i,r}} q^{z_i+kt_i}\right)} \\ &= \prod_{i=1}^{m} \frac{(-1)^{(\delta_i+1)\upsilon_i}}{\left(a_1^{\gamma_{i,1}} a_2^{\gamma_{i,2}} \cdots a_r^{\gamma_{i,r}}\right)^{\upsilon_i} q^{z_i\upsilon_i + \binom{\upsilon_i}{2} t_i}}.\end{aligned} \tag{2.9}$$

Combining the contiguous relations (2.6) and (2.9), we find that

$$\frac{(-1)^{\rho_i}}{a_1^{w_{i,1}} a_2^{w_{i,2}} \cdots a_r^{w_{i,r}} q^{s_i}} = \prod_{i=1}^{m} \frac{(-1)^{(\delta_i+1)\upsilon_i}}{\left(a_1^{\gamma_{i,1}} a_2^{\gamma_{i,2}} \cdots a_r^{\gamma_{i,r}}\right)^{\upsilon_i} q^{z_i\upsilon_i + \binom{\upsilon_i}{2} t_i}}.$$

Hence we arrive at

$$w_i = \upsilon_1\gamma_1 + \upsilon_2\gamma_2 + \cdots + \upsilon_m\gamma_m, \tag{2.10}$$

where w_i and γ_i are defined as before. This leads to the conclusion that $V_\theta \subseteq U_\theta$.

To prove that $U_\theta \subseteq V_\theta$, let v be a vector in U_θ. By Lemma 2.3, we see that there exists a nonzero vector $(x_1, x_2, \ldots, x_r)$ and a constant $\varepsilon \neq 0$ such that εv is the exponent vector of a in the denominator of the contiguous relation

$$\frac{\theta(a_1q^{x_1}, a_2q^{x_2}, \ldots, a_rq^{x_r})}{\theta(a_1, a_2, \ldots, a_r)}$$

as given in (2.3). Therefore εv belongs to the vector space V_θ, that is, $v \in V_\theta$. This yields $U_\theta \subseteq V_\theta$, and hence the proof is complete. ∎

3. Stanley's lemma and an upper bound of $\dim V_C$

In this section, we consider the solution space of a system of linearly independent contiguous relations. Let C be a system of linearly independent contiguous relations in the form of (1.10). Denote by $W = \{w_1, w_2, \ldots, w_d\}$ the exponent vectors associated with the contiguous relations in C. Let V_C denote the vector space spanned by the multiple theta functions as given in (1.8) that satisfy the contiguous relations in C.

Based on Stanley's lemma [14, Lemma 4.5.7 (i)] on the fundamental parallelepiped, we obtain an upper bound of the dimension of V_C. Stanley's lemma is stated as follows.

Lemma 3.1. *Let $v_1, v_2, \ldots, v_d$ be linearly independent integer vectors over $\mathbb{Q}$ and let F be the set of linear combinations of $v_1, v_2, \ldots, v_d$ with rational coefficients, namely,*

$$F = \{\gamma \in \mathbb{Z}^r \mid \gamma = c_1v_1 + c_2v_2 + \cdots + c_dv_d,\ c_i \in \mathbb{Q} \text{ for } 1 \le i \le d\}.$$

Suppose that

$$\Pi = \{\lambda_1v_1 + \lambda_2v_2 + \cdots + \lambda_dv_d \mid 0 \le \lambda_i < 1 \text{ for } 1 \le i \le d\}$$

is the fundamental parallelepiped generated by $v_1, v_2, \ldots, v_d$. Then every element $\gamma \in F$ can be expressed uniquely in the following form

$$\gamma = \beta + b_1v_1 + b_2v_2 + \cdots + b_dv_d, \tag{3.1}$$

where $\beta \in \Pi \cap \mathbb{Z}^r$ and $b_1, b_2, \ldots, b_d$ are integers. Conversely, any vector γ in the form (3.1) belongs to F.

The following theorem gives an upper bound of the dimension of V_C.

Theorem 3.2. *The dimension of V_C is bounded by $|\Pi_W|$, where*

$$\Pi_W = \{\lambda_1w_1 + \lambda_2w_2 + \cdots + \lambda_dw_d \mid 0 \le \lambda_i < 1,\ 1 \le i \le d\} \cap \mathbb{Z}^r. \tag{3.2}$$

Proof. Write $w_i = (w_{i,1}, w_{i,2}, \ldots, w_{i,r})$ for $1 \le i \le d$. Note that V_θ stands for the vector space spanned by $w_1, w_2, \ldots, w_d$. Assume that $\theta(a_1, a_2, \ldots, a_r) \in V_C$. Let

$$\theta(a_1, a_2, \ldots, a_r) = \sum_{(\eta_1,\eta_2,\ldots,\eta_r)\in\mathbb{Z}^r} h_{\eta_1,\eta_2,\ldots,\eta_r} a_1^{\eta_1} a_2^{\eta_2} \cdots a_r^{\eta_r}. \tag{3.3}$$

By Jacobi's triple product identity (1.3), we get

$$\begin{aligned} f_i(a_1, a_2, \ldots, a_r) &= [(-1)^{\delta_i} a_1^{\gamma_{i,1}} a_2^{\gamma_{i,2}} \cdots a_r^{\gamma_{i,r}} q^{z_i}; q^{t_i}]_\infty \\ &= \frac{1}{(q^{t_i}; q^{t_i})_\infty} \sum_{n_i=-\infty}^{\infty} (-1)^{(1+\delta_i)n_i} q^{t_i\binom{n_i}{2}+z_in_i} \left(a_1^{\gamma_{i,1}} a_2^{\gamma_{i,2}} \cdots a_r^{\gamma_{i,r}}\right)^{n_i}. \end{aligned} \tag{3.4}$$

Hence

$$\begin{aligned} \theta(a_1, a_2, \ldots, a_r) &= \prod_{i=1}^{m} \frac{1}{(q^{t_i}; q^{t_i})_\infty} \sum_{n_i=-\infty}^{\infty} (-1)^{(1+\delta_i)n_i} q^{t_i\binom{n_i}{2}+z_in_i} \left(a_1^{\gamma_{i,1}} a_2^{\gamma_{i,2}} \cdots a_r^{\gamma_{i,r}}\right)^{n_i} \\ &= \sum_{m_i\in\mathbb{Z}} \left(\sum_{(n_1,n_2,\ldots,n_m)} \frac{(-1)^{(1+\delta_i)n_i} q^{t_i\binom{n_i}{2}+z_in_i}}{(q^{t_i}; q^{t_i})_\infty} \right) a_1^{m_1} a_2^{m_2} \cdots a_r^{m_r}, \end{aligned} \tag{3.5}$$

where the inner sum ranges over $(n_1, n_2, \ldots, n_m) \in \mathbb{Z}^m$ such that $n_1\gamma_1 + n_2\gamma_2 + \cdots + n_m\gamma_m = (m_1, m_2, \ldots, m_r)$. Comparing (3.3) and (3.5), we get

$$h_{\eta_1,\eta_2,\ldots,\eta_r} = \sum_{(n_1,n_2,\ldots,n_m)} \frac{(-1)^{(1+\delta_i)n_i} q^{t_i\binom{n_i}{2}+z_in_i}}{(q^{t_i}; q^{t_i})_\infty},$$

where the sum ranges over $(n_1, n_2, \ldots, n_m) \in \mathbb{Z}^m$ such that

$$\eta = (\eta_1, \eta_2, \ldots, \eta_r) = n_1\gamma_1 + n_2\gamma_2 + \cdots + n_m\gamma_m. \tag{3.6}$$

Since $\gamma_1, \gamma_2, \ldots, \gamma_m$ are linearly independent, we see that for a given exponent vector η of a nonzero term in the expansion (3.3) there is a unique vector $(n_1, n_2, \ldots, n_m)$ in $\mathbb{Z}^m$ that satisfies (3.6). It follows that $\eta \in U_\theta$. By Theorem 2.1, that is, $U_\theta = V_\theta$, we have $\eta \in V_\theta$.

Since $w_1, w_2, \ldots, w_d$ form a basis of V_θ, η as given in (3.6) can be expressed as

$$\eta = g_1 w_1 + g_2 w_2 + \cdots + g_d w_d, \tag{3.7}$$

where $g_i \in \mathbb{Q}$. According to Lemma 3.1, η can be uniquely written as

$$\eta = \beta + b_1 w_1 + b_2 w_2 + \cdots + b_d w_d, \tag{3.8}$$

where $b_1, b_2, \ldots, b_d$ are integers and $\beta \in \Pi_W$.

Next we show that V_C is isomorphic to a vector space whose dimension does not exceed $|\Pi_W|$. Let $d_w = |\Pi_W|$ and $\beta_1, \beta_2, \ldots, \beta_{d_w}$ be the elements of Π_W listed in any order. Let

$$H_\theta = (h_{\beta_1}, h_{\beta_2}, \ldots, h_{\beta_{d_w}}), \tag{3.9}$$

where h_{β_i} is the coefficient of a^{β_i} in the expansion (3.3) of $\theta(a)$. Let

$$H = \{H_\theta \,|\, \theta \in V_C\},$$

and let V_H be the vector space spanned by H. It is clear that $\dim V_H \le d_w$. Define a linear map $\phi\colon V_C \to V_H$ by $\phi(\theta) = H_\theta$. We proceed to show that ϕ is a bijection so that $\dim V_C = \dim V_H \le d_w$.

To prove that ϕ is a bijection, it suffices to show that there is a unique $\theta \in V_C$ such that $\phi(\theta) = H_\theta$. Assume to the contrary that $\phi(\theta_1) = \phi(\theta_2) = H_\theta$, where $\theta_1, \theta_2 \in V_C$ and $\theta_1 \neq \theta_2$. Let $\theta = \theta_1 - \theta_2$. Then θ is in V_C and $\phi(\theta) = 0$. Assume that θ is expanded as (3.3) with h_η being the coefficient of a^η. Since $\phi(\theta) = (h_{\beta_1}, h_{\beta_2}, \ldots, h_{\beta_{d_w}}) = 0$, we have $h_{\beta_i} = 0$ for $\beta_i \in \Pi_W$.

It remains to show that $\theta = 0$. Recall that for any vector $w_i = (w_{i,1}, w_{i,2}, \ldots, w_{i,r})$ in W, it corresponds to a contiguous relation in C, namely,

$$\frac{\theta(a_1 q^{\alpha_{i,1}}, a_2 q^{\alpha_{i,2}}, \ldots, a_r q^{\alpha_{i,r}})}{\theta(a_1, a_2, \ldots, a_r)} = \frac{(-1)^{\delta_i}}{a_1^{w_{i,1}} a_2^{w_{i,2}} \cdots a_r^{w_{i,r}} q^{s_i}}. \tag{3.10}$$

Rewriting the above contiguous relation as

$$\theta(a_1, a_2, \ldots, a_r) = (-1)^{\delta_i} a_1^{w_{i,1}} a_2^{w_{i,2}} \cdots a_r^{w_{i,r}} q^{s_i} \theta(a_1 q^{\alpha_{i,1}}, a_2 q^{\alpha_{i,2}}, \ldots, a_r q^{\alpha_{i,r}})$$

and equating the coefficients of the terms with exponent vector $\eta = (\eta_1, \eta_2, \ldots, \eta_r)$ on both sides, we obtain the following relation

$$h_\eta = (-1)^{\delta_i} q^{s_i + \alpha_i(\eta - w_i)} h_{\eta - w_i},$$

where $\alpha_i = (\alpha_{i,1}, \alpha_{i,2}, \dots, \alpha_{i,r})$. Iterating the above relation b_i times, we deduce that

$$h_\eta = (-1)^{b_i\delta_i} q^{b_i s_i + b_i \alpha_i \eta - \binom{b_i+1}{2}\alpha_i w_i} h_{\eta - b_i w_i}. \tag{3.11}$$

Iterating (3.11) for $i = 1, 2, \dots, d$, we find that

$$h_\eta = (-1)^{\sum_{i=1}^d b_i\delta_i} q^{\sum_{i=1}^d \left(b_i s_i - \binom{b_i+1}{2}\alpha_i w_i + b_i(\alpha_i\eta - \sum_{j=1}^{i-1} b_j \alpha_i w_j)\right)} h_{\eta - b_1 w_1 - b_2 w_2 - \cdots - b_d w_d}. \tag{3.12}$$

Substituting (3.8) into (3.12), we get

$$\begin{aligned} h_\eta &= (-1)^{\sum_{i=1}^d b_i\delta_i} q^{\sum_{i=1}^d \left(b_i s_i - \binom{b_i+1}{2}\alpha_i w_i + b_i(\alpha_i\beta + b_i\alpha_i w_i + \sum_{j=i+1}^{d} b_j \alpha_i w_j)\right)} h_\beta \\ &= (-1)^{\sum_{i=1}^d b_i\delta_i} q^{\sum_{i=1}^d \left(\frac{\alpha_i w_i}{2} b_i^2 - \frac{\alpha_i w_i}{2} b_i + b_i s_i + b_i(\alpha_i\beta + \sum_{j=i+1}^{d} b_j \alpha_i w_j)\right)} h_\beta, \end{aligned} \tag{3.13}$$

where β belongs to Π_W. But $h_\beta = 0$ for any $\beta \in \Pi_W$, so we deduce that $h_\eta = 0$ for any exponent vector η of a nonzero term of θ in the expansion (3.3). Thus we obtain $\theta = 0$, namely, $\theta_1 = \theta_2$, a contradiction to the assumption, and so the proof is complete. ∎

4. Multiple theta function identities

In this section, we demonstrate how to apply the vector space approach to prove multiple theta function identities. In general, we consider a multiple theta function identity in the form of (1.11), that is,

$$\sum_{k=1}^{m} c_k \theta_k(a_1, a_2, \dots, a_r) = 0, \tag{4.1}$$

where $1 < m \le r$. Recall that

$$\theta(a_1, a_2, \dots, a_r) = \prod_{i=1}^{d} f_i(a_1, a_2, \dots, a_r),$$

where

$$f_i(a_1, a_2, \dots, a_r) = [(-1)^{\delta_i} a_1^{\gamma_{i,1}} a_2^{\gamma_{i,2}} \cdots a_r^{\gamma_{i,r}} q^{z_i}; q^{t_i}]_\infty.$$

The first example is Bailey's generalization of the quintuple product identity,

$$\begin{aligned} &(b/a, aq/b, df/a, aq/df, ef/a, aq/ef, bde/a, aq/bde; q)_\infty \\ &\quad = (f/a, aq/f, bd/a, aq/bd, be/a, aq/be, def/a, aq/def; q)_\infty \\ &\qquad - \frac{b}{a}(d, q/d, e, q/e, f/b, bq/f, bdef/a^2, a^2q/bdef; q)_\infty, \end{aligned} \tag{4.2}$$

see [3]. Clearly, (4.2) is of the form (4.1), that is, $m = 3$, $d = 4$, and

$$\theta_1 - \theta_2 + \theta_3 = 0, \tag{4.3}$$

where

$$\theta_1 = (b/a, aq/b, df/a, aq/df, ef/a, aq/ef, bde/a, aq/bde; q)_\infty,$$

$$\theta_2 = (f/a, aq/f, bd/a, aq/bd, be/a, aq/be, def/a, aq/def; q)_\infty,$$

$$\theta_3 = \frac{b}{a}(d, q/d, e, q/e, f/b, bq/f, bdef/a^2, a^2q/bdef; q)_\infty.$$

An identity of the form (4.1) can be verified via three steps.

Step 1. For any $1 \le k \le m$, let U_k be the vector space spanned by exponent vectors $(\gamma_{i,1}, \gamma_{i,2}, \ldots, \gamma_{i,r})$ of $(a_1, a_2, \ldots, a_r)$ in the factors of θ_k and let V_k be the vector space consisting of exponent vectors $(w_{i,1}, w_{i,2}, \ldots, w_{i,r})$ of all contiguous relations of θ_k in the form (1.10), namely,

$$\frac{\theta(a_1 q^{\alpha_{i,1}}, a_2 q^{\alpha_{i,2}}, \ldots, a_r q^{\alpha_{i,r}})}{\theta(a_1, a_2, \ldots, a_r)} = \frac{(-1)^{\rho_i}}{a_1^{w_{i,1}} a_2^{w_{i,2}} \cdots a_r^{w_{i,r}} q^{s_i}}. \tag{4.4}$$

Since the exponent vectors $(\gamma_{i,1}, \gamma_{i,2}, \ldots, \gamma_{i,r})$ in the factors of θ are supposed to be linearly independent, the dimension of the vector space U_k is d. Moreover, one can verify that $U_1 = U_2 = \cdots = U_m$. By Theorem 2.1, we have $U_k = V_k$, which implies that $\dim V_k = d$. We begin with d linearly independent contiguous relations satisfied by any θ_k, which can be found by using the procedure given in Lemma 2.3.

For the above identity (4.2), it can be easily checked that $U_1 = U_2 = U_3$. The exponent vectors in the factors of θ_1 are linearly independent, so that $\dim U_1 = 4$. By Theorem 2.1, we have $\dim V_1 = 4$. Following the procedure given in Lemma 2.3, we find that θ_1 satisfies the following four linearly independent contiguous relations

$$\frac{\theta(aq, b, d, e, fq)}{\theta(a, b, d, e, f)} = \frac{b^2 de}{a^2 q^2},$$

$$\frac{\theta(aq, bq, dq, e, f)}{\theta(a, b, d, e, f)} = \frac{f}{bdq},$$

$$\frac{\theta(aq, b, dq, eq, f)}{\theta(a, b, d, e, f)} = \frac{1}{deq},$$

$$\frac{\theta(aq, q, dq, e, fq)}{\theta(a, b, d, e, f)} = \frac{b}{dfq}. \tag{4.5}$$

In fact, since $U_1 = U_2 = U_3$, we see that θ_2 and θ_3 also satisfy the above contiguous relations.

Step 2. Let W denote the set of exponent vectors in the d linearly independent contiguous relations obtained in Step 1. Applying Stanley's lemma, it can be shown that all the multiple theta functions in the form of (1.8) that satisfy the contiguous relations associated to W form a vector space, denoted by V_C. By Theorem 3.2, the dimension of V_C is bounded by $|\Pi_W|$, where Π_W is the set of integer vectors in the fundamental parallelepiped associated with W, as given by (3.2). Let $\Pi_W = \{\beta_1, \beta_2, \ldots, \beta_{d_w}\}$.

For example, for the contiguous relations in (4.5), we have

$$W = \{(2, -2, -1, -1, 0), (0, 1, 1, 0, -1), (0, 0, 1, 1, 0), (0, -1, 1, 0, 1)\},$$

and

$$\Pi_W = \{\lambda_1(2, -2, -1, -1, 0) + \lambda_2(0, 1, 1, 0, -1) + \lambda_3(0, 0, 1, 1, 0)$$

$$+ \lambda_4(0,-1,1,0,1) \mid 0 \le \lambda_i < 1, i = 1,2,3,4\} \cap \mathbb{Z}^5$$
$$= \big\{(0,0,0,0,0),(0,0,1,0,0),(1,-1,0,0,0),(1,-1,1,0,0)\big\}.$$

Let $\beta_1 = (0,0,0,0,0)$, $\beta_2 = (0,0,1,0,0)$, $\beta_3 = (1,-1,0,0,0)$ and $\beta_4 = (1,-1,1,0,0)$. Since $d_w = 4$, by Theorem 3.2, we have $\dim V_C \le 4$.

Step 3. We use the upper bound of $\dim V_C$ to transform identity (4.1) into a finite number of relations satisfied by the coefficients $c_1, c_2, \ldots, c_m$. More precisely, for $1 \le k \le m$, let

$$H_{\theta_k} = (h_{k,\beta_1}, h_{k,\beta_2}, \ldots, h_{k,\beta_{d_w}}),$$

where h_{k,β_i} is the coefficient of the term with exponent vector $\beta_i \in \Pi_W$ in the expansion (3.3) of θ_k. Since $\theta_1, \theta_2, \ldots, \theta_m$ satisfy the same linearly independent contiguous relations associated with W, the coefficients of each θ_k satisfy the same recurrence relation (3.13). Following the procedure to prove Theorem 3.2, any coefficient in the expansion of θ_k can be iteratively computed from H_{θ_k}. This implies that identity (4.1) can be proved by verifying the following relation

$$\sum_{k=1}^{m} c_k H_{\theta_k} = 0,$$

or equivalently, for any $\beta_i \in \Pi_W$

$$\sum_{k=1}^{m} c_k h_{k,\beta_i} = 0.$$

For example, (4.2) can be reduced to the relation

$$H_{\theta_1} - H_{\theta_2} + H_{\theta_3} = 0,$$

where

$$H_{\theta_k} = (h_{k,\beta_1}, h_{k,\beta_2}, h_{k,\beta_3}, h_{k,\beta_4})$$

for $1 \le k \le 3$. That is to say that for any $\beta_i \in \Pi_W$

$$h_{1,\beta_i} - h_{2,\beta_i} + h_{3,\beta_i} = 0. \tag{4.6}$$

In summary, identity (4.2) holds if the above relations can be verified for $1 \le i \le 4$.

Let us consider the case $k = 1$ and $i = 1$. In this case, $h_{1,\beta_1} = [a^0 b^0 d^0 e^0 f^0]\,\theta_1$. By Jacobi's triple product identity (1.3), we have

$$\begin{aligned}
[a^0b^0d^0e^0f^0]\,\theta_1 &= [a^0b^0d^0e^0f^0]\,(b/a, aq/b, df/a, aq/df, ef/a, aq/ef, bde/a, aq/bde; q)_\infty \\
&= [a^0b^0d^0e^0f^0]\,\frac{1}{(q;q)_\infty^4} \sum_{n=-\infty}^{\infty} (-1)^n q^{\binom{n}{2}} (b/a)^n \sum_{n=-\infty}^{\infty} (-1)^n q^{\binom{n}{2}} (df/a)^n \\
&\qquad \times \sum_{n=-\infty}^{\infty} (-1)^n q^{\binom{n}{2}} (ef/a)^n \sum_{n=-\infty}^{\infty} (-1)^n q^{\binom{n}{2}} (bde/a)^n \\
&= \frac{1}{(q;q)_\infty^4} \sum_{(n_1,n_2,n_3,n_4)\in\mathbb{Z}^4} (-1)^{n_1+n_2+n_3+n_4} q^{\binom{n_1}{2}+\binom{n_2}{2}+\binom{n_3}{2}+\binom{n_4}{2}},
\end{aligned} \tag{4.7}$$

where the summation in (4.7) ranges over all $(n_1, n_2, n_3, n_4) \in \mathbb{Z}^4$ such that

$$\begin{cases} n_1 + n_2 + n_3 + n_4 = 0, \\ n_1 + n_4 = 0, \\ n_2 + n_4 = 0, \\ n_3 + n_4 = 0, \\ n_2 + n_3 = 0. \end{cases}$$

Since the above system of equations has a unique solution $n_1 = n_2 = n_3 = n_4 = 0$, we get

$$[a^0b^0d^0e^0f^0]\,\theta_1 = \frac{1}{(q;q)_\infty^4}.$$

Following the same procedure, we get

$$[a^0b^0d^0e^0f^0]\,\theta_2 = \frac{1}{(q;q)_\infty^4}, \qquad [a^0b^0d^0e^0f^0]\,\theta_3 = 0,$$

and hence relation (4.6) holds for $i = 1$. Similarly, for $i = 2, 3, 4$, we obtain

$$\begin{array}{lll} [a^0b^0d^1e^0f^0]\,\theta_1 = 0 & [a^0b^0d^1e^0f^0]\,\theta_2 = 0 & [a^0b^0d^1e^0f^0]\,\theta_3 = 0 \\ [a^1b^{-1}d^0e^0f^0]\,\theta_1 = -\frac{q}{(q;q)_\infty^4}, & [a^1b^{-1}d^0e^0f^0]\,\theta_2 = 0, & [a^1b^{-1}d^0e^0f^0]\,\theta_3 = \frac{q}{(q;q)_\infty^4}, \\ [a^1b^{-1}d^1e^0f^0]\,\theta_1 = 0, & [a^1b^{-1}d^1e^0f^0]\,\theta_2 = -\frac{q^2}{(q;q)_\infty^4}, & [a^1b^{-1}d^1e^0f^0]\,\theta_3 = -\frac{q^2}{(q;q)_\infty^4}. \end{array}$$

Thus (4.6) is verified for any i. This proves identity (4.2).

The next example is concerned with the extended Riemann identity on theta functions proved by Malekar and Bhate [10, Theorem 3.1] by using eigenvectors corresponding to the discrete Fourier transform $\Phi(2)$, which can be reformulated as follows.

Example 4.1 (Extended Riemann Identity)**.** *We have*

$$\begin{aligned} &4qxyuv[-q^2x^2, -q^2y^2, -q^2u^2, -q^2v^2; q^2]_\infty + 4q^{\frac{1}{2}}xy[-q^2x^2, -q^2y^2, -qu^2, -qv^2; q^2]_\infty \\ &\quad + 4q^{\frac{1}{2}}uv[-qx^2, -qy^2, -q^2u^2, -q^2v^2; q^2]_\infty + 4[-qx^2, -qy^2, -qu^2, -qv^2; q^2]_\infty \\ &= \frac{(q^{\frac{1}{2}}; q^{\frac{1}{2}})_\infty^4}{(q^2; q^2)_\infty^4}\big([-q^{\frac{1}{4}}x, -q^{\frac{1}{4}}y, -q^{\frac{1}{4}}u, -q^{\frac{1}{4}}v; q^{\frac{1}{2}}]_\infty + [q^{\frac{1}{4}}x, q^{\frac{1}{4}}y, -q^{\frac{1}{4}}u, -q^{\frac{1}{4}}v; q^{\frac{1}{2}}]_\infty \\ &\quad + [-q^{\frac{1}{4}}x, -q^{\frac{1}{4}}y, q^{\frac{1}{4}}u, q^{\frac{1}{4}}v; q^{\frac{1}{2}}]_\infty + [q^{\frac{1}{4}}x, q^{\frac{1}{4}}y, q^{\frac{1}{4}}u, q^{\frac{1}{4}}v; q^{\frac{1}{2}}]_\infty\big). \end{aligned} \tag{4.8}$$

Proof. It is easy to check that each multiple theta function in (4.8) satisfies the following four linearly independent contiguous relations

$$\frac{f(xq, y, u, v)}{f(x, y, u, v)} = \frac{1}{x^2q},$$
$$\frac{f(x, yq, u, v)}{f(x, y, u, v)} = \frac{1}{y^2q},$$

$$\frac{f(x,y,uq,v)}{f(x,y,u,v)} = \frac{1}{u^2q},$$
$$\frac{f(x,y,u,vq)}{f(x,y,u,v)} = \frac{1}{v^2q}.$$

Then we have

$$W = \{(2,0,0,0),(0,2,0,0),(0,0,2,0),(0,0,0,2)\}$$

and

$$\begin{aligned}\Pi_W &= \{\lambda_1(2,0,0,0)+\lambda_2(0,2,0,0)+\lambda_3(0,0,2,0)\\ &\qquad + \lambda_4(0,0,0,2) \mid 0\le \lambda_i < 1, 1\le i\le 4\} \cap \mathbb{Z}^4\\ &= \{(a_1,a_2,a_3,a_4) \mid a_i = 0 \text{ or } 1, 1 \le i \le 4\}.\end{aligned}$$

By Theorem 3.2, (4.8) can be verified by showing that it holds only for the the terms $x^{\eta_1}y^{\eta_2}u^{\eta_3}v^{\eta_4}$ with $(\eta_1,\eta_2,\eta_3,\eta_4) \in \Pi_W$. In view of the symmetries of x,y and u,v, it is sufficient to show that it holds for the terms with exponent vectors in

$$\Pi'_W = \{(0,0,0,0),(1,0,0,0),(1,1,0,0),(1,0,1,0),(1,1,1,0),(1,1,1,1)\}$$

In fact, in each case, the required equality can be justified by Jacobi's triple product identity (1.3).

For example, let $(\eta_1,\eta_2,\eta_3,\eta_4) = (0,0,0,0)$. Denote the multiple theta functions on both sides by L_1,L_2,L_3,L_4 and R_1,R_2,R_3,R_4, respectively. For the left hand side, it is clear that

$$\begin{aligned}&[x^0y^0u^0v^0]\,4\,(L_1+L_2+L_3+L_4)\\ &\quad = [x^0y^0u^0v^0]\,4\,[-qx^2,-qy^2,-qu^2,-qv^2;q^2]_\infty\\ &\quad = [x^0y^0u^0v^0]\,\frac{4}{(q^2;q^2)_\infty^4}\sum_{n\in\mathbb{Z}} q^{2\binom{n}{2}}(qx^2)^n \sum_{n\in\mathbb{Z}} q^{2\binom{n}{2}}(qy^2)^n \sum_{n\in\mathbb{Z}} q^{2\binom{n}{2}}(qu^2)^n \sum_{n\in\mathbb{Z}} q^{2\binom{n}{2}}(qv^2)^n\\ &\quad = \frac{4}{(q^2;q^2)_\infty^4}.\end{aligned}$$

Similarly, we have

$$[x^0y^0u^0v^0]\,\frac{(q^{\frac12};q^{\frac12})_\infty^4}{(q^2;q^2)_\infty^4}(R_1+R_2+R_3+R_4) = \frac{(q^{\frac12};q^{\frac12})_\infty^4}{(q^2;q^2)_\infty^4}\frac{4}{(q^{\frac12};q^{\frac12})_\infty^4} = \frac{4}{(q^2;q^2)_\infty^4}.$$

Hence identity (4.8) holds with respect to the coefficients of the term $x^0y^0u^0v^0$. The other cases can be verified in the same way. ∎

The following identity was derived by Chu [6], which can be verified by our approach.

Example 4.2. *We have*

$$[d/b, c/b, cd/\alpha, \alpha; q]_\infty - [c, d, \alpha/b, cd/b\alpha; q]_\infty = c[cd/b, 1/b, d/\alpha, \alpha/c; q]_\infty. \tag{4.9}$$

Proof. Denote the three terms on both sides of the identity by L_1, L_2 and R, respectively. It is easy to check that L_1, L_2 and R satisfy the following four linearly independent contiguous relations

$$\frac{f(bq,cq,d,\alpha)}{f(b,c,d,\alpha)} = \frac{\alpha}{bcq},$$
$$\frac{f(bq,c,dq,\alpha)}{f(b,c,d,\alpha)} = \frac{\alpha}{bdq},$$
$$\frac{f(b,cq,d,\alpha q)}{f(b,c,d,\alpha)} = \frac{b}{c\alpha},$$
$$\frac{f(bq,c,d,\alpha)}{f(b,c,d,\alpha)} = \frac{cd}{b^2 q}.$$

Thus we have

$$W = \{(1,1,0,-1),(1,0,1,-1),(-1,1,0,1),(2,-1,-1,0)\}$$

and

$$\begin{aligned}\Pi_W = \{&\lambda_1(1,1,0,-1)+\lambda_2(1,0,1,-1)+\lambda_3(-1,1,0,1)\\ &+\lambda_4(2,-1,-1,0) \mid 0 \le \lambda_i < 1, 1 \le i \le 4\} \cap \mathbb{Z}^4\\ =& \big\{(0,0,0,0),(1,0,0,0),(0,1,0,0),(2,0,0,-1)\big\}.\end{aligned}$$

Using Jacobi's triple product identity (1.3), we have

$$\begin{aligned}
&[b^0c^0d^0\alpha^0]\,L_1 = \tfrac{1}{(q;q)_\infty^4}, && [b^0c^0d^0\alpha^0]\,L_2 = -\tfrac{1}{(q;q)_\infty^4}, && [b^0c^0d^0\alpha^0]\,R = 0,\\
&[b^1c^0d^0\alpha^0]\,L_1 = 0 && [b^1c^0d^0\alpha^0]\,L_2 = 0 && [b^1c^0d^0\alpha^0]\,R = 0\\
&[b^0c^1d^0\alpha^0]\,L_1 = 0, && [b^0c^1d^0\alpha^0]\,L_2 = \tfrac{1}{(q;q)_\infty^4}, && [b^0c^1d^0\alpha^0]\,R = \tfrac{1}{(q;q)_\infty^4},\\
&[b^2c^0d^0\alpha^{-1}]\,L_1 = -\tfrac{q^2}{(q;q)_\infty^4}, && [b^2c^0d^0\alpha^{-1}]\,L_2 = 0, && [b^2c^0d^0\alpha^{-1}]\,R = -\tfrac{q^2}{(q;q)_\infty^4}.
\end{aligned}$$

Using Theorem 3.2, we obtain (4.9). ∎

5. Addition formulas for multiple theta functions

Addition formulas for theta functions play an important role in the theory of elliptic functions, see, for example, [4, 9, 13, 16]. In this section, we show that some classical addition formulas for theta functions can be verified by using our vector space approach, such as Riemann's addition formula and the addition formulas for the Jacobi theta functions.

First, let us consider the Riemann's addition formula, see, for example, Krattenthaler [9]. Weierstrass showed that it is equivalent to an identity on sigma functions, see Whittaker and Watson [16, p. 451].

Example 5.1 (Riemann's addition formula)**.** *Let* $\theta(z) = (z, q/z; q)_\infty$. *We have*

$$\begin{aligned}\theta(xy)\theta(x/y)\theta(uv)\theta(u/v) - \theta(xv)\theta(x/v)\theta(uy)\theta(u/y)\\ = \frac{u}{y}\theta(yv)\theta(y/v)\theta(xu)\theta(x/u).\end{aligned} \tag{5.1}$$

Proof. Denote the three multiple theta functions in (5.1) by L_1, L_2 and R, respectively. It can be checked that L_1, L_2 and R satisfy the following four linearly independent contiguous relations

$$\frac{f(xq, y, u, v)}{f(x, y, u, v)} = \frac{1}{x^2},$$
$$\frac{f(x, yq, u, v)}{f(x, y, u, v)} = \frac{1}{y^2 q},$$
$$\frac{f(x, y, uq, v)}{f(x, y, u, v)} = \frac{1}{u^2},$$
$$\frac{f(x, y, u, vq)}{f(x, y, u, v)} = \frac{1}{v^2}.$$

Then we have

$$W = \{(2,0,0,0), (0,2,0,0), (0,0,2,0), (0,0,0,2)\}$$

and

$$\begin{aligned}\Pi_W &= \{\lambda_1(2,0,0,0) + \lambda_2(0,2,0,0) + \lambda_3(0,0,2,0)\\ &\quad + \lambda_4(0,0,0,2) \mid 0 \le \lambda_i < 1, 1 \le i \le 4\} \cap \mathbb{Z}^4\\ &= \{(a_1, a_2, a_3, a_4) \mid a_i = 0 \text{ or } 1, 1 \le i \le 4\}.\end{aligned}$$

Owing to the symmetries of the parameters x, y, u and v, it is suffices to show that identity (5.1) holds for the terms with exponent vectors in

$$\Pi'_W = \{(0,0,0,0), (1,0,0,0), (1,1,0,0), (1,1,1,0), (1,1,1,1)\}.$$

Using Jacobi's triple product identity (1.3), we have

$$\begin{aligned}
&[x^0y^0u^0v^0]\, L_1 = \frac{1}{(q;q)_\infty^4}, &&[x^0y^0u^0v^0]\, L_2 = \frac{1}{(q;q)_\infty^4}, &&[x^0y^0u^0v^0]\, R = 0,\\
&[x^1y^0u^0v^0]\, L_1 = 0, &&[x^1y^0u^0v^0]\, L_2 = 0, &&[x^1y^0u^0v^0]\, R = 0,\\
&[x^1y^1u^0v^0]\, L_1 = \frac{-1}{(q;q)_\infty^4}, &&[x^1y^1u^0v^0]\, L_2 = 0, &&[x^1y^1u^0v^0]\, R = \frac{-1}{(q;q)_\infty^4},\\
&[x^1y^1u^1v^0]\, L_1 = 0, &&[x^1y^1u^1v^0]\, L_2 = 0, &&[x^1y^1u^1v^0]\, R = 0,\\
&[x^1y^1u^1v^1]\, L_1 = \frac{1}{(q;q)_\infty^4}, &&[x^1y^1u^1v^1]\, L_2 = \frac{1}{(q;q)_\infty^4}, &&[x^1y^1u^1v^1]\, R = 0.
\end{aligned}$$

Using Theorem 3.2, we complete the proof. ∎

Our approach is also applicable to addition formulas on Jacobi theta functions (5.2) due to Whittaker and Watson [16, Chapter XXI]. Recall that the four Jacobi theta functions are given by

$$
\begin{aligned}
\vartheta_1(z,q) &= \sum_{n=-\infty}^{\infty} (-1)^{n-\frac{1}{2}} q^{(n+\frac{1}{2})^2} e^{(2n+1)iz}, \\
\vartheta_2(z,q) &= \sum_{n=-\infty}^{\infty} q^{(n+\frac{1}{2})^2} e^{(2n+1)iz}, \\
\vartheta_3(z,q) &= \sum_{n=-\infty}^{\infty} q^{n^2} e^{2niz}, \\
\vartheta_4(z,q) &= \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} e^{2niz}. \qquad (5.2)
\end{aligned}
$$

Let

$$
\begin{aligned}
2w' &= -w + x + y + z, \\
2x' &= w - x + y + z, \\
2y' &= w + x - y + z, \\
2z' &= w + x + y - z,
\end{aligned}
$$

see Whittaker and Watson [16, Chapter XXI].

For example, we consider two addition formulas given by Whittaker and Watson [16].

Example 5.2 (Whittaker and Watson [16, p. 468])**.** *We have*

$$
\begin{aligned}
&\vartheta_1(w)\vartheta_1(x)\vartheta_1(y)\vartheta_1(z) + \vartheta_2(w)\vartheta_2(x)\vartheta_2(y)\vartheta_2(z) \\
&\quad = \vartheta_1(w')\vartheta_1(x')\vartheta_1(y')\vartheta_1(z') + \vartheta_2(w')\vartheta_2(x')\vartheta_2(y')\vartheta_2(z'). \qquad (5.3)
\end{aligned}
$$

Proof. Making the substitutions $e^{iw} \to w$, $e^{ix} \to x$, $e^{iy} \to y$, $e^{iz} \to z$ and $q^2 \to q$, (5.3) becomes

$$
\begin{aligned}
&\theta(w^2q;q)\theta(x^2q;q)\theta(y^2q;q)\theta(z^2q;q) + \theta(-w^2q;q)\theta(-x^2q;q)\theta(-y^2q;q)\theta(-z^2q;q) \\
&\quad = \theta(qxyz/w;q)\theta(qwyz/x;q)\theta(qwxz/y;q)\theta(qwxy/z;q) \\
&\qquad + \theta(-qxyz/w;q)\theta(-qwyz/x;q)\theta(-qwxz/y;q)\theta(-qwxy/z;q), \qquad (5.4)
\end{aligned}
$$

where $\theta(z) = (z, q/z; q)_\infty$. Denote the multiple theta functions in (5.4) by L_1, L_2 and R_1, R_2, respectively. It is easy to check that L_1, L_2 and R_1, R_2 satisfy the following four linearly independent contiguous relations

$$
\begin{aligned}
\frac{f(wq, xq, y, z)}{f(w, x, y, z)} &= \frac{1}{w^2x^2q^2}, \\
\frac{f(wq, x, yq, z)}{f(w, x, y, z)} &= \frac{1}{w^2y^2q^2},
\end{aligned}
$$

$$\frac{f(wq,x,yq,zq)}{f(w,x,y,z)} = \frac{1}{w^2z^2q^2},$$
$$\frac{f(w,xq,yq,z)}{f(w,x,y,z)} = \frac{1}{x^2y^2q^2}.$$

Hence
$$W = \{(2,2,0,0),(2,0,2,0),(2,0,0,2),(0,2,2,0)\}$$
and
$$\begin{aligned}\Pi_W &= \{\lambda_1(2,2,0,0)+\lambda_2(2,0,2,0)+\lambda_3(2,0,0,2)+\lambda_4(0,2,2,0)\\ &\qquad \mid 0\le\lambda_i<1, i=1,2,3,4\}\cap\mathbb{Z}^4\\ &= \big\{(0,0,0,0),(1,1,0,0),(1,0,1,0),(1,0,0,1),(0,1,1,0),(2,1,1,0),\\ &\quad (2,1,0,1),(1,2,1,0),(2,0,1,1),(1,1,2,0),(1,1,1,1),(3,1,1,1),\\ &\quad (2,2,1,1),(2,2,2,0),(2,1,2,1),(3,2,2,1),(1,1,1,0),(2,2,1,0),\\ &\quad (2,1,2,0),(2,1,1,1),(1,2,2,0),(3,2,2,0),(3,2,1,1),(2,3,2,0),\\ &\quad (3,1,2,1),(2,2,3,0),(2,2,2,1),(4,2,2,1),(3,3,2,1),(3,3,3,0),\\ &\quad (3,2,3,1),(4,3,3,1)\big\}.\end{aligned}$$

Notice that any nonzero term on both sides of identity (5.4) has even powers in w, x, y and z. Hence it is sufficient to show that identity (5.4) holds only for the terms with exponent vectors belonging to
$$\Pi'_W = \{(0,0,0,0),(2,2,2,0)\}.$$
By Jacobi's triple product identity (1.3), we obtain that

$$\begin{aligned}
&[w^0z^0y^0z^0]\,L_1 = 1/(q;q)_\infty^4, && [w^0z^0y^0z^0]\,L_2 = 1/(q;q)_\infty^4,\\
&[w^0z^0y^0z^0]\,R_1 = 1/(q;q)_\infty^4, && [w^0z^0y^0z^0]\,R_2 = 1/(q;q)_\infty^4,\\
&[w^2z^2y^2z^0]\,L_1 = -q^3/(q;q)_\infty^4, && [w^2z^2y^2z^0]\,L_2 = q^3/(q;q)_\infty^4,\\
&[w^2z^2y^2z^0]\,R_1 = 0, && [w^2z^2y^2z^0]R_2 = 0.
\end{aligned}$$

This completes the proof. ∎

Example 5.3 (Whittaker and Watson [16, p. 468])**.** *We have*

$$\begin{aligned}2\vartheta_3(w)\vartheta_3(x)\vartheta_3(y)\vartheta_3(z) =& -\vartheta_1(w')\vartheta_1(x')\vartheta_1(y')\vartheta_1(z') + \vartheta_2(w')\vartheta_2(x')\vartheta_2(y')\vartheta_2(z')\\ &+\vartheta_3(w')\vartheta_3(x')\vartheta_3(y')\vartheta_3(z') + \vartheta_4(w')\vartheta_4(x')\vartheta_4(y')\vartheta_4(z'). \quad (5.5)\end{aligned}$$

Proof. By the substitutions $e^{iw}\to w,\ e^{ix}\to x,\ e^{iy}\to y,\ e^{iz}\to z$, we can rewrite (5.5) as follows

$$\begin{aligned}&2\theta(-qw^2;q^2)\theta(-qx^2;q^2)\theta(-qy^2;q^2)\theta(-qz^2;q^2)\\ &\quad = -qwxyz\theta(q^2xyz/w;q^2)\theta(q^2wyz/x;q^2)\theta(q^2wxz/y;q^2)\theta(q^2wxy/z;q^2)\end{aligned}$$

$$+ qwxyz\theta(-q^2xyz/w;q^2)\theta(-q^2wyz/x;q^2)\theta(-q^2wxz/y;q^2)\theta(-q^2wxy/z;q^2)$$
$$+ \theta(-qxyz/w;q^2)\theta(-qwyz/x;q^2)\theta(-qwxz/y;q^2)\theta(-qwxy/z;q^2)$$
$$+ \theta(qxyz/w;q^2)\theta(qwyz/x;q^2)\theta(qwxz/y;q^2)\theta(qwxy/z;q^2), \tag{5.6}$$

where $\theta(z) = (z, q/z; q)_\infty$.

Denote the terms in the above identity by L, R_1, R_2, R_3, R_4, respectively. It is easy to check that L, R_1, R_2, R_3, R_4 satisfy the following four linearly independent contiguous relations

$$\frac{f(wq^2, xq^2, y, z)}{f(w, x, y, z)} = \frac{1}{w^2x^2q^2},$$
$$\frac{f(wq^2, x, yq^2, z)}{f(w, x, y, z)} = \frac{1}{w^2y^2q^2},$$
$$\frac{f(wq^2, x, y, zq^2)}{f(w, x, y, z)} = \frac{1}{w^2z^2q^2},$$
$$\frac{f(w, x, yq^2, zq^2)}{f(w, x, y, z)} = \frac{1}{y^2z^2q^2}.$$

Then we have

$$W = \{(2,2,0,0), (2,0,2,0), (2,0,0,2), (0,0,2,2)\}$$

and

$$\begin{aligned}\Pi_W &= \{\lambda_1(2,2,0,0) + \lambda_2(2,0,2,0) + \lambda_3(2,0,0,2) + \lambda_4(0,0,2,2) \mid 0 \le \lambda_i < 1\} \cap \mathbb{Z}^4 \\ &= \big\{(0,0,0,0), (1,1,0,0), (1,0,1,0), (1,0,0,1), (0,0,1,1), (2,1,1,0), \\ &\quad (2,1,0,1), (1,1,1,1), (2,0,1,1), (1,0,2,1), (1,0,1,2), (3,1,1,1), \\ &\quad (2,1,1,2), (2,1,2,1), (2,0,2,2), (3,1,2,2), (1,0,1,1), (2,1,1,1), \\ &\quad (2,0,2,1), (2,0,1,2), (1,0,2,2), (3,1,2,1), (3,1,1,2), (2,1,2,2) \\ &\quad (3,0,2,2), (2,0,3,2), (2,0,2,3), (4,1,2,2), (3,1,2,3), (3,1,3,2), \\ &\quad (3,0,3,3), (4,1,3,3)\big\}.\end{aligned}$$

Since (5.6) only contains terms with even powers in w, x, y, z, (5.4) can be justified by showing that it holds only for the terms with exponent vectors in

$$\Pi'_W = \{(0,0,0,0), (2,0,2,2)\}.$$

By Jacobi's triple product identity (1.3), we get

$$[w^0z^0y^0z^0]\, L = 2/(q^2;q^2)_\infty^4, \qquad [w^2x^0y^2z^2]\, L = 2q^3/(q^2;q^2)_\infty^4,$$
$$[w^0x^0y^0z^0]\, R_1 = 0, \qquad [w^0x^0y^0z^0]\, R_2 = 0,$$
$$[w^0x^0y^0z^0]\, R_3 = 1/(q^2;q^2)_\infty^4, \qquad [w^0x^0y^0z^0]\, R_4 = 1/(q^2;q^2)_\infty^4,$$
$$[w^2x^0y^2z^2]\, R_1 = q^3/(q^2;q^2)_\infty^4, \qquad [w^2x^0y^2z^2]\, R_2 = q^3/(q^2;q^2)_\infty^4,$$

$$[w^2x^0y^2z^2]\, R_3 = 0, \qquad\qquad [w^2x^0y^2z^2]\, R_4 = 0.$$

Hence the proof is complete. ∎

To conclude this paper, we remark that our vector space approach can be used to discover multiple theta function identities. For example, consider the following three linearly independent contiguous relations

$$\begin{aligned} \frac{f(aq,bq,cq)}{f(a,b,c)} &= \frac{1}{abc}, \\ \frac{f(aq^2,b,c)}{f(a,b,c)} &= \frac{1}{a^2q}, \\ \frac{f(a,bq^2,c)}{f(a,b,c)} &= \frac{1}{b^2q}. \end{aligned} \tag{5.7}$$

Then we have $W = \{(1,1,1),(2,0,0),(0,2,0)\}$ and

$$\Pi_W = \{(0,0,0),(1,0,0),(0,1,0),(1,1,0)\}.$$

Thus the dimension of V_C is bounded by four. By comparing the coefficients of (a,b,c) with exponent vectors in Π_W, we find the following identity which might be new

$$\begin{aligned} &(q;q^2)_\infty^2 \big([a,b,-c;q]_\infty + [a,-b,c;q]_\infty + [-a,b,c;q]_\infty\big) \\ &\quad = (a,q/a;q)_\infty [bc,bq/c;q^2]_\infty + (b,q/b;q)_\infty [ac,aq/c;q^2]_\infty + (c,q/c;q)_\infty [ab,aq/b;q^2]_\infty, \end{aligned} \tag{5.8}$$

which reduces to (1.13) by setting $c = -1$.

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