Cubic Equations Through the Looking Glass of Sylvester

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Abstract

One can hardly believe that there is still something to be said about cubic equations. To dodge this doubt, we will instead try and say something about Sylvester. He doubtless found a way to solve cubic equations. As mentioned by Rota, it was the only method in this vein that he could remember. We realize that Sylvester's magnificent approach for reduced cubic equations boils down to an easy identity.

A special case of a remarkable discovery of Sylvester [6] states that in the generic case, a cubic binary form can be represented as a sum of two third powers of linear forms. This result has been presented in the contexts of invariant theory of binary forms, canonical forms, the apolarity of polynomials and the umbral method, see [1–5]. However, it is a little surprising that no account for this ingenious idea has been given in the simplest terms, or down to the level of high school algebra. It is even more surprising that the formulation of such a marvelous approach for reduced cubic polynomials seems to have never been touched upon before. Let us get to the point.

As is well known, by substituting x with $x - a_1$, a cubic polynomial

$$x^3 + 3a_1x^2 + 3a_2x + a_3 \tag{1}$$

can be written in the reduced form

$$f(x) = x^3 - 3px + q. \tag{2}$$

First, for the case $p = 0$, the equation can be readily solved. From now on, we assume that $p \neq 0$. For the moment, please do not ask why. Let us write $f(x)$ as

$$f(x) = x^3 - 3rsx + rs(r + s). \tag{3}$$

Hence the parameters r and s are determined by the relations

$$rs = p, \quad rs(r + s) = q.$$

In other words, r and s are the roots of the quadratic equation

$$x^2 - \frac{q}{p}x + p = 0. \tag{4}$$

If $q^2 = 4p^3$, the above equation has a double root, that is,

$$r = \frac{q}{2p}.$$

If this is the case, we claim that r is a root of $f(x)$. To wit,

$$f(r) = \left(\frac{q}{2p}\right)^3 - 3p\left(\frac{q}{2p}\right) + q = 0. \tag{5}$$

Once a root of $f(x)$ is found, the remaining two will be at finger tips. Indeed, one can further deduce that r is a double root of $f(x)$ because r is also a root of $f'(x) = 3(x^2 - p)$.

Now we are only left with the general case when the quadratic equation (4) has two distinct roots r and s. Under this circumstance, here emerges an identity:

$$x^3 - 3rsx + rs(r+s) = \frac{s}{s-r}(x-r)^3 + \frac{r}{r-s}(x-s)^3, \tag{6}$$

which spells out why we choose to write $f(x)$ in the form of (3). This relation also ensures that the cubic equation $f(x) = 0$ can be solved with ease.

Last but not least, we should not forget that we owe our thanks to Sylvester.

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References

[1] R. Ehrenborg and G.-C. Rota, Apolarity and canonical forms for homogeneous polynomials, European J. Combin., 14 (1993), 157-181.

[2] J.P.S. Kung and G.-C. Rota, The theory of binary forms, Bull. Amer. Math. Soc. (N.S.), 10 (1984), 27-85.

[3] J.P.S. Kung, Canonical forms of binary forms: Variations on a theme of Sylvester, In: D. Stanton, Ed., Invariant Theory and Tableaux, Springer-Verlag, 1989, pp. 46-58.

[4] J.P.S. Kung, G.-C. Rota and C.H.F. Yan, Combinatorics: The Rota Way, Cambridge University Press, New York, 2009.

[5] Gian-Carlo Rota, Two turning points of invariant theory, Math. Intelligencer, 21 (1999), 20-27.

[6] J.J. Sylvester, On a remarkable discovery in the theory of canonical forms and of hyperdeterminants, Math. Mag. 2 (1851), 391-410.