A Grammar of Dumont and a Theorem of Diaconis-Evans-Graham

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Abstract

We come across an unexpected connection between a remarkable grammar of Dumont for the joint distribution of $(\mathrm{exc}, \mathrm{fix})$ over S_n and a beautiful theorem of Diaconis-Evans-Graham on successions and fixed points of permutations. With the grammar in hand, we demonstrate the advantage of the grammatical calculus in deriving the generating functions, where the constant property plays a substantial role. In consideration of left successions of a permutation, we present a grammatical approach to the joint distribution investigated by Roselle. Moreover, we obtain a left succession analogue, or a shifted version, of the Diaconis-Evans-Graham theorem, exemplifying the idea of a grammar assisted bijection. The grammatical labelings give rise to an equidistribution of $(\mathrm{jump}, \mathrm{des})$ and $(\mathrm{exc}, \mathrm{drop})$ restricted to the set of left successions and the set of fixed points, where jump is the statistic defined to be the number ascents minus the number of left successions.

Keywords: Context-free grammars, increasing binary trees, the Diaconis-Evans-Graham theorem, successions, fixed points.

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1 Introduction

This paper is concerned with a beautiful theorem of Diaconis-Evans-Graham [5] on the correspondence between successions and fixed points of permutations. Unlike a typical equidistribution property, this theorem possesses an attractive feature that the bijection can be restricted to a specific set of successions and the same set of fixed points, and so it says more than just an equidistribution.

The topic of the enumeration of successions of permutations has a rich history. Dumont referred to the work of Roselle [9] on the joint distribution of the number of ascents and the number of successions. In fact, the grammar proposed by Dumont [6] is meant to deal with the joint distribution of the number of excedances, the number of drops and the number of fixed points of a permutation. His argument can be rephrased in the language of a grammatical labeling of complete increasing binary trees. We will show that this grammar is related to the Diaconis-Evans-Graham theorem, even though it does not look so at first sight. It is worth mentioning that Dumont's citation to Roselle was not accurate, nevertheless such an incident was somehow just to the point. Indeed, this work would not have come into being without the lucky pointer of Dumont.

First, we come to the realization that the grammar of Dumont can be adapted to a problem of Roselle. We need to pay attention to the notion of a left succession, analogous to that of a left peak of a permutation. As remarked by Roselle, the consideration of a left succession at position 1 was regarded convenient for the computation of the generating function for interior successions. The definition of a left succession assumes that a zero is patched at the beginning of a permutation and that it should be taken into account for the counting. In contrast to a left succession, a usual succession is called an interior succession.

Once the grammar is proposed, a formal justification is needed to endow it with a combinatorial significance. This is usually done by a grammatical labelings. Then there is no fear in performing the grammatical calculus for the purpose of deriving the generating functions. We will how the grammar of Dumont works for the joint distribution of $(\mathrm{exc}, \mathrm{fix})$. Further, we give a different labeling scheme for permutations which shows that the same grammar of Dumont suits equally well for the joint distribution of $(\mathrm{jump}, \mathrm{lsuc})$, where jump and lsuc denote the number of jumps and the number of left successions of a permutation, respectively, whose definitions will be given later. It is no surprise that the constant property plays a substantial role in the computation.

While the grammar is instrumental in establishing an equidistribution, it is not clear whether one can take a step forward in obtaining a Diaconis-Evans-Graham type theorem in regard with a given set left successions and the same set of fixed points. Fortunately, the answer is yes. In fact, it is exactly where the idea of a grammar assisted bijection comes on the scene.

Let us see what will evolve then.

2 A grammar of Dumont

In this section, we recall a remarkable grammar of Dumont [6] for the joint distribution of the statistics $(\mathrm{exc}, \mathrm{drop}, \mathrm{fix})$ over S_n, the set of permutations of $[n] = \{1, 2, \ldots, n\}$, where $n \geq 1$. For a permutation $\sigma \in S_n$, an index $1 \leq i \leq n$ is called an excedance if $\sigma_i > i$, or a drop if $\sigma_i < i$, or a fixed point if $\sigma_i = i$. The number of excedances, the number of drops and the number of fixed points of σ are denoted by $\mathrm{exc}(\sigma)$, $\mathrm{drop}(\sigma)$ and $\mathrm{fix}(\sigma)$, respectively. A drop of a permutation is also called an anti-excedance.

The joint distribution of $(\mathrm{exc}, \mathrm{fix})$ was determined by Foata-Schützenberger [7], see also Shin-Zeng [10]. For $n \geq 1$, define

$$F_n(x, z) = \sum_{\sigma \in S_n} x^{\mathrm{exc}(\sigma)} z^{\mathrm{fix}(\sigma)}.$$

Then

$$\sum_{n=0}^{\infty} F_n(x, z) \frac{t^n}{n!} = \frac{(1-x)e^{zt}}{e^{xt} - xe^t}. \tag{2.1}$$

Putting

$$F_n(x, y, z) = \sum_{\sigma \in S_n} x^{\mathrm{exc}(\sigma)} y^{\mathrm{drop}(\sigma)} z^{\mathrm{fix}(\sigma)},$$

the formula (2.1) can be converted into the homogeneous form

$$\sum_{n=0}^{\infty} F_n(x, y, z) \frac{t^n}{n!} = \frac{(y-x)e^{zt}}{ye^{xt} - xe^{yt}}. \tag{2.2}$$

The grammar of Dumont reads

$$G = \{a \rightarrow az,\ z \rightarrow xy,\ x \rightarrow xy,\ y \rightarrow xy\}. \tag{2.3}$$

Let D be the formal derivative with respect to G.

Theorem 2.1 (Dumont)**.** *The following relation is valid for $n \geq 0$,*

$$D^n(a) = aF_n(x, y, z). \tag{2.4}$$

Dumont's argument can be understood as a description of the procedure of recursively generating permutations in the cycle notation. Recall that each cycle is written with the smallest element at the beginning and the cycles are arranged in the increasing order of the smallest elements, see Stanley [11]. Here we give an explanation in the language of a grammatical labeling, which we call the (a, x, y, z)-labeling, both for permutations and for increasing binary trees.

Given a permutation σ of $[n]$, represent it in the cycle notation. Use a to signify a position where a new cycle may be formed. If i is in a 1-cycle, we label it by z. If (i, j) is an arc in the cycle notation, that is, $\sigma(i) = j$, we label it by x if $i < j = \sigma(i)$, that is, i is an excedance, or by y if $i > j$, that is, i is a drop. Then an insertion of $n+1$ into σ can be formally described with the aid of the grammar rules.

With the above grammar at disposal, one can build a complete increasing binary tree to record the insertion process of generating a permutation of $[n+1]$ from a permutation of $[n]$, in the cycle notation, of course. With the help of the correspondence between permutations and complete increasing binary trees, see, Stanley [11], here is a labeling scheme: The rightmost leaf is labeled by a, each fixed point has a left leaf z, an x-leaf corresponds to an excedance and a y-leaf corresponds to a drop.

Below is a permutation in the cycle notation with labels, corresponding to the tree in Figure 1,

$$(y\,1\,x\,8\,y\,4\,x\,9\,y\,6)\ (z\,2)\ (y\,3\,x\,5)\ (z\,7)\ a.$$

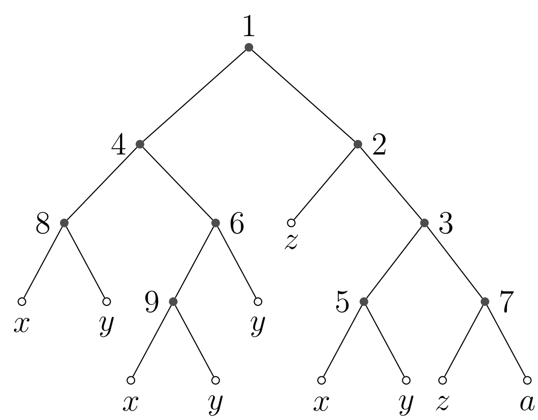

Figure 1: The (a, x, y, z)-labeling for $(\mathrm{exc}, \mathrm{drop}, \mathrm{fix})$.

To recover a permutation σ from a complete increasing binary tree T, we may decompose T into a forest of planted increasing binary trees by removing the edges from the root to the a-leaf and deleting the a-leaf. Each component is either a single root or a tree for which the root has only one child. Such a tree is called a planted increasing binary tree. For example, the tree in Figure 1 has the decomposition as given in Figure 2.

Clearly, a planted increasing binary tree can be regarded as a representation of a cycle since a cycle can be expressed in the form of the minimum element followed by a permutation. But the permutation after the minimum element in turn corresponds to an increasing binary tree. In this way, the permutation

$$(1\,8\,4\,9\,6)\ (2)\ (3\,5)\ (7)$$

can be recovered from the tree in Figure 1.

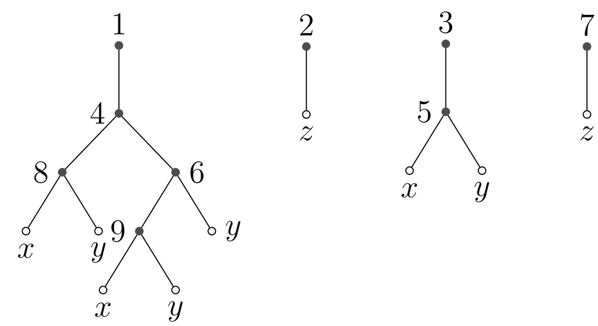

Figure 2: A forest of planted increasing binary trees.

A grammatical derivation of the generating function of the Eulerian polynomials $A_n(x, y)$ was given in [4]. The same reasoning can be carried over to the computation of the generating function of $F_n(x, y, z)$. Bear in mind that the generating function with respect to the formal derivative D is defined by

$$\mathrm{Gen}(w, t) = \sum_{n=0}^{\infty} D^n(w) \frac{t^n}{n!},$$

with w is a Laurent polynomial in the variables a, x, y, z. Note that the generating functions with respect to D permits the multiplication property, which is equivalent to the Leibniz rule, see [4] and references therein.

Theorem 2.2. *We have*

$$\mathrm{Gen}(a, t) = \frac{a(y - x)e^{zt}}{ye^{xt} - xe^{yt}}. \tag{2.5}$$

Proof. In virtue of the rules

$$x \to xy, \quad y \to xy,$$

we have the generating function

$$\mathrm{Gen}(x, t) = \frac{x - y}{1 - yx^{-1}e^{(x-y)t}},$$

see [4]. Since $D(z - y) = xy - xy = 0$, i.e., $z - y$ is a constant relative to D, we find that

$$D^n(ax^{-1}) = D^{n-1}\left(ax^{-1}(z - y)\right) = ax^{-1}(z - y)^n,$$

and hence

$$\mathrm{Gen}(ax^{-1}, t) = \sum_{n=0}^{\infty} D^n(ax^{-1}) \frac{t^n}{n!} = ax^{-1}e^{(z-y)t}. \tag{2.6}$$

By the Leibniz rule or the product rule, we infer that

$$\mathrm{Gen}(a, t) = \mathrm{Gen}(x \cdot ax^{-1}, t) = \mathrm{Gen}(x, t)\,\mathrm{Gen}(ax^{-1}, t) = \frac{a(y - x)e^{zt}}{ye^{xt} - xe^{yt}},$$

as required. ∎

Setting $a = 1$, we arrive at the relation (2.2). Setting $z = 0$ yields the generating function of the derangement polynomials, see Brenti [1].

3 The joint distribution of Roselle

In this section, we give an account of the generating function of Roselle [9] for the joint distribution of the number of ascents and the number of successions in a nutshell. Starting with recurrence relations, Roselle employed the symbolic method to accomplish the task of computation. Such an antiquate mechanism is rarely in demand these days, but perhaps it should not be completely forgotten, even though it might seem obscure or dubious and even if its extinction is inevitable.

3.1 The formulas of Roselle

Let us recall some definitions. Let $n \geq 1$, and let σ be a permutation of $[n]$. We assume that $\sigma_0 = 0$. An ascent or a rise of σ is an index $0 \leq i \leq n-1$ such that $\sigma_i < \sigma_{i+1}$. The number of ascents of σ is denoted by $\operatorname{asc}(\sigma)$. An index i $(1 \leq i \leq n-1)$ is called a descent of σ if $\sigma_i > \sigma_{i+1}$. In this definition, the index n is not counted as a descent. The number of descents of σ is denoted by $\operatorname{des}(\sigma)$. An index i $(1 \leq i \leq n-1)$ of σ is called a succession, or an interior succession, if $\sigma_i + 1 = \sigma_{i+1}$. We call an index i $(1 \leq i \leq n)$ a left succession if $\sigma_{i-1} + 1 = \sigma_i$. Mind the subtlety here concerning the choice of the index for a left succession.

In order to single out ascents that are not left successions, we say that an index $1 \leq i \leq n$ of σ is a jump if $i-1$ is an ascent but i is not a left succession, that is, $\sigma_i \geq \sigma_{i-1} + 2$. The number of jumps of σ is denoted by $\operatorname{jump}(\sigma)$.

For $2 \leq i \leq n$, if i is jump, then $i-1$ is called a big ascent by Ma-Qi-Yeh-Yeh [8], and the number of big ascents of σ is denoted by $\operatorname{basc}(\sigma)$. It should be noted that the position 1 is not considered a big ascent in any event.

Let $P(n, r, s)$ denote the number of permutations of $[n]$ with r ascents and s (interior) successions. For example, $P(3, 2, 0) = 2$. The two permutations of $\{1, 2, 3\}$ with two ascents and no successions are 132, 213. Nevertheless, 132 has a left succession. While the term left succession is not manifestly put to use, one can find a clue through the generating function for the number of permutations of $[n]$ with r ascents and with no left successions, see Roselle [9]. As a matter of fact, he introduced the polynomial

$$P_n^*(x) = \sum_{r=1}^{n-1} P^*(n, r) x^r,$$

where $P^*(n, r)$ is number of permutations on $[n]$ with the first element greater than 1, and with r rises, no successions. Notice that $P^*(n, r)$ is the number of permutations on $[n]$ with r rises and no left successions. By means of $P_n^*(x)$, he defined the polynomial

$$P_n^*(z, x) = \sum_{j=0}^{n} \binom{n}{j} P_j^*(x) z^{n-j},$$

and derived the generating function for the joint distribution of number of jumps and number of left successions.

Theorem 3.1 (Roselle)**.** *We have*

$$\sum_{n=0}^{\infty} P_n^*(z,x)\frac{t^n}{n!} = \frac{(1-x)e^{zt}}{e^{xt}-xe^t}. \tag{3.1}$$

Notice that this formula coincides with (2.1) for the joint distribution of $(\mathrm{exc}, \mathrm{fix})$. But this is by no means a coincidence. As it happens, we will have the same grammar and so we ought to have the same story.

Define

$$P_n(z,x) = \sum_{r=1}^{n}\sum_{s=0}^{r-1} P(n,r,s)x^r z^s.$$

Roselle showed that

$$P_n(z,x) = P_n^*(xz,x) + x(1-z)P_{n-1}^*(xz,x). \tag{3.2}$$

Combining the formula (3.1) and the relation (3.2) yields the generating function of $P_n(z,x)$ in the following form, see [8].

Corollary 3.2.

$$\sum_{n=0}^{\infty} P_{n+1}(z,x)\frac{t^n}{n!} = \frac{x(1-x)^2e^{(xz+1)t}}{(e^{xt}-xe^t)^2}. \tag{3.3}$$

Proof. In light of (3.1), we find that

$$\begin{aligned}\sum_{n=9}^{\infty} P_n^*(xz,x)\frac{t^n}{n!} &= \frac{(1-x)e^{xzt}}{e^{xt}-xe^t},\\ \sum_{n=0}^{\infty} P_{n+1}^*(xz,x)\frac{t^n}{n!} &= \left(\frac{(1-x)e^{xzt}}{e^{xt}-xe^t}\right)'.\end{aligned}$$

Owing to (3.2), an easy computation reveals (3.3). ∎

3.2 A grammatical labeling for left successions

As alluded by the grammar of Dumont, we tend to believe that the notion of a left succession should be considered as a legitimate object of the subject, but it does not seem to have gained enough recognition.

Define

$$L_n(x,y,z) = \sum_{\sigma\in S_n} x^{\mathrm{jump}(\sigma)}y^{\mathrm{des}(\sigma)}z^{\mathrm{lsuc}(\sigma)}.$$

For $n = 0$, set $L_0(x, y, z) = 1$.

The following theorem shows that the polynomials $L_n(x, y, z)$ can be generated by the grammar G in (2.3) of Dumont, that is,

$$G = \{a \to az,\ z \to xy,\ x \to xy,\ y \to xy\}.$$

Theorem 3.3. *Let D be the formal derivative with respect to G. For $n \geq 0$, we have*

$$D^n(a) = aL_n(x, y, z). \tag{3.4}$$

The above theorem can be confirmed by a labeling scheme of permutations. Assume that σ is a permutation of $[n]$. Consider the position after each element σ_i for $i = 0, 1, \ldots, n$, with $\sigma_0 = 0$. First of all, label the position after the maximum element n by a. For the remaining positions, if i is a jump, then label the position on the right of σ_i by x; if i is a left succession, then label the position on the left of σ_i by z, if i is a descent and $\sigma_i \neq n$, label the position on the right of σ_i by y. Below is an example for such a labeling:

$$x\ 2\ x\ 6\ y\ 3\ z\ 4\ y\ 1\ x\ 5\ x\ 8\ z\ 9\ a\ 7\ y. \tag{3.5}$$

Write $*$ for the element $n+1$ to be inserted into σ. The change of labels can be described as follows. Assume that $*$ is to be inserted at the position between σ_i and σ_{i+1}, where $0 \leq i \leq n$.

1. If $*$ is inserted at a position a, that is, $\sigma_i = n$, then we get $n\, z\, *\, a\sigma_{i+1}$ in the neighborhood, this operation is captured by the rule $a \to az$.

2. If $*$ is inserted at a position x, then we see the update of σ: $\sigma_i\, x\, \sigma_{i+1} \to \sigma_i\, x\, *\, a\, \sigma_{i+1}$. In the meantime, the label a after n in σ, wherever it is, will be switched to y, because $*$ is not inserted after n. This change of labels is reflected by the rule $x \to xy$.

3. If $*$ is inserted at a position y, since $\sigma_i \neq n$, the update of σ can be described by $\sigma_i\, y\, \sigma_{i+1} \to \sigma_i\, x\, *\, a\, \sigma_{i+1}$. In the meantime, the label a after n in the labeling of σ, wherever it is, will be switched to y. This change of labels is governed by the rule $y \to xy$.

4. If $*$ is inserted at a position z, then we have the update $\sigma_i\, z\sigma_{i+1} \to \sigma_i\, x\, *\, a\, \sigma_{i+1}$. In the meantime, the label a after n in the labeling of σ, wherever it is, will be switched to y. This change of labels is in compliance with rule $z \to xy$.

We now have the same grammar for two occasions. Thus we reach an equidistribution.

Theorem 3.4. *For $n \geq 1$, the statistics* $(\mathrm{jump}, \mathrm{des}, \mathrm{lsuc})$ *and the statistics* $(\mathrm{exc}, \mathrm{drop}, \mathrm{fix})$ *are equidistributed over the set of permutations of* $[n]$.

In fact, we are going to pursue a stronger version of the above theorem, that is, a left succession analogue of the Diaconis-Evans-Graham theorem. As can be seen, while a grammar might be sufficient to guarantee an equidistribution of two sets of statistics, it does not tell us explicitly how to form a bijection. Nevertheless, there are occasions that the grammar could be helpful in establishing a correspondence even with a specific constraint.

3.3 Back to interior successions

Returning to the original formulation of the joint distribution of Roselle, let $R_n(x, y, z)$ denote the homogeneous form of $P_n(z, x)$, that is,

$$R_n(x,y,z) = \sum_{\sigma \in S_n} x^{\mathrm{jump}(\sigma)} y^{\mathrm{des}(\sigma)} z^{\mathrm{suc}(\sigma)}, \tag{3.6}$$

which we call the Roselle polynomials.

Using the same reasoning for the grammatical labeling for left successions together with a slight alteration of the grammar, a grammatical calculus can be carried out for the Roselle polynomials. Suppose that we are working with the grammar for left successions, but we would like to avoid 1 being counted as a left succession, which is labeled by z. This requirement can be easily met by turning to an additional label b as a substitute of the label z. That is to say, the rule $z \to xy$ should be recast as $b \to xy$. For example, we should start with the initial labeling $0b1a$ instead of $0z1a$. As for the original labels a, x, y, z, their roles will remain unchanged. Thus we meet with the mended grammar

$$G = \{a \to az,\ b \to xy,\ x \to xy,\ y \to xy, z \to xy\}. \tag{3.7}$$

Let D be the formal derivative of G in (3.7). We have

$$D(ab) = abz + axy,$$

which is the sum of weights of the two permutations

$$0\ b\ 1\ z\ 2\ a\ 0, \quad 0\ x\ 2\ a\ 1\ y\ 0.$$

In general, for $n \geq 0$, the following relation holds

$$R_n(x,y,z) = D^{n-1}(ab)\big|_{a=1,b=1}. \tag{3.8}$$

The grammatical calculus shows that the generating function for the Roselle polynomials is essentially a product of the generating function of $L_n(x, y, z)$ and the generating function of the bivariate Eulerian polynomials.

Theorem 3.5. *We have*

$$\mathrm{Gen}(ab,t) = \frac{a(y-x)e^{zt}}{ye^{xt} - xe^{yt}} \left(\frac{x-y}{1 - yx^{-1}e^{(x-y)t}} - x + b \right). \tag{3.9}$$

Proof. By the Leibniz rule, we get

$$\mathrm{Gen}(ab,t)=\sum_{n=0}^{\infty}D^n(ab)\frac{t^n}{n!}=\mathrm{Gen}(a,t)\mathrm{Gen}(b,t).$$

Since $D(b)=D(x)=xy$, it follows that

$$\mathrm{Gen}(b,t)=\mathrm{Gen}(x,t)-x+b=\frac{x-y}{1-yx^{-1}e^{(x-y)t}}-x+b,$$

which, together with Theorem 2.2, yields (3.9). ∎

Setting $a=1$, $y=1$, $b=x$, $z=xz$, and treating the initial two elements $0\,1$ of a permutation as an ascent, we see that

$$D^{n-1}(ab)|_{a=1,y=1,b=x,z=xz}=x\sum_{\sigma\in S_n}x^{\mathrm{jump}(\sigma)}(xz)^{\mathrm{suc}(\sigma)}=\sum_{\sigma\in S_n}x^{\mathrm{asc}}z^{\mathrm{suc}},\tag{3.10}$$

which is the generating function for the joint distribution of $(\mathrm{asc},\mathrm{suc})$ over permutations on $[n]$, that is,

$$P_n(z,x)=D^{n-1}(ab)|_{a=1,y=1,b=x,z=xz}.$$

Moreover,

$$\sum_{n=0}^{\infty}P_{n+1}(z,x)\frac{t^n}{n!}=\mathrm{Gen}(ab,t)|_{a=1,y=1,b=x,z=xz}=\frac{x(1-x)^2e^{(z+1)t}}{(e^{xt}-xe^t)^2},$$

which is in accordance with (3.3).

We finish this section with a relation between $R_n(x,y,z)$ and $L_n(x,y,z)$.

Theorem 3.6. *For $n\geq 0$, we have*

$$R_{n+1}(x,y,z)=L_n(x,y,z)+\sum_{k=1}^{n}\binom{n}{k}A_k(x,y)L_{n-k}(x,y,z),\tag{3.11}$$

where for $k\geq 1$, $A_k(x,y)$ are the bivariate Eulerian polynomials.

This relation admits a combinatorial interpretation. Let T be a complete increasing binary tree on $[n+1]$. Suppose that we wish to interpret $R_{n+1}(x,y,z)$ in terms of complete increasing binary trees. We may adopt the following labeling for $L_{n+1}(x,y,z)$, except that if the root of T has a z-leaf, we should label it by 1 rather than z. If this is the case, then the subtree of T can be viewed a complete increasing tree on $[n]$ with the labeling for $L_n(x,y,z)$. If the root of T has a nonempty left subtree, then this subtree does not have any z-leaves, which can be reckoned as a labeling for the Eulerian polynomials, and so we are done.

4 A analogue of the Diaconis-Evans-Graham theorem

The main result of this paper is a left succession analogue of the Diaconis-Evan-Graham theorem on successions and fixed points of permutations. The grammar of Dumont can be utilized to produce a bijection, which implies an equidistribution of the statistics $(\mathrm{jump}, \mathrm{des})$ and $(\mathrm{exc}, \mathrm{drop})$ restricted to a given set of left successions and the same set of fixed points. Consequences and applications will be discussed.

For a permutation $\sigma \in S_n$, define

$$\begin{aligned} M(\sigma) &= \{i \,|\, 1 \leq i \leq n-1,\ \sigma_i + 1 = \sigma_{i+1}\}, \\ G(\sigma) &= \{i \,|\, 1 \leq i \leq n-1,\ \sigma_i = i\}, \\ F(\sigma) &= \{i \,|\, 1 \leq i \leq n,\ \sigma_i = i\}. \end{aligned}$$

It should be stressed that the index n is excluded in the definition of $G(\sigma)$. Given a subset $I \subseteq [n-1]$, denote by $M_n(I)$ the set of permutations of $[n]$ with I being the set of (interior) successions, and denote by $G_n(I)$ the set of permutations $\sigma \in S_n$ such that $G(\sigma) = I$. Similarly, $F_n(I)$ denotes the set of permutations σ of $[n]$ such that $F(\sigma) = I$.

Theorem 4.1 (Diaconis-Evans-Graham)**.** *Let $n \geq 1$ and $I \subseteq [n-1]$. Then there is a bijection between $M_n(I)$ and $G_n(I)$.*

For the special case $I = \emptyset$, a permutation without successions is called a relative derangement. Let D_n denote the number of derangements of $[n]$, and let Q_n denote the number of relative derangements of $[n]$. Roselle [9] and Brualdi [2] deduced that

$$Q_n = D_n + D_{n-1}. \tag{4.1}$$

A bijective proof of this relation was given in [3], appealing to the first fundamental transformation. Taking $I = \emptyset$, a permutation in $G_n(I)$ may or may not have n as a fixed point. The permutations in these two cases are counted by D_{n-1} and D_n, respectively. Thus the $I = \emptyset$ case of the proof of the Diaconis-Evans-Graham theorem yields a combinatorial interpretation of (4.1).

Here comes the question of what happens for left successions. Define the set of left successions of a permutations as

$$L(\sigma) = \{\sigma_i \,|\, 1 \leq i \leq n,\ \sigma_{i-1} + 1 = \sigma_i\}.$$

For any $I \subseteq [n]$, denote by $L_n(I)$ the set of permutations $\sigma \in S_n$ such that $L(\sigma) = I$. We establish the following correspondence, and the proof is a grammar assisted bijection.

Theorem 4.2. *For $n \geq 1$ and any $I \subseteq [n]$, there is a bijection from $L_n(I)$ to $F_n(I)$ that maps the pair of statistics $(\mathrm{jump}, \mathrm{des})$ to the pair of statistics $(\mathrm{exc}, \mathrm{drop})$.*

Proof. Given a permutation σ on $[n]$, we wish to construct a complete increasing binary T with the (a, x, y, z)-labeling. The map can be described as a recursive procedure. For $n = 1$, the permutation $z1a$ is mapped to the complete increasing tree with one internal vertex 1.

Assume that $\sigma = \sigma_1\sigma_2 \cdots \sigma_n$ is a permutation of $[n]$ and that T is the tree corresponding to σ. For $1 \leq i \leq n$, the position i is referred to the position immediately before σ_i, whereas the position $n + 1$ is meant to be the position after σ_n.

To build a bijection, we need to inductively maintain a coordinated property of σ and T. Besides having the same weight, they should be synchronized in a certain sense to keep the process running till the completion of the task. More precisely, we say that the labeling of σ is coherent with the labeling of T provided that the following conditions are satisfied. In fact, these properties should be preserved after each update.

- If the position i in σ is labeled by x, then the vertex σ_i in T has a x-leaf;
- If the position i in σ is labeled by y, then the vertex $\sigma_{i-1} + 1$ in T has a y-leaf;
- If the position i in σ is labeled by z, then the vertex σ_i in T has a z-leaf.

Suppose that $* = n + 1$ is to be inserted into σ. It is necessary to find out how to update the tree T accordingly. Now that there are $n + 1$ (insertion) positions for σ and there are $n + 1$ leaves for T, it suffices to define a map from the set of positions to the set of leaves of T with the understanding that when $*$ is inserted at position, say i, T will be updated to T' by turning the corresponding leaf of T into an internal vertex $*$. Denote by σ' the permutation produced from σ by inserting $*$ at the position i. There are four cases on the ground of the four rules of the grammar.

1. If $*$ is inserted at a position labeled by a, we add $*$ to T at the position of the a-leaf. This operation is consistent with the rule $a \to az$.

2. For a label z at the position i, by the induction hypothesis, we know that the vertex σ_i in T' has a z-leaf, so we apply the rule $z \to xy$ to this z-leaf to update T. Notice that when $*$ is inserted, the label a on the right of n in σ will be switched to y. In view of the construction, this y-label corresponds to the y-leaf of $*$ in T'. By inspection, we find that the labeling of σ' is coherent with the labeling of T'.

3. When the insertion occurs at position i labeled by x, by the induction hypothesis, we know that the vertex σ_i in T has a x-leaf. Then we apply the rule $x \to xy$ to this leaf. Notice that the y-leaf of $*$ in T' corresponds to the y-label on the right of n by the construction. Again, it can be checked that the labeling of σ' is coherent with the labeling of T'.

4. For a position i labeled by y, by the induction hypothesis, we know that the vertex $\sigma_{i-1}+1$ in T has a y-leaf. Then we apply the rule $y \to xy$ to this leaf. In this case, the labeling of σ' remains coherent with the labeling of T'.

So far we have provided a procedure to update T depending on where the element $*$ is inserted into σ. Moreover, every stage of this procedure is reversible. The detailed examination is omitted. Notice that the grammar ensures that the map is weight-preserving, that is, the weight of σ equals that of T.

It should be added that a left succession i is created in σ whenever a vertex σ_i with a left z-leaf is created in T. Meanwhile, a left succession i is destroyed in σ whenever a z-leaf with parent σ_i is destroyed. This completes the proof. ∎

Taking the following permutation σ with 2 jumps and 1 descent as an example,

$$z\ 1\ x\ 3\ y\ 2\ x\ 4\ z\ 5\ a,$$

where $L(\sigma) = \{1, 5\}$. Figure 3 illustrates how to build the corresponding trees step by step, where a label in boldface indicates where an insertion takes place.

We remark that in principle one can reformulate the above bijection in a way without resorting to increasing binary trees as an intermediate structure. In doing so, all we need is a comparative scrutiny of the two recursive procedures to generate permutations. One relies on the linear representation and the other on the cycle notation.

For $n = 3$, the correspondence is given in the table below. The cases when $F(I) = \emptyset$ are not listed, such as $I = \{1, 2\}$.

$I \subseteq [n]$	$L_n(I) \to F_n(I)$	(jump, des) $\to$ (exc, drop)
$\emptyset$	$2\ 1\ 3 \to (123)$ $3\ 2\ 1 \to (132)$	$(2, 1)$ $(1, 2)$
$\{1\}$	$1\ 3\ 2 \to (1)(23)$	$(1, 1)$
$\{2\}$	$3\ 1\ 2 \to (13)(2)$	$(1, 1)$
$\{3\}$	$2\ 3\ 1 \to (12)(3)$	$(1, 1)$
$\{1, 2, 3\}$	$1\ 2\ 3 \to (1)(2)(3)$	$(0, 0)$

We conclude with discussions about consequences of the above correspondence. Given a subset I of $[n]$, denote by $F_n(I)$ the set of permutations of $[n]$ with I being the set of fixed points. Let $L_n(I)$ stands for the set of permutations of $[n]$ with I being the set of elements whose indices are left successions.

The special case $I = \emptyset$ of Theorem 4.1 is related to the notion of a skew derangement as termed in [3]. Returning to the definition of $F(\sigma)$, as it is not required that $\sigma(n) \neq n$, we

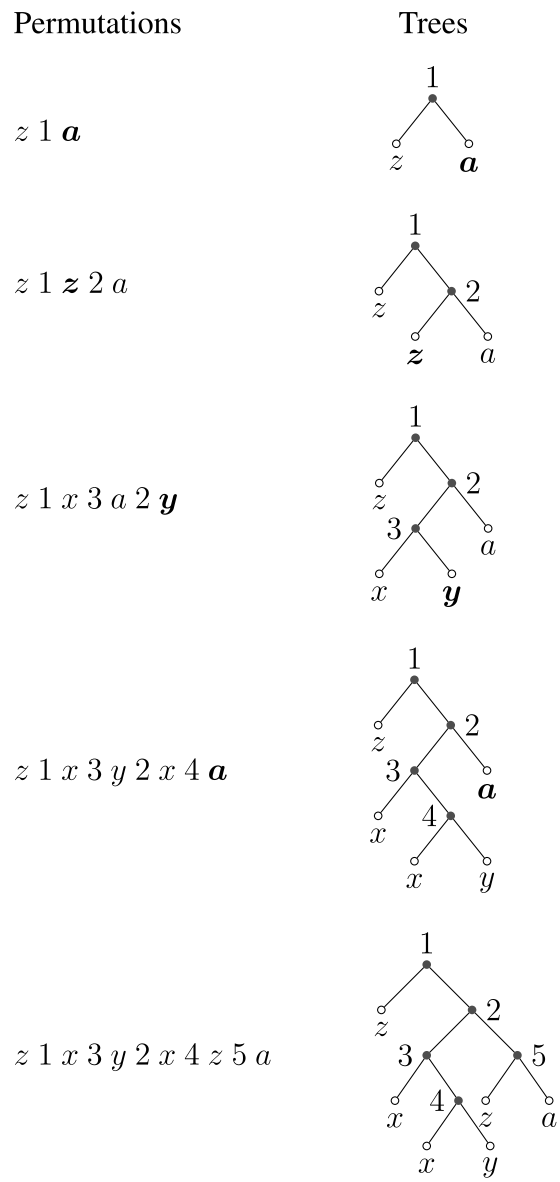

Figure 3: An example.

may regard σ as a one-to-one map f from $[n]$ to the set $\{0, 1, \ldots, n-1\}$. In the case $I = \emptyset$, a permutation σ is in $F_n(I)$ if and only if $f(i) \neq i$ for any i, that is, f is a skew derangement. Specializing Theorem 4.2 to $I = \emptyset$, we obtain another combinatorial interpretation of (4.1), along with the following property.

Corollary 4.3. *There is a one-to-one correspondence between the set of permutations on* $[n]$ *without left successions and the set of permutations without fixed points such that the statistics* $(\mathrm{jump}, \mathrm{des})$ *are transformed into the statistics* $(\mathrm{exc}, \mathrm{drop})$.

Notice that a relative derangement has a left succession if and only if it begins with the element 1. In this case, we may treat the element 1 as zero and treat the rest of the permutation as a permutation of $[n-1]$. This yields a permutation of $[n-1]$ without left successions.

Thus Theorem 4.2 does offer an alternative way to establish (4.1) combinatorially. Strictly speaking, the set of relative derangements of $[n]$ can be divided into two classes. The first consists of those with no left successions, as counted by D_n, and the second consists of those beginning with 1, as counted by D_{n-1}.

By comparing Theorem 4.1 with Theorem 4.2, we are led to a correspondence between left successions and interior successions.

Corollary 4.4. *Let $I \subseteq [n-1]$. Write $L_n(I)$ for the set of permutations of $[n]$ with I being the set of elements whose indices are left successions and write $M_n(I)$ for the set of permutations of $[n]$ with I being the set of interior successions. Then there exists a bijection between $L_n(I)$ and $M_n(I)$.*

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