Context-free Grammars and Multivariate Stable Polynomials over Stirling Permutations

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Abstract

Recently, Haglund and Visontai established the stability of the multivariate Eulerian polynomials as the generating polynomials of the Stirling permutations, which serves as a unification of some results of Bóna, Brenti, Janson, Kuba, and Panholzer concerning Stirling permutations. Let $B_n(x)$ be the generating polynomials of the descent statistic over Legendre-Stirling permutations, and let $T_n(x) = 2^n C_n(x/2)$, where $C_n(x)$ are the second-order Eulerian polynomials. Haglund and Visontai proposed the problems of finding multivariate stable refinements of the polynomials $B_n(x)$ and $T_n(x)$. We obtain context-free grammars leading to multivariate stable refinements of the polynomials $B_n(x)$ and $T_n(x)$. Moreover, the grammars enable us to obtain combinatorial interpretations of the multivariate polynomials in terms of Legendre-Stirling permutations and marked Stirling permutations. Such stable multivariate polynomials provide solutions to two problems posed by Haglund and Visontai.

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1 Introduction

This paper presents an approach to the construction of stable combinatorial polynomials from the perspective of context-free grammars. The framework of using context-free grammars to generate combinatorial polynomials was proposed by Chen [6]. More specifically, we introduce the structure of marked Stirling permutations, and we find context-free grammars that lead to multivariate stable polynomials over marked Stirling permutations and Legendre-Stirling permutations. These multivariate stable polynomials provide solutions to two problems posed by Haglund and Visontai [13] in their study of multivariate stable refinements of the second-order Eulerian polynomials.

Let us first review some backgrounds on the second-order Eulerian polynomials. These polynomials were first introduced by Gessel and Stanley [10], which are defined as the generating functions of the descent statistic over Stirling permutations. Recall that a Stirling permutation of order n is a permutation $\pi = \pi_1\pi_2\cdots\pi_{2n-1}\pi_{2n}$ of the multiset $\{1, 1, 2, 2, \ldots, n, n\}$, denoted by $[n]_2$, which satisfies the following condition: if $\pi_i = \pi_j$ then $\pi_k > \pi_i$ whenever $i < k < j$. For $1 \le i \le 2n$, we say that i is a descent of π if $i = 2n$ or $\pi_i > \pi_{i+1}$. Analogously, i is called an ascent of π if $i = 1$ or $\pi_{i-1} < \pi_i$. Let Q_n denote the set of Stirling permutations of order n. Let $C(n, k)$ be the number of Stirling permutations of $[n]_2$ with k descents, and let

$$C_n(x) = \sum_{k=1}^{n} C(n,k)x^k.$$

Gessel and Stanley [10] showed that

$$\sum_{n=0}^{\infty} S(n+k,k)x^n = \frac{C_n(x)}{(1-x)^{2k+1}},$$

where $S(n,k)$, as usual, denotes the Stirling number of the second kind. The numbers $C(n,k)$ are called the second-order Eulerian numbers by Graham, Knuth and Patashnik [11], and accordingly the polynomials $C_n(x)$ are called the second-order Eulerian polynomials by Haglund and Visontai [13].

The Stirling permutations were further studied by Bóna [1], Brenti [5], Janson [14] and Janson, Kuba and Panholzer [15]. Bóna [1] introduced a statistic, called plateau, on stirling permutations, and proved that ascents, descents and plateaux have the same distribution over Q_n. Given a Stirling permutation $\pi = \pi_1\pi_2\ldots\pi_{2n} \in Q_n$, the index i is called a plateau of π if $\pi_{i-1} = \pi_i$. Analogous to that of the classical Eulerian polynomials, Bóna [1] obtained the real-rootedness of the second-order Eulerian polynomials $C_n(x)$.

Theorem 1.1 *For any positive integer n, the roots of the polynomial $C_n(x)$ are all real, distinct, and non-positive.*

It should be noted that the real-rootedness of $C_n(x)$ is essentially the real rootedness of the generating function of generalized Stirling permutations obtained by Brenti [5]. A permutation π of the multiset $\{1^{r_1}, 2^{r_2}, \ldots, n^{r_n}\}$ is called a generalized Stirling permutation of rank n if π satisfies the same condition as for a Stirling permutation. Let Q_n^* denote the set of generalized Stirling permutations of rank n. In particular, if $r_1 = r_2 = \cdots = r_n = r$ for some r, then π is called an r-Stirling permutation of order n. Let $Q_n(r)$ denote the set of r-Stirling permutations of order n. It is clear that 1-Stirling permutations are ordinary permutations and 2-Stirling permutations are the Stirling permutations. Brenti [5] showed that the descent generating polynomials over Q_n^* have only real roots.

Janson [14] defined the following trivariate generating function

$$C_n(x,y,z) = \sum_{\pi\in Q_n} x^{\mathrm{des}(\pi)} y^{\mathrm{asc}(\pi)} z^{\mathrm{plat}(\pi)},$$

where $\mathrm{des}(\pi)$, $\mathrm{asc}(\pi)$, and $\mathrm{plat}(\pi)$ denote the numbers of descents, the number of ascents, and the number of plateaux of π, respectively, and proved that $C_n(x,y,z)$ is symmetric in x, y, z. This implies the equidistribution of these three statistics derived by Bóna.

The symmetric property of $C_n(x,y,z)$ was further extended to r-Stirling permutations by Janson, Kuba and Panholzer [15]. For an r-Stirling permutation, they introduced the notion of a j-plateau. For an r-Stirling permutation $\pi = \pi_1\pi_2\ldots\pi_{nr}$ and an integer $1 \leqslant j \leqslant r-1$, a number $1 \leqslant i < nr$ is called a j-plateau of π if $\pi_i = \pi_{i+1}$ and there are $j-1$ indices $l < i$ such that $\pi_l = \pi_i$, i.e., the number π_i appears j times up to the i-th position of π. Let j-$\mathrm{plat}(\pi)$ denote the number of j-plateaux of π. Meanwhile, define a descent and an ascent of π

similar as ordinary permutations, and let $\mathrm{des}(\pi)$ and $\mathrm{asc}(\pi)$ denote the number of descents and ascents of π. Janson, Kuba and Panholzer [15] showed that the distribution of (des, 1-plat, 2-plat, ..., $(r-1)$-plat, asc) is symmetric over the set of r-Stirling permutations.

Based on the theory of multivariate stable polynomials recently developed by Borcea and Brändén [2–4], Haglund and Visontai [13] presented a unified approach to the stability of the generating functions of Stirling permutations and r-Stirling permutations. A polynomial $f(\mathbf{z}) \in \mathbb{C}[\mathbf{z}] = \mathbb{C}[z_1, z_2, \ldots, z_m]$ is said to be stable, if whenever the imaginary part $\mathrm{Im}(z_i) > 0$ for all i then $f(\mathbf{z}) \neq 0$. Clearly, a univariate polynomial $f(z) \in \mathbb{R}[z]$ has only real roots if and only if it is stable.

For the case of univariate real polynomials, Pólya and Schur [16] characterized all diagonal operators preserving stability or real-rootedness. Recently, Borcea and Brändén [2–4] characterized all linear operators preserving stability of multivariate polynomials, see also the survey of Wagner [18]. This implies a characterization of linear operators preserving stability of univariate polynomials.

A multivariate polynomial is called multiaffine if the degree of each variable is at most 1. Borcea and Brändén showed that each of the operators preserving stability of multiaffine polynomials has a simple form. Using this property, Haglund and Visontai [13] obtained a stable multiaffine refinement of the second-order Eulerian polynomial $C_n(x)$. Given a Stirling permutation $\pi = \pi_1\pi_2\cdots\pi_{2n} \in Q_n$, let

$$\begin{aligned} A(\pi) &= \{i|\pi_{i-1} < \pi_i\}, \\ D(\pi) &= \{i|\pi_i > \pi_{i+1}\}, \\ P(\pi) &= \{i|\pi_{i-1} = \pi_i\} \end{aligned}$$

denote the set of ascents, the set of descents and the set plateaux of π, respectively. Define

$$C_n(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \sum_{\pi \in Q_n} \prod_{i \in D(\pi)} x_{\pi_i} \prod_{i \in A(\pi)} y_{\pi_i} \prod_{i \in P(\pi)} z_{\pi_i}.$$

Haglund and Visontai [13] proved the stability of $C_n(\mathbf{x}, \mathbf{y}, \mathbf{z})$.

Theorem 1.2 *The polynomial $C_n(\mathbf{x}, \mathbf{y}, \mathbf{z})$ is stable.*

It is worth mentioning that, as observed by Haglund and Visontai, the recurrence relation between $C_{n-1}(\mathbf{x}, \mathbf{y}, \mathbf{z})$ and $C_n(\mathbf{x}, \mathbf{y}, \mathbf{z})$ can be used to derive the symmetry of $C_n(\mathbf{x}, \mathbf{y}, \mathbf{z})$, which implies the symmetry of $C_n(x, y, z)$ obtained by Janson, Kuba and Panholzer [15].

Moreover, Haglund and Visontai extended the stability of $C_n(\mathbf{x}, \mathbf{y}, \mathbf{z})$ to generating polynomials of r-Stirling permutations by taking the j-plateau statistic into consideration. Let $P_j(\pi)$ denote the set of j-plateaux of π. Haglund and Visontai [13] obtained the following multivariate stable polynomial over r-Stirling permutations

$$E_n(\mathbf{x}, \mathbf{y}, \mathbf{z}_1, \ldots, \mathbf{z}_{r-1}) = \sum_{\pi \in Q_n(r)} \left(\prod_{i \in D(\pi)} x_{\pi_i} \right) \left(\prod_{i \in A(\pi)} y_{\pi_i} \right) \prod_{j=1}^{r-1} \left(\prod_{i \in P_j(\pi)} z_{j,\pi_i} \right).$$

They also obtained a similar multivariate stable polynomial for generalized Stirling permutations.

In view of the real-rootedness of $C_n(x)$ and its multivariate stable refinement $C_n(\mathbf{x},\mathbf{y},\mathbf{z})$, Haglund and Visontai posed the problem of finding multivariate stable polynomials as refinements of the generating polynomials of the descent statistic over Legendre-Stirling permutations. The Legendre-Stirling permutations were introduced by Egge [9] as a generalization of Stirling permutations in the study of Legendre-Stirling numbers of the second kind. For any $n \geq 1$, let M_n be the multiset $\{1,1,\bar{1},2,2,\bar{2},\cdots,n,n,\bar{n}\}$. A permutation $\pi=\pi_1\pi_2\ldots\pi_{3n}$ on M_n is called a Legendre-Stirling permutation if whenever $i<j<k$ and $\pi_i=\pi_k$ are both unbarred, then $\pi_j>\pi_i$. For a Legendre-Stirling permutation π on M_n, we say that i is a descent if either $i=3n$ or $\pi_i>\pi_{i+1}$. Let $B_{n,k}$ denote the number of Legendre-Stirling permutations of M_n with k descents. Define

$$B_n(x)=\sum_{k=1}^{2n-1} B_{n,k}x^k.$$

Egge proved the real-rootedness of $B_n(x)$.

Theorem 1.3 *For $n \geq 1$, $B_n(x)$ has distinct, real, non-positive roots.*

In order to derive a multivariate stable refinement of $B_n(x)$, we introduce an approach of generating stable polynomials by a sequence of grammars. Based on the Stirling grammar given by Chen and Fu [7], we find a sequence $G_1,G_2,\ldots$ of context-free grammars to generate Legendre-Stirling permutations. We show that the formal derivative with respect to G_n preserves stability by applying Borcea and Brändén's characterization of linear operators preserving stability. This leads to a multivariate stable refinement $B_n(\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{u},\mathbf{v})$ of $B_n(x)$. On the other hand, according to the grammars, we obtain the following combinatorial interpretation

$$B_n(\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{u},\mathbf{v})=\sum_{\pi}\prod_{i\in X(\pi)}x_{\pi_i}\prod_{i\in Y(\pi)}y_{\pi_i}\prod_{i\in Z(\pi)}z_{\pi_i}\prod_{i\in U(\pi)}u_{\pi_i}\prod_{i\in V(\pi)}v_{\pi_i}.$$

The real-rootedness of $B_n(x)$ is a consequence of the stability of $B_n(\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{u},\mathbf{v})$ by setting $v_i=y_i=y$ and $x_i=z_i=u_i=1$ for $0\leq i\leq n$.

Haglund and Visontai also posed the problem of finding multivariate stable refinements of the polynomials $T_n(x)$, which are given by

$$T_n(x)=2^nC_n\left(\frac{x}{2}\right)=\sum_{k}2^{n-k}C(n,k)x^k, \tag{1.1}$$

where $C(n,k)$ and $C_n(x)$, as before, denote the second-order Eulerian numbers and the second-order Eulerian polynomials respectively. The polynomials $T_n(x)$ were introduced by Riordan [17].

In light of the relation (1.1) between $T_n(x)$ and $C_n(x)$, we introduce the structure of marked Stirling permutations and the following multivariate polynomials

$$T_n(\mathbf{x},\mathbf{y},\mathbf{z})=\sum_{\pi}\prod_{i\in D(\pi)}x_{\pi_i}\prod_{i\in A(\pi)}y_{\pi_i}\prod_{i\in P(\pi)}z_{\pi_i},$$

where π ranges over marked Stirling permutations of $[n]_2$. We shall show that the polynomials $T_n(\mathbf{x},\mathbf{y},\mathbf{z})$ are stable. The polynomial $T_n(x)$ becomes the specialization of $T_n(\mathbf{x},\mathbf{y},\mathbf{z})$ by setting $x_i=z_i=1$ and $y_i=x$ for $0\leq i\leq n$. This implies that $T_n(x)$ is real-rooted.

This paper is organized as follows. In Section 2, we give an overview of differential operators associated with context-free grammars. We find context-free grammars to generate the polynomials $C_n(\mathbf{x}, \mathbf{y}, \mathbf{z})$. In Section 3, we obtain context-free grammars that lead to the multivariate generating polynomials $B_n(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{u}, \mathbf{v})$. In Section 4, we introduce the structure of marked Stirling permutations, and we give context-free grammars to generate the multivariate polynomials $T_n(\mathbf{x}, \mathbf{y}, \mathbf{z})$. In Section 5, based on Borcea and Brändén's characterization of stability preserving linear operators, we present an approach to proving the stability of polynomials generated by context-free grammars. In particular, we prove the stability of multivariate polynomials $B_n(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{u}, \mathbf{v})$ and $T_n(\mathbf{x}, \mathbf{y}, \mathbf{z})$.

2 Context-free grammars

In this section, we give an overview of the idea of using context-free grammars G to generate combinatorial polynomials and combinatorial structures as developed by Chen [6]. A context-free grammar G over an alphabet A is defined to be a set of production rules. Roughly speaking, a production rule means to substitute a letter in the alphabet A by a polynomial in A over a field. Given a context-free grammar, one may define a formal derivative D as a linear operator on polynomials in A, where the action of D on a letter is defined by the substitution rule of the grammar and the action of D on a product of two polynomials u and v is defined by the Leibnitz rule, that is,

$$D(uv) = D(u)v + uD(v).$$

Many combinatorial polynomials can be generated by context-free grammars. Meanwhile, context-free grammars can be used to generate combinatorial structures. More precisely, one may use a word on an alphabet to label a combinatorial structure such that the context-free grammar serves as the procedure to recursively generate the combinatorial structures. Such a labeling of a combinatorial structure is called a grammatical labeling in [7].

For example, the grammar

$$G = \{a \to ab,\, b \to b\}$$

is used in [6] to generate the set of partitions of $[n]$ and the Stirling polynomials,

$$S_n(x) = \sum_{i=0}^{n} S(n,k)x^k,$$

where $S(n,k)$ denotes the Stirling number of the second kind. For a partition P, we label a block of P by letter b and label the partition itself by letter a, and we define the weight of a partition by the product of its labels. So a partition P with k blocks has the weight $w(P) = ab^k$. For example, the partition $\{\{1,2\},\{3\}\}$ is labeled as follows

$$\overset{\{1,2\}}{b}\;\overset{\{3\}}{b}\;a\,.$$

In the above notation, we write a partition $P = \{P_1, P_2, \ldots, P_k\}$ of $[n]$ in such a way that the blocks are ordered in the increasing order of their minimum elements. Moreover, we put the letter a at the end of the partition.

Using the above grammatical labeling of a partition, we deduce that

$$D^n(a) = \sum_P w(P) = \sum_{k=1}^{n} S(n,k)ab^k. \tag{2.1}$$

Many properties of the Stirling polynomials follow from the above expression in terms of the differential operator D with respect to the grammar G.

Let us explain how the grammar works for the generation of partitions. For $n = 1$, there is one partition of $[1]$, that is, $\{\{1\}\}$, whose label is ab. Assume that we have generated all the partitions of $[n-1]$ by applying the operator D^{n-2} to $\{\{1\}\}$ with the initial grammatical labeling.

Let us give an example to demonstrate the action of the differential operator D with respect to the grammar G to a partition of $[n]$ with the aforementioned grammatical labeling. Consider the following partition of $\{1,2,3,4,5,6\}$

$$\overset{\{1,3,6\}}{b}\overset{\{2,5\}}{b}\overset{\{4\}}{b}\,a\,.$$

If we apply the substitution rule to the letter a, then we get ab which we rewrite as ba, where a still serves as the label of the new partition, and b stands for a new block $\{7\}$. In this case, we get a partition

$$\overset{\{1,3,6\}}{b}\overset{\{2,5\}}{b}\overset{\{4\}}{b}\overset{\{7\}}{b}\,a\,.$$

If we apply the substitution rule to the second letter b, then we get b. In this case, we insert the element 7 in the second block, and we are led to the following partition with consistent grammatical labeling

$$\overset{\{1,3,6\}}{b}\overset{\{2,5,7\}}{b}\overset{\{4\}}{b}\,a\,.$$

Starting with the empty partition with label a, we get

$$\begin{aligned}
D(a) &= \overset{\{1\}}{b}\,a,\\
D^2(a) &= \overset{\{1\}}{b}\overset{\{2\}}{b}\,a + \overset{\{1,2\}}{b}\,a,\\
D^3(a) &= \overset{\{1\}}{b}\overset{\{2\}}{b}\overset{\{3\}}{b}\,a + \overset{\{1\}}{b}\overset{\{2,3\}}{b}\,a + \overset{\{1,3\}}{b}\overset{\{2\}}{b}\,a + \overset{\{1,2\}}{b}\overset{\{3\}}{b}\,a + \overset{\{1,2,3\}}{b}\,a\,.
\end{aligned}$$

Without considering the combinatorial structures during the applications of the differential operator D, we may directly compute $D^n(x)$ to derive the Stirling polynomials $S_n(x)$.

As the second example, we consider the context-free grammar

$$G = \{x \to xy,\ y \to xy\}$$

introduced by Dumont [8] which is used to compute the Eulerian polynomials. For a permutation $\pi = \pi_1\pi_2\ldots\pi_n$ of $[n]$, let

$$\begin{aligned}
A(\pi) &= \{i \,|\, \pi_{i-1} < \pi_i\},\\
D(\pi) &= \{i \,|\, \pi_i > \pi_{i+1}\}
\end{aligned}$$

denote the set of ascents and the set of descents of π, respectively. Here we set $\pi_0 = \pi_{n+1} = 0$. In other words, for any permutation π of $[n]$, 1 is always an ascent and n is always a descent. An element π_i is called a descent top of π if $i \in D(\pi)$, and π_i is called an ascent top if $i \in A(\pi)$, see Haglund and Visontai [13].

The grammatical labeling of a permutation π is defined as follows. If π_i is an ascent top of π, then we label π_{i-1} with the letter x. If π_i is a descent top, then we label π_i by the letter y. For this labeling, the weight of π is given by

$$w(\pi) = x^{|A(\pi)|} y^{|D(\pi)|}.$$

Then for $n \geqslant 1$, we have

$$D^n(x) = \sum_{\pi \in S_n} w(\pi) = \sum_{m=1}^{n} A(n,m) y^m x^{n+1-m},$$

where $A(n,m)$ is the Eulerian number, namely, the number of permutations of $[n]$ with m descents, see Dumont [8].

For $n = 1$, there is only one permutation of [1], that is 1, whose label is xy. Assume that we have generated all the permutations of $[n-1]$ by applying the operator D^{n-2} to 1.

Next we give an example to illustrate the action of D on a permutation of [6]. Take a permutation

$$x\,\overset{3}{y}\,\overset{2}{x}\,\overset{5}{x}\,\overset{6}{y}\,\overset{4}{y}\,\overset{1}{y}\;.$$

If we apply the substitution rule $x \to xy$ to the third letter x, we insert 7 after 5. As for the grammatical labeling, we keep all the labels and assign the element 7 a new label y as if it comes from the substitution rule $x \to xy$. Indeed, it is easily checked that what we get is a permutation with a consistent grammatical labeling, namely,

$$x\,\overset{3}{y}\,\overset{2}{x}\,\overset{5}{x}\,\overset{7}{y}\,\overset{6}{y}\,\overset{4}{y}\,\overset{1}{y}\;.$$

Similarly, if we apply the substitution rule $y \to xy$ to the second letter y, then we insert 7 after 6. In this case, we need to change the label of 6 from y into x, and assign y to the new element 7. In other words, the label y becomes xy just like the substitution rule. So we get the following permutation with a grammatical labeling,

$$x\,\overset{3}{y}\,\overset{2}{x}\,\overset{5}{x}\,\overset{6}{x}\,\overset{7}{y}\,\overset{4}{y}\,\overset{1}{y}\;.$$

Indeed, the above examples indicate that permutations of $[n]$ and the Eulerian polynomials $A_n(x)$ can be generated by the operator D associated with the grammar G.

In order to generate combinatorial structures with more parameters, we may use a sequence of grammars. Let us consider the the multivariate refinement of Eulerian polynomials $A_n(\mathbf{x}, \mathbf{y})$ introduced by Haglund and Visontai [13], which involve the sets of ascent tops and descent tops, not just the numbers of ascents and descents. More precisely,

$$A_n(\mathbf{x}, \mathbf{y}) = \sum_{\pi \in S_n} \prod_{i \in A(\pi)} x_{\pi_i} \prod_{i \in D(\pi)} y_{\pi_i}.$$

We shall introduce a sequence of grammars $\{G_n\}$ to generate the multivariate polynomials $A_n(\mathbf{x}, \mathbf{y})$.

For $n \geq 1$, define

$$G_n = \{x_i \rightarrow x_n y_n, y_i \rightarrow x_n y_n, 0 \leq i < n\},$$

and denote by D_n the formal differential operator with respect to G_n. The multivariate polynomials $A_n(\mathbf{x}, \mathbf{y})$ can be generated by the sequence of grammars G_n.

Theorem 2.1 *For $n \geq 1$, we have*

$$D_n D_{n-1} \cdots D_1(x_0) = A_n(\mathbf{x}, \mathbf{y}).$$

Proof. We define the grammatical labeling of a permutation π as follows. For a permutation π, if π_i is an ascent top, we label π_{i-1} by the letter x_{π_i}; if π_i is a descent top, we label π_i by the letter y_{π_i}. So the weight of π is given by

$$w(\pi) = \prod_{i \in A(\pi)} x_{\pi_i} \prod_{i \in D(\pi)} y_{\pi_i}.$$

We proceed to show by induction that $D_n D_{n-1} \cdots D_1(x_0)$ equals the sum of the weights of permutations of $[n]$. For $n = 1$, the theorem is valid since the weight of the permutation 1 is $x_1 y_1$. Assume that the theorem holds for $n - 1$, that is,

$$D_{n-1} \cdots D_1(x_0) = \sum_{\pi \in S_{n-1}} w(\pi).$$

We now use an example to illustrate the action of D_n on a permutation of $[n-1]$. Let $\pi = 325641$. The grammatical labeling is as follows

$$x_3 \overset{3}{y_3} \overset{2}{x_5} \overset{5}{x_6} \overset{6}{y_6} \overset{4}{y_4} \overset{1}{y_1}\ .$$

If we apply the substitution rule $x_6 \rightarrow x_7 y_7$ to the letter x_6, we define the action as the insertion of 7 immediately after 5. The labels of 5 and 7 will be changed to x_7 and y_7 as given by the grammar. It is not hard to see that the permutation we obtain has a consistent grammatical labeling,

$$x_3 \overset{3}{y_3} \overset{2}{x_5} \overset{5}{x_7} \overset{7}{y_7} \overset{6}{y_6} \overset{4}{y_4} \overset{1}{y_1}\ .$$

Similarly, if we apply the substitution rule $y_6 \rightarrow x_7 y_7$ to the letter y_6, we obtain a permutation with a consistent grammatical labeling

$$x_3 \overset{3}{y_3} \overset{2}{x_5} \overset{5}{x_6} \overset{6}{x_7} \overset{7}{y_7} \overset{4}{y_4} \overset{1}{y_1}\ .$$

It is clear that all permutations of $[n]$ can be obtained this way. So we conclude that

$$D_n D_{n-1} \cdots D_1(x_0) = D_n \left(\sum_{\pi \in S_{n-1}} w(\pi) \right) = \sum_{\sigma \in S_n} w(\sigma).$$

Hence the theorem holds for all positive numbers n by induction. ∎

For $n = 0$, the empty permutation is labeled by x_0. The values of $A_n(\mathbf{x}, \mathbf{y})$ for $n = 1, 2, 3$ are given below.

$$\begin{aligned}
D_1(x_0) &= x_1 \overset{1}{y_1}, \\
D_2 D_1(x_0) &= x_2 \overset{2}{y_2} \overset{1}{y_1} + x_1 \overset{1}{x_2} \overset{2}{y_2}, \\
D_3 D_2 D_1(x_0) &= x_3 \overset{3}{y_3} \overset{2}{y_2} \overset{1}{y_1} + x_2 \overset{2}{x_3} \overset{3}{y_3} \overset{1}{y_1} + x_2 \overset{2}{y_2} \overset{1}{x_3} \overset{3}{y_3} + x_3 \overset{3}{y_3} \overset{1}{x_2} \overset{2}{y_2} \\
&\quad + x_1 \overset{1}{x_3} \overset{3}{y_3} \overset{2}{y_2} + x_1 \overset{1}{x_2} \overset{2}{x_3} \overset{3}{y_3}\ .
\end{aligned}$$

Let us now consider the grammar to generate Stirling permutations. Chen and Fu [7] showed that the grammar

$$G = \{x \to x^2 y, y \to x^2 y\}$$

can be used to generate Stirling permutations. Let D denote the differential operator associated with the grammar G. It has been shown in [7] that

$$D^n(x) = x \sum_{m=1}^{n} C(n,m) x^{2n-m} y^m,$$

where $C(n,m)$ denotes the second-order Eulerian number. Notice that

$$D^n(x) \,|_{\,x=1} = C_n(y),$$

where $C_n(y)$ is the second-order Eulerian polynomial.

The grammatical labeling of a Stirling permutation is defined as follows. For a Stirling permutation π, if $i \in D(\pi)$, we label π_i by y; if $i \in A(\pi)$ or $i \in P(\pi)$, we label π_{i-1} by x. For example, the Stirling permutation $\pi = 233211$ has the following grammatical labeling

$$x \, {}^{2} \, x \, {}^{3} \, x \, {}^{3} \, y \, {}^{2} \, y \, {}^{1} \, x \, {}^{1} \, y \, .$$

Next we show that one can use a refinement of the grammar G to derive the multivariate polynomials $C_n(\mathbf{x}, \mathbf{y}, \mathbf{z})$ of Haglund and Visontai [13]. Recall that

$$C_n(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \sum_{\pi \in Q_n} \prod_{i \in A(\pi)} x_{\pi_i} \prod_{i \in D(\pi)} y_{\pi_i} \prod_{i \in P(\pi)} z_{\pi_i}.$$

As a refinement of the grammar G, we define

$$G_n = \{x_i \to x_n y_n z_n, y_i \to x_n y_n z_n, z_i \to x_n y_n z_n, 0 \le i < n\}.$$

and we denote by D_n the differential operator associated with the grammar G_n.

Theorem 2.2 *For $n \geqslant 1$, we have*

$$D_n D_{n-1} \cdots D_1(z_0) = C_n(\mathbf{x}, \mathbf{y}, \mathbf{z}).$$

Proof. First, let us define the grammatical labeling of a Stirling permutation π. For a Stirling permutation π, if π_i is an ascent top, we label π_{i-1} by the letter x_{π_i}; if π_i is a descent top, we label π_i by the letter y_{π_i}; and if π_i is a plateau, we label π_{i-1} by the letter z_{π_i}. For this labeling, the weight of π is given by

$$w(\pi) = \prod_{i \in A(\pi)} x_{\pi_i} \prod_{i \in D(\pi)} y_{\pi_i} \prod_{i \in P(\pi)} z_{\pi_i}.$$

We aim to show that $D_n D_{n-1} \cdots D_1(z_0)$ equals the sum of weights of Stirling permutations of $[n]_2$. Let us use induction on n. The theorem is obvious for $n = 0$ since the weight of the empty permutation is z_0. Assume that the theorem holds for $n - 1$, that is,

$$D_{n-1} \cdots D_1(z_0) = \sum_{\pi \in Q_{n-1}} w(\pi).$$

Let us use an example to demonstrate the action of D on a Stirling permutation of $[n-1]_2$. Let $\pi = 233211$. The grammatical labeling of π is as follows

$$x_2 \overset{2}{x_3} \overset{3}{z_3} \overset{3}{y_3} \overset{2}{y_2} \overset{1}{z_1} \overset{1}{y_1}\ .$$

In general, if we apply a substitution rule of G_4 to any letter in π, we get $x_4y_4z_4$. Here we insert the two elements 44 after the element whose label is replaced by the substitution rule, and we use the labels x_4, y_4 and z_4 to relabel the three elements that are affected by the substitution. For example, if we apply the substitution rule $x_2 \to x_4y_4z_4$ to the above Stirling permutation, then we get a Stirling permutation with the following grammatical labeling

$$x_4 \overset{4}{z_4} \overset{4}{y_4} \overset{2}{x_3} \overset{3}{z_3} \overset{3}{y_3} \overset{2}{y_2} \overset{1}{z_1} \overset{1}{y_1}\ .$$

It is easily seen that the application of any substitution rule of G_n to any Stirling permutation of $[n-1]_2$ leads to a Stirling permutation of $[n]_2$ with a consistent grammatical labeling. Hence we deduce that

$$D_nD_{n-1}\cdots D_1(z_0) = D_n\left(\sum_{\pi\in Q_{n-1}} w(\pi)\right) = \sum_{\sigma\in Q_n} w(\sigma).$$

Thus, the theorem holds for n. This completes the proof. ∎

For $n=0$, the empty permutation is labeled by z_0. The values of the polynomials $C_n(\mathbf{x},\mathbf{y},\mathbf{z})$ for $n=1,2$ are as follows,

$$\begin{aligned} D_1(z_0) &= x_1 \overset{1}{z_1} \overset{1}{y_1}, \\ D_2D_1(z_0) &= x_2 \overset{2}{z_2} \overset{2}{y_2} \overset{1}{z_1} \overset{1}{y_1} + x_1 \overset{1}{x_2} \overset{2}{z_2} \overset{2}{y_2} \overset{1}{y_1} + x_1 \overset{1}{z_1} \overset{1}{x_2} \overset{2}{z_2} \overset{2}{y_2}\ . \end{aligned}$$

We shall give further refinements of the above two sequences of grammars as solutions to the problems of Haglund and Visontai [13]. On one hand, we use these refined grammars to construct multivariate polynomials for Legendre-Stirling permutations and marked Stirling permutations. On the other hand, we use the grammars to construct stability preserving operators leading to the stability of the multivariate polynomials.

3 Legendre-Stirling permutations

In this section, we introduce several statistics on Legendre-Stirling permutations of

$$M_n = \{1, 1, \bar{1}, 2, 2, \bar{2}, \cdots, n, n, \bar{n}\}.$$

In terms of these statistics, we obtain multivariate polynomials $B_n(\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{u},\mathbf{v})$ as refinements of $B_n(x)$. In fact, the combinatorial construction of the multivariate polynomials $B_n(\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{u},\mathbf{v})$ is obtained from further refinements of the grammars to generate permutations and Stirling permutations with respect to the numbers of descents. Using these grammars, we derive the combinatorial interpretation by giving a suitable grammatical labeling. In Section 5, we shall use grammars to prove the stability of $B_n(\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{u},\mathbf{v})$. This leads to a solution to the problem of Haglund and Visontai.

Let L_n denote the set of Legendre-Stirling permutations of M_n. For a Legendre-Stirling permutation $\pi \in L_n$, define

$$\begin{aligned}
X(\pi) &= \{i \,|\, \pi_{i-1} \leqslant \pi_i, \pi_i \text{ is unbarred and appears the first time}\}, \\
Y(\pi) &= \{i \,|\, \pi_i > \pi_{i+1} \text{ and } \pi_i \text{ is unbarred}\}, \\
Z(\pi) &= \{i \,|\, \pi_{i-1} \leqslant \pi_i, \pi_i \text{ is unbarred and appears the second time}\}, \\
U(\pi) &= \{i \,|\, \pi_{i-1} \leqslant \pi_i \text{ and } \pi_i \text{ is barred}\}, \\
V(\pi) &= \{i \,|\, \pi_i > \pi_{i+1} \text{ and } \pi_i \text{ is barred}\}.
\end{aligned}$$

Here we set $\pi_0 = \pi_{3n+1} = 0$.

For example, let $\pi = \bar{1}1\bar{2}2332\bar{3}1$. Then we have $X(\pi) = \{2, 4, 5\}$, $Y(\pi) = \{6, 9\}$, $Z(\pi) = \{6\}$, $U(\pi) = \{1, 3, 8\}$ and $V(\pi) = \{8\}$.

Define

$$B_n(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{u}, \mathbf{v}) = \sum_{\pi \in L_n} \prod_{i \in X(\pi)} x_{\pi_i} \prod_{i \in Y(\pi)} y_{\pi_i} \prod_{i \in Z(\pi)} z_{\pi_i} \prod_{i \in U(\pi)} u_{\pi_i} \prod_{i \in V(\pi)} v_{\pi_i}.$$

We define the grammars $\{G_n\}$ as follows,

$$\begin{aligned}
G_{2n-1} &= \{x_i, y_i, z_i, u_i, v_i \to u_n v_n, 0 \leqslant i < n\}, \\
G_{2n} &= \{x_i, y_i, z_i, u_i, v_i \to x_n y_n z_n, 0 \leqslant i < n; \\
&\qquad u_n \to x_n z_n u_n, v_n \to x_n y_n z_n\}.
\end{aligned}$$

Notice that G_{2n-1} is a refinement of the grammar

$$G = \{x \to xy, y \to xy\}.$$

and G_{2n} is a refinement of the grammar

$$G = \{x \to x^2 y, y \to x^2 y\}$$

The grammatical labeling of a Legendre-Stirling permutation is defined as follows. Let π be a Legendre-Stirling permutation on M_n. For $i \in X(\pi)$, $i \in Z(\pi)$ or $i \in U(\pi)$, we label π_{i-1} by the letter x_{π_i}, z_{π_i} or u_{π_i}, respectively; for $i \in Y(\pi)$ or $i \in V(\pi)$, we label π_i by the letter y_{π_i} or v_{π_i}, respectively. For example, the above Legendre-Stirling permutation $\pi = \bar{1}1\bar{2}2332\bar{3}1$ has the following grammatical labeling

$$u_1 \overset{\bar{1}}{x_1} \overset{1}{u_2} \overset{\bar{2}}{x_2} \overset{2}{x_3} \overset{3}{z_3} \overset{3}{y_3} \overset{2}{u_3} \overset{\bar{3}}{v_3} \overset{1}{y_1} \ .$$

The following theorem shows that the polynomials $B_n(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{u}, \mathbf{v})$ can be generated by the grammars G_n.

Theorem 3.1 *For $n \geqslant 1$, we have*

$$D_{2n} D_{2n-1} \cdots D_1(x_0) = B_n(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{u}, \mathbf{v}). \tag{3.1}$$

Proof. We use induction on n. The case for $n=0$ is obvious since the empty permutation is labeled by x_0. Assume that the theorem holds for $n-1$, that is,

$$D_{2n-2}\cdots D_1(x_0)=\sum_{\pi\in L_{n-1}} w(\pi). \tag{3.2}$$

Note that any Legendre-Stirling permutation of M_n can be obtained from a Legendre-Stirling permutation of M_{n-1} through two operations: (1) Insert a barred element $\bar{n}$; (2) Insert two elements nn. We use an example to show that the operators D_{2n-1} and D_{2n} correspond to these two operations.

Consider the Legendre-Stirling permutation $\pi=\bar{1}1\bar{2}2332\bar{3}1$, whose grammatical labeling is given by

$$u_1 \overset{\bar{1}}{x_1} \overset{1}{u_2} \overset{\bar{2}}{x_2} \overset{2}{x_3} \overset{3}{z_3} \overset{3}{y_3} \overset{2}{u_3} \overset{\bar{3}}{v_3} \overset{1}{y_1}\ .$$

The first operation is just the procedure of generating permutations. In general, if we apply a substitution rule of G_7 to π, we always get u_4v_4. Here we insert $\bar{4}$ after the element whose label is replaced by the substitution rule. At the same time, we relabel the two involved elements by the letters u_4 and v_4. For example, if we apply the substitution rule $z_3\to u_4v_4$ to π, then we obtain a Legendre-Stirling permutation with a consistent grammatical labeling

$$u_1 \overset{\bar{1}}{x_1} \overset{1}{u_2} \overset{\bar{2}}{x_2} \overset{2}{x_3} \overset{3}{u_4} \overset{\bar{4}}{v_4} \overset{3}{y_3} \overset{2}{u_3} \overset{\bar{3}}{v_3} \overset{1}{y_1}\ .$$

As for the second operation, consider the above Legendre-Stirling permutation $\sigma=\bar{1}1\bar{2}23\bar{4}32\bar{3}1$. The two substitution rules $u_4\to x_4z_4u_4$ and $v_4\to x_4y_4z_4$ of G_8 correspond to the operations of inserting two elements 44 before $\bar{4}$ or after $\bar{4}$, respectively. So we get two Legendre-Stirling permutations $\bar{1}1\bar{2}2344\bar{4}32\bar{3}1$ or $\bar{1}1\bar{2}23\bar{4}4432\bar{3}1$.

Next we consider the rest of substitution rules of G_8. If we apply any of the remaining substitution rules of G_8 to σ, we always get $x_4y_4z_4$. Here we insert two elements 44 into σ between π_i and π_{i+1}, which are elements less than 4. For example, by applying the production rule $u_2\to x_4y_4z_4$, we obtain the Legendre-Stirling permutation

$$u_1 \overset{\bar{1}}{x_1} \overset{1}{x_4} \overset{4}{z_4} \overset{4}{y_4} \overset{\bar{2}}{x_2} \overset{2}{x_3} \overset{3}{u_4} \overset{\bar{4}}{v_4} \overset{3}{y_3} \overset{2}{u_3} \overset{\bar{3}}{v_3} \overset{1}{y_1}\ .$$

It can be checked that any of applications of the substitution rules of G_8 to σ leads to consistent grammatical labelings. Moreover, it can be verified that the action of $D_{2n}D_{2n-1}$ on the Legendre-Stirling permutations of M_{n-1} generates all the Legendre-Stirling permutations of M_n. So we conclude that

$$D_{2n}D_{2n-1}\cdots D_1(x_0)=\sum_{\pi\in L_n} w(\pi).$$

Then the theorem follows by induction. ∎

For $n=0$, the empty permutation is labeled by x_0, and $B_1(\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{u},\mathbf{v})$ is calculated as follows,

$$\begin{aligned} D_1(x_0) &= u_1 \overset{\bar{1}}{v_1}, \\ D_2D_1(x_0) &= x_1 \overset{1}{z_1} \overset{1}{u_1} \overset{\bar{1}}{v_1} + u_1 \overset{\bar{1}}{x_1} \overset{1}{z_1} \overset{1}{y_1}\ . \end{aligned}$$

4 Marked Stirling permutations

In this section, we introduce the structure of marked Stirling permutations, and we define several statistics in order to construct multivariate polynomials $T_n(\mathbf{x}, \mathbf{y}, \mathbf{z})$ as refinements of $T_n(x)$. We also give a sequence of grammars to generate $T_n(\mathbf{x}, \mathbf{y}, \mathbf{z})$ as well as marked Stirling permutations with suitable grammatical labelings. By using the grammars, the stability of $T_n(\mathbf{x}, \mathbf{y}, \mathbf{z})$ can be established in Section 5. This gives a solution to the problem of Haglund and Visontai concerning a stable refinement of $T_n(x)$.

A marked Stirling permutation is defined by the following marking rule. Given a Stirling permutation $\pi = \pi_1\pi_2\cdots\pi_{2n}$, if π_i is an element of π such that π_i occurs the second time in π and $\pi_i < \pi_{i+1}$, then we may mark the element π_i. We denote a marked element i by $\bar{i}$. A marked Stirling permutation is a Stirling permutation with some elements marked according to the above rule.

For example, there is only one marked Stirling permutation of $[1]_2$: 11, whereas there are four marked Stirling permutations of $[2]_2$:

$$2211, 1221, 1122, 1\bar{1}22.$$

Let $\bar{Q}_n$ denote the set of marked Stirling permutations of $[n]_2$. We use $A(\pi)$, $D(\pi)$, $P(\pi)$ to denote the set of descents, the set of ascents and the set of plateaux of π. More precisely, given a marked Stirling permutation $\pi = \pi_1\pi_2\ldots\pi_{2n} \in \bar{Q}_n$, let

$$\begin{aligned} A(\pi) &= \{i \,|\, \pi_{i-1} < \pi_i\}, \\ D(\pi) &= \{i \,|\, \pi_i > \pi_{i+1}\}, \\ P(\pi) &= \{i \,|\, \pi_{i-1} = \pi_i\} \end{aligned}$$

denote the set of ascents, the set of descents and the set of plateaux of π, respectively. Let $T(n, m)$ be the number of marked Stirling permutations of $[n]_2$ with m descents. It follows from relation (1.1) that

$$T_n(x) = \sum_{m=1}^{n} T(n, m)x^m.$$

Note that Riordan [17] introduced the polynomials $T_n(x)$ and proved that $T_n(1)$ equals the Schröder number, namely, the number of series-reduced rooted trees with $n + 1$ labeled leaves.

We shall prove that the polynomials $T_n(x)$ can be generated by the grammar G defined by

$$G = \{x \to x^2y, y \to 2x^2y\}.$$

The grammatical labeling of a marked Stirling permutation can be described as follows. Let π be a marked Stirling permutation of $[n]_2$. If $i \in D(\pi)$, we label π_i by y. If $i \in A(\pi)$ or $i \in P(\pi)$, we label π_{i-1} by x. The weight of a marked Stirling permutation π of $[n]_2$ with m descents is given by

$$w(\pi) = x^{2n+1-m}y^m.$$

Theorem 4.1 *For $n \geqslant 1$, we have*

$$D^n(x) = \sum_{m=1}^{n} T(n, m)x^{2n-m+1}y^m.$$

Setting $x = 1$, we have

$$D^n(x)|_{x=1} = T_n(y).$$

Proof. We aim to show that $D^n(x)$ equals the sum of the weights of marked Stirling permutations of $[n]_2$. We use induction on n. The case for $n = 0$ follows from the fact that the weight of the empty permutation is x. Assume that the theorem holds for $n - 1$, that is,

$$D^{n-1}(x) = \sum_{\pi \in \bar{Q}_{n-1}} w(\pi).$$

We now use an example to demonstrate the action of D on a marked Stirling permutation of $[n-1]_2$. Let $\pi = 12\bar{2}331$ with the following grammatical labeling

$$x \overset{1}{x} \overset{2}{x} \overset{\bar{2}}{x} \overset{3}{x} \overset{3}{y} \overset{1}{y}\ .$$

If we apply the substitution rule $x \to x^2y$ to the fourth letter x, then we insert the two elements 44 after $\bar{2}$. We keep all the labels except that we assign the labels x and y to the two new letters 44. It is not difficult to see that the generated marked Stirling permutation has a consistent grammatical labeling

$$x \overset{1}{x} \overset{2}{x} \overset{\bar{2}}{x} \overset{4}{x} \overset{4}{y} \overset{3}{x} \overset{3}{y} \overset{1}{y}\ .$$

If we apply the substitution rule $y \to 2x^2y$ to the first letter y, then we insert 44 after the second element 3. We change the label of the second element 3 from y into x and assign x and y to the two new elements 44. According to the marking rule, the second element 3 may be marked or unmarked. These two choices correspond the coefficient 2 in the substitution rule $y \to 2x^2y$. So we are led to the following two marked Stirling permutations with consistent grammatical labelings,

$$x \overset{1}{x} \overset{2}{x} \overset{\bar{2}}{x} \overset{3}{x} \overset{3}{x} \overset{4}{x} \overset{4}{y} \overset{1}{y},$$

and

$$x \overset{1}{x} \overset{2}{x} \overset{\bar{2}}{x} \overset{3}{x} \overset{\bar{3}}{x} \overset{4}{x} \overset{4}{y} \overset{1}{y}\ .$$

It can be verified that the above process generates all marked Stirling permutations of $[n]_2$. It follows that

$$D^n(x) = D(D^{n-1}(x)) = D\left(\sum_{\pi \in \bar{Q}_{n-1}} w(\pi)\right) = \sum_{\sigma \in \bar{Q}_n} w(\sigma).$$

Hence the proof is complete by induction. ∎

As a multivariate refinement of $T_n(x)$, we define the following generating polynomial of marked Stirling permutations of $[n]_2$,

$$T_n(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \sum_{\pi \in \bar{Q}_n} \prod_{i \in A(\pi)} x_{\pi_i} \prod_{i \in D(\pi)} y_{\pi_i} \prod_{i \in P(\pi)} z_{\pi_i}.$$

Let

$$G_n = \{x_i \to x_n y_n z_n, y_i \to 2x_n y_n z_n, z_i \to x_n y_n z_n, 0 \le i < n\}.$$

The grammatical labeling of a marked Stirling permutation can be described as follows. For a marked Stirling permutation π of $[n]_2$, if $i \in A(\pi)$, we label π_{i-1}

by x_i; if $i \in D(\pi)$, we label π_i by y_i; and if $i \in P(\pi)$, we label π_{i-1} by z_i. Then the weight of π equals

$$w(\pi) = \prod_{i \in A(\pi)} x_{\pi_i} \prod_{i \in D(\pi)} y_{\pi_i} \prod_{i \in P(\pi)} z_{\pi_i}.$$

The following theorem shows that the polynomials $T_n(\mathbf{x}, \mathbf{y}, \mathbf{z})$ can be generated by the grammars G_n.

Theorem 4.2 *For $n \geq 1$, we have*

$$D_n D_{n-1} \cdots D_1(z_0) = T_n(\mathbf{x}, \mathbf{y}, \mathbf{z}).$$

The proof of the above theorem is analogous to that of Theorem 4.1. Hence the details are omitted. Here we use an example to illustrate the action of D_4 to the above marked Stirling permutation $\pi = 12\bar{2}331$ with the grammatical labeling

$$x_1 \stackrel{1}{x_2} \stackrel{2}{z_2} \stackrel{\bar{2}}{x_3} \stackrel{3}{z_3} \stackrel{3}{y_3} \stackrel{1}{y_1}\,.$$

If we apply the substitution rule $x_3 \to x_4 y_4 z_4$ of G_4 to the letter x_3, then we insert the two elements 44 after $\bar{2}$ to get a marked Stirling permutation with the following consistent grammatical labeling

$$x_1 \stackrel{1}{x_2} \stackrel{2}{z_2} \stackrel{\bar{2}}{x_4} \stackrel{4}{z_4} \stackrel{4}{y_4} \stackrel{3}{z_3} \stackrel{3}{y_3} \stackrel{1}{y_1}\,.$$

If we apply the substitution rule $y_3 \to 2x_4 y_4 z_4$ of G_4 to the letter y_3, then we insert 44 after the second element 3 to get the following two marked Stirling permutations with consistent grammatical labelings

$$x_1 \stackrel{1}{x_2} \stackrel{2}{z_2} \stackrel{\bar{2}}{x_3} \stackrel{3}{z_3} \stackrel{3}{x_4} \stackrel{4}{z_4} \stackrel{4}{y_4} \stackrel{1}{y_1},$$

and

$$x_1 \stackrel{1}{x_2} \stackrel{2}{z_2} \stackrel{\bar{2}}{x_3} \stackrel{3}{z_3} \stackrel{\bar{3}}{x_4} \stackrel{4}{z_4} \stackrel{4}{y_4} \stackrel{1}{y_1}\,.$$

For $n = 0$, the empty permutation is labeled by z_0. For $n = 1, 2$, $T_n(\mathbf{x}, \mathbf{y}, \mathbf{z})$ are given below,

$$\begin{aligned}
D_1(z_0) &= x_1 \stackrel{1}{z_1} \stackrel{1}{y_1}, \\
D_2 D_1(z_0) &= x_2 \stackrel{2}{z_2} \stackrel{2}{y_2} \stackrel{1}{z_1} \stackrel{1}{y_1} + x_1 \stackrel{1}{x_2} \stackrel{2}{z_2} \stackrel{2}{y_2} \stackrel{1}{y_1} + x_1 \stackrel{1}{z_1} \stackrel{1}{x_2} \stackrel{2}{z_2} \stackrel{2}{y_2} \\
&\quad + x_1 \stackrel{1}{z_1} \stackrel{\bar{1}}{x_2} \stackrel{2}{z_2} \stackrel{2}{y_2}\,.
\end{aligned}$$

5 Grammars preserving stability

In this section, we prove the stability of the multivariate polynomials $B_n(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{u}, \mathbf{v})$ and $T_n(\mathbf{x}, \mathbf{y}, \mathbf{z})$ based on context-free grammars and the characterization of stability preserving linear operators due to Borcea and Brändén [3].

Our idea of proving the stability of the polynomials by a sequence of context-free grammars $\{G_n\}$ goes as follows. Since the initial polynomial x is stable, if $D_1, D_2, \ldots, D_n, \ldots$ preserve stability, then $D_n D_{n-1} \ldots D_1(x)$ is stable. If D_n is not stability preserving, then we try to find a sequence of stability preserving operator $\{T_n\}$ such that

$$T_n T_{n-1} \ldots T_1(x) = D_n D_{n-1} \ldots D_1(x).$$

If such operators T_n exist, then we reach the conclusion that the multivariate polynomials $D_n D_{n-1} \ldots D_1(x)$ are stable.

Note that $B_n(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{u}, \mathbf{v})$ and $T_n(\mathbf{x}, \mathbf{y}, \mathbf{z})$ are all multiaffine polynomials in the sense that the degree in each variable is at most 1. In order to construct the stability preserving operators T_n based on the grammars G_n, we consider some equivalent forms of production rules when we restrict our attention to multiaffine polynomials.

For example, let

$$G_n = \{a \to ab_n, b_i \to b_n, 0 \le i < n\}.$$

Observe that as far as the computation is concerned, the formal differential operator D_n with respect to G_n is in accordance with the following operator

$$T_n = b_n(1 + \sum_{i=1}^{n} \partial/\partial_{b_i}),$$

when they are applied to certain polynomials. Thus we obtain

$$T_n T_{n-1} \ldots T_1(a) = D_n D_{n-1} \ldots D_1(a).$$

However, D_n and T_n are different operator in general, since

$$D_n(a + b_1) \ne T_n(a + b_1).$$

For multiaffine polynomials, the characterization of stability preserving operators is simpler than that for the general case. For the purpose of this paper, we only need the following sufficient condition to prove the stability of $B_n(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{u}, \mathbf{v})$ and $T_n(\mathbf{x}, \mathbf{y}, \mathbf{z})$, see Borcea and Brändén [3].

Lemma 5.1 *Let $f \in \mathbb{C}[z_1, z_2, \ldots, z_n]$ be a stable multiaffine polynomial and let T denote a linear operator acting on the polynomials in $\mathbb{C}[z_1, z_2, \ldots, z_n]$. If*

$$T\left(\prod_{i=1}^{n}(z_i + w_i)\right) \in \mathbb{C}[z_1, z_2, \ldots, z_n, w_1, \ldots, w_n]$$

is stable, then $T(f)$ is either stable or identically 0.

Next we show how to prove the stability of polynomials generated by context-free grammars. Let us consider the multiaffine polynomials $C_n(\mathbf{x}, \mathbf{y}, \mathbf{z})$ defined by Haglund and Visontai [13]. Let

$$G_n = \{x_i \to x_n y_n z_n, y_i \to x_n y_n z_n, z_i \to x_n y_n z_n, 0 \le i < n\},$$

and let G_n denote the differential operator associated with the grammar G_n. Let $f_n = D_n D_{n-1} \cdots D_1(z_0)$. From the grammatical labelings, it is clear that f_n is multiaffine. We wish to prove the stability of f_n by induction on n. Since z_0 is stable, it suffices to prove that the operator D_{n+1} preserves stability of multiaffine polynomials.

Let

$$F = \prod_{i=0}^{n} (x_i + w_i)(y_i + v_i)(z_i + u_i).$$

By Lemma 5.1, it suffices to check the stability of $D_{n+1}(F)$, that is,

$$D_{n+1}(F) = x_{n+1}y_{n+1}z_{n+1}F \sum_{i=0}^{n} \left(\frac{1}{x_i + w_i} + \frac{1}{y_i + v_i} + \frac{1}{z_i + u_i} \right)$$

is stable.

If x_i, y_i, z_i, w_i, v_i and u_i have positive imaginary parts for all $0 \le i \le n$, then

$$\xi = \sum_{i=0}^{n} \left(\frac{1}{x_i + w_i} + \frac{1}{y_i + v_i} + \frac{1}{z_i + u_i} \right)$$

has negative imaginary part. Thus,

$$D_{n+1}(F) = x_{n+1}y_{n+1}z_{n+1}F\xi \neq 0.$$

Hence $D_{n+1}(F)$ is stable. By Lemma 5.1, we find that $D_{n+1}(f_n)$ is a stable polynomial. So we conclude that

$$f_{n+1} = D_{n+1}D_nD_{n-1}\cdots D_1(z_0)$$

is stable.

The stability of $A_n(\mathbf{x}, \mathbf{y})$ can be proved in the same way. Indeed, let

$$G_n = \{x_i \to x_ny_n, y_i \to x_ny_n, 0 \leqslant i < n\},$$

and let D_n denote the differential operator with respect to G_n. It turns out that the operator D_n preserves the stability of multiaffine polynomials.

It is worth mentioning that the formal differential operators used in the above two examples are essentially equivalent to the operators given by Haglund and Visontai [13] in their proofs of the stability of $C_n(\mathbf{x}, \mathbf{y}, \mathbf{z})$ and $A_n(\mathbf{x}, \mathbf{y})$.

Next we construct stable multivariate refinements of $S_n(x)$, the Stirling polynomials. Recall that the grammar

$$G = \{a \to ab, b \to b\}$$

generates the polynomials $S_n(x)$. Define

$$G_n = \{a \to ab_n, b_i \to b_n, 1 \le i < n\},$$

and let D_n denote the formal differential operator associated with G_n. We define the grammatical labeling of a partition as follows. For a partition $P = \{P_1, P_2, \ldots, P_k\}$, we label the partition itself by the letter a and label a block P_i by the letter b_m, where m is the maximum element in P_i. Then the weight of P is given by

$$w(P) = a \prod_{i=1}^{k} b_{m_i},$$

where m_i is the maximum element in P_i. Denote by $S_n(a, \boldsymbol{b})$ the sum of weights of partitions of $[n]$. The next theorem shows that $S_n(a, \boldsymbol{b})$ can be generated by G_n. However, in this case, the differential operator D_n associated with G_n is not stability preserving even for multiaffine polynomials. Instead, we shall find an equivalent operator T_n that preserves stability for multiaffine polynomials.

Theorem 5.2 *For $n \geq 1$, we have*

$$D_n D_{n-1} \dots D_1(a) = S_n(a, \boldsymbol{b}).$$

The proof of the above theorem is analogous to that of (2.1). Here we use the same example to demonstrate the action of D_7 on a partition of [6]. Consider the partition of $\{1, 2, 3, 4, 5, 6\}$ with the following grammatical labeling

$$\underset{b_6}{\{1,3,6\}}\underset{b_5}{\{2,5\}}\underset{b_4}{\{4\}}\ a\ .$$

If we apply the substitution rule $a \to ab_7$ of G_7 to the letter a, then we get a partition with a consistent labeling

$$\underset{b_6}{\{1,3,6\}}\underset{b_5}{\{2,5\}}\underset{b_4}{\{4\}}\underset{b_7}{\{7\}}\ a\ .$$

If we apply the substitution rule $b_5 \to b_7$ of G_7 to the letter b_5, then we get the following partition with a consistent grammatical labeling

$$\underset{b_6}{\{1,3,6\}}\underset{b_7}{\{2,5,7\}}\underset{b_4}{\{4\}}\ a\ .$$

Theorem 5.3 *For $n \geq 1$, the multivariate polynomial $S_n(a, \boldsymbol{b})$ is stable.*

Proof. From the grammatical labelings, we see that $S_n(a, \boldsymbol{b})$ is multiaffine. Note that $S_n(a, \boldsymbol{b})$ is multiaffine in $a, b_1, b_2, \dots, b_n$ with every term containing a as a factor. Since

$$G_{n+1} = \{a \to ab_{n+1}, b_i \to b_{n+1}, 1 \leq i \leq n\},$$

for each multiaffine monomial of $S_n(a, \boldsymbol{b})$ which is of the form ah, we have

$$D_{n+1}(ah) = ab_{n+1}h + aD_{n+1}(h).$$

It follows that

$$\begin{aligned} S_{n+1}(a, \boldsymbol{b}) &= D_{n+1}(S_n(a, \boldsymbol{b})) \\ &= b_{n+1} S_n(a, \boldsymbol{b}) + b_{n+1} \sum_{i=1}^{n} \partial/\partial_{b_i}(S_n(a, \boldsymbol{b})). \end{aligned}$$

Define

$$T_{n+1} = b_{n+1} \left(1 + \sum_{i=1}^{n} \partial/\partial_{b_i} \right).$$

Then we have $S_{n+1}(a, \boldsymbol{b}) = T_{n+1}(S_n(a, \boldsymbol{b}))$.

We proceed to prove the stability of $S_n(a, \boldsymbol{b})$ by induction on n. Since a is stable, we only need to show that the linear operator T_{n+1} preserves stability of multiaffine polynomials.

Let

$$F = (a + w) \prod_{i=1}^{n} (b_i + v_i).$$

Then we have

$$
\begin{aligned}
T_{n+1}(F) &= b_{n+1}F + b_{n+1}\sum_{i=1}^{n} \partial/\partial_{b_i}(F) \\
&= b_{n+1}F + b_{n+1}F\sum_{i=1}^{n} \frac{1}{b_i+v_i} \\
&= b_{n+1}F\left(1+\sum_{i=1}^{n}\frac{1}{b_i+v_i}\right),
\end{aligned}
$$

To prove that $T_{n+1}(F)$ is stable, we assume that a, w, b_i and v_i have positive imaginary parts for all $1 \le i \le n+1$. Consequently,

$$
\xi = 1+\sum_{i=1}^{n}\frac{1}{b_i+v_i}
$$

is nonzero since it has negative imaginary part. Moreover, each factor of F has positive imaginary part, and so does b_{n+1}. This yields that F and b_{n+1} do not vanish. It follows that

$$
T_{n+1}(F) = b_{n+1}F\xi \neq 0.
$$

Hence $T_{n+1}(F)$ is stable. In view of Lemma 5.1, we see that $T_{n+1}(S_n(a,\boldsymbol{b}))$ is stable. This completes the proof. ∎

It is worth mentioning that we use the operator T_{n+1} instead of D_{n+1} in the above proof because the operator D_{n+1} does not satisfy the condition in Lemma 5.1. Take D_2 as an example. It can be seen that $D_2((a+w)(b_1+u))$ is not stable. Note that

$$
D_2((a+w)(b_1+u)) = b_2(a(b_1+u+1)+w).
$$

Let $a=\frac{i-1}{2}, b_1=\frac{i}{2}-1, u=\frac{i}{2}-1$ and $w=i$. But we have $D_2((a+w)(b_1+u)) = 0$. This implies that $D_2((a+w)(b_1+u))$ is not stable.

Next we prove the stability of $B_n(\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{u},\mathbf{v})$.

Theorem 5.4 *For $n \geq 1$, the multivariate polynomial $B_n(\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{u},\mathbf{v})$ is stable.*

Proof. Let $f_n = D_nD_{n-1}\ldots D_1(x_0)$. From the grammatical labelings, it can be seen that f_n is multiaffine. We proceed to prove the stability of $D_{2n}D_{2n-1}\ldots D_1(x_0)$ by induction on n. The stability of x_0 is evident.

We now assume that f_{2n-2} is stable. Let us consider the actions of D_{2n-1} and D_{2n}. By using the argument in the proof of the stability of $C_n(\mathbf{x},\mathbf{y},\mathbf{z})$, it can be shown that the operator D_{2n-1} preserves stability of multiaffine polynomials. This leads to the stability of f_{2n-1} since $f_{2n-1} = D_{2n-1}(f_{2n-2})$.

Recall that

$$
\begin{aligned}
G_{2n} = \{&x_i, y_i, z_i, u_i, v_i \rightarrow x_ny_nz_n, 0 \leqslant i < n; \\
&u_n \rightarrow x_nz_nu_n, v_n \rightarrow x_ny_nz_n\}.
\end{aligned}
$$

Let B denote the following alphabet

$$
\{x_i, y_i, z_i, u_i, v_i, 0 \leqslant i < n\} \cup \{v_n\}.
$$

Since f_{2n-1} is multiaffine and each term in f_{2n-1} contains u_n, we may write a monomial of f_{2n-1} in the form $u_n h$. Then we have

$$D_{2n}(u_n h) = (x_n z_n u_n)h + x_n y_n z_n D_{2n}(h).$$

Thus,

$$\begin{aligned} f_{2n} &= D_{2n}(f_{2n-1}) \\ &= x_n z_n f_{2n-1} + x_n y_n z_n \sum_{w \in B} \partial/\partial_w(f_{2n-1}). \end{aligned}$$

Hence we may write f_{2n} as $T(f_{2n-1})$, where T is a linear operator as given by

$$T = x_n z_n + x_n y_n z_n \sum_{w \in B} \partial/\partial_w.$$

It remains to show that T preserves the stability of multiaffine polynomials. Let

$$F = (u_n + r_{u_n}) \prod_{w \in B} (w + r_w).$$

By Lemma 5.1, it suffices to verify the stability of the following polynomial

$$\begin{aligned} T(F) &= x_n z_n F + x_n y_n z_n F \sum_{w \in B} \frac{1}{w + r_w} \\ &= x_n y_n z_n F \left(\frac{1}{y_n} + \sum_{w \in B} \frac{1}{w + r_w} \right). \end{aligned}$$

Suppose that all the variables $x_i, y_i, z_i, u_i, v_i, r_{x_i}, r_{y_i}, r_{z_i}, r_{u_i}$ and r_{v_i} have positive imaginary parts for $0 \le i \le n$. Then

$$\xi = \frac{1}{y_n} + \sum_{w \in B} \frac{1}{w + r_w}$$

has negative imaginary part, and so it is nonzero. Meanwhile, every factor of F is nonzero since its imaginary part is positive. Note that under the above assumption, x_n, y_n and z_n have positive imaginary parts, and hence they are nonzero. Consequently, $T(F) = x_n y_n z_n F \xi$ does not vanish. This leads to the stability of $T(F)$.

In light of Lemma 5.1, we deduce that f_{2n} is stable. This completes the proof. ∎

The proof of the stability of $C_n(\mathbf{x}, \mathbf{y}, \mathbf{z})$ applies to the stability of $T_n(\mathbf{x}, \mathbf{y}, \mathbf{z})$. The details are omitted.

Theorem 5.5 *For $n \geqslant 1$, the multivariate polynomial $T_n(\mathbf{x}, \mathbf{y}, \mathbf{z})$ is stable.*

Multivariate stable polynomials can be reduced to real-rooted univariate polynomials by diagonalization and specialization, see Wagner [18]. More precisely, if $f \in \mathbb{R}[z_1, z_2, \ldots, z_n]$ is stable, then $f(z_1, \ldots, z_n)|_{z_i = z_j}$ and $f(z_1, \ldots, z_n)|_{z_i = a}$ are also stable, where $1 \leqslant i \neq j \leqslant n$ and $a \in \mathbb{R}$. For example, setting $a = 1$ and $b_1 = b_2 = \cdots = b_n = x$ in $S_n(a, \boldsymbol{b})$ leads to the real-rootedness of $S_n(x)$, see Harper [12].

For the multivariate stable polynomials $B_n(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{u}, \mathbf{v})$, applying the diagonalization $y_i = v_i = x$ and the specialization $x_i = u_i = z_i = 1$ for all $0 \le i \le n$, we are led to Theorem 1.3.

Let $M(n, k)$ denote the number of Legendre-Stirling permutations of M_n with k barred descents. By setting $x_i = u_i = y_i = z_i = 1$, and $v_i = x$ for all $0 \le i \le n$, we obtain the real-rootedness of the generating function of $M(n, k)$.

Corollary 5.6 *For $n \geqslant 1$, the polynomial*

$$M_n(x) = \sum_{k=1}^{n} M(n, k) x^k$$

has only real roots.

For the multivariate stable polynomials $T_n(\mathbf{x}, \mathbf{y}, \mathbf{z})$, by setting $x_i = z_i = 1$ and $y_i = y$ for all $0 \leqslant i \leqslant n$, we are led to the real-rootedness of $T_n(y)$, which is equivalent to the real-rootedness of $C_n(x)$.

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