A Context-free Grammar for the *e*-Positivity of the Trivariate Second-order Eulerian Polynomials

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Abstract. Ma-Ma-Yeh made a beautiful observation that a change of the grammar of Dumont instantly leads to the γ-positivity of the Eulearian polynomials. We notice that the transformed grammar bears a striking resemblance to the grammar for 0-1-2 increasing trees also due to Dumont. The appearance of the factor of two fits perfectly in a grammatical labeling of 0-1-2 increasing plane trees. Furthermore, the grammatical calculus is instrumental to the computation of the generating functions. This approach can be adapted to study the e-positivity of the trivariate second-order Eulerian polynomials first introduced by Dumont in the context of ternary trees, and independently defined by Janson, in connection with the joint distribution of the numbers of ascents, descents and plateaux over Stirling permutations.

Keywords: Context-free grammars, 0-1-2 increasing plane trees, 0-1-2-3 increasing plane trees, Stirling permutations, Eulerian polynomials, Second-order Eulerian polynomials, γ-positivity, e-positivity

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1 Introduction

The objective of this paper is to present a context-free grammar to derive the e-positivity of the trivariate second-order Eulerian polynomials $C_n(x,y,z)$ defined on Stirling permutations, first introduced by Dumont [8] in terms of ternary trees and rediscovered by Janson [17] associated with Stirling permutations.

This work was inspired by a beautiful observation of Ma-Ma-Yeh [20] that a change or a transformation of a context-free grammar found by Dumont [9] instantly leads to the γ-positivity of the Eulerian polynomials. We find that the transformed grammar not only implies the γ-positivity, it also provides a combinatorial interpretation of the γ-coefficients in terms of a certain kind of increasing plane trees.

For $n \geq 1$, let $[n] = \{1, 2, \ldots, n\}$ and let S_n denote the set of permutations of $[n]$. For a permutation $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in S_n$, we assume that a zero is patched at the beginning and at the end, that is, $\sigma_0 = \sigma_{n+1} = 0$. An index $1 \leq i \leq n$ is said to be a descent (ascent) of a permutation $\sigma \in S_n$ if $\sigma_i > \sigma_{i+1}$ $(\sigma_{i-1} < \sigma_i)$. The number of permutations of $[n]$ with k descents $(1 \leq k \leq n)$ is often denoted by $A(n,k)$, or sometimes by $\left\langle {n \atop k} \right\rangle$. The Eulerian polynomials $A_n(x)$ are defined by $A_0(x) = 1$ and for $n \geq 1$,

$$A_n(x) = \sum_{\sigma \in S_n} x^{\operatorname{des}(\sigma)} = \sum_{k=1}^{n} A(n,k)x^k, \tag{1.1}$$

where $\operatorname{des}(\sigma)$ denotes the number of descents of a permutation σ. A bivariate version of the Eulerian polynomials $A_n(x,y)$ is given by

$$A_n(x,y) = \sum_{\sigma \in S_n} x^{\operatorname{des}(\sigma)} y^{\operatorname{asc}(\sigma)} = \sum_{k=1}^{n} A(n,k)x^k y^{n+1-k}, \tag{1.2}$$

where $n \geq 1$ and $\operatorname{asc}(\sigma)$ stands for the number of ascents of σ. Bear in mind that for any permutation $\sigma \in S_n$, we have

$$\operatorname{des}(\sigma) + \operatorname{asc}(\sigma) = n + 1. \tag{1.3}$$

The first few values of $A_n(x,y)$ are given below,

$$\begin{aligned}
A_1(x,y) &= xy,\\
A_2(x,y) &= xy^2 + x^2y,\\
A_3(x,y) &= xy^3 + 4x^2y^2 + x^3y,\\
A_4(x,y) &= xy^4 + 11x^2y^3 + 11x^3y^2 + x^4y,\\
A_5(x,y) &= xy^5 + 26x^2y^4 + 66x^3y^3 + 26x^4y^2 + x^5y,\\
A_6(x,y) &= xy^6 + 57x^2y^5 + 302x^3y^4 + 302x^4y^3 + 57x^5y^2 + x^6y.
\end{aligned}$$

A celebrated theorem of Foata and Schützenberger [10] states that for $n \geq 1$, the Eulerian polynomial $A_n(x)$ can be expanded uniquely in the following form

$$A_n(x) = \sum_{k=1}^{[(n+1)/2]} \gamma_{n,k} x^k (1+x)^{n-2k+1} \tag{1.4}$$

with nonnegative coefficients $\gamma_{n,k}$. The above expression (1.4) is called the γ-expansion of $A_n(x)$, which can be restated as for $A_n(x,y)$:

$$A_n(x,y) = \sum_{k=1}^{[(n+1)/2]} \gamma_{n,k}\,(xy)^k (x+y)^{n-2k+1}. \tag{1.5}$$

Remarkably, Foata and Schützenberger discovered a combinatorial interpretation of the coefficients $\gamma_{n,k}$, that is, for $n \geq 1$ and $1 \leq k \leq [(n+1)/2]$, $\gamma_{n,k}$ equals the number of permutations of $[n]$ with k descents, but no double descents. Here a double descent of a permutation $\sigma \in S_n$ is defined to be an index $1 \leq i \leq n-1$ such that $\sigma_i > \sigma_{i+1} > \sigma_{i+2}$.

The nonnegativity of the coefficients $\gamma_{n,k}$ has been referred to as the γ-positivity. This property of the Eulerian polynomials and other polynomials along with the q-analogues has been extensively studied ever since, see, for example, [1, 3, 7, 10, 11, 13, 16, 18, 19, 20, 21, 22, 23, 24, 25, 26, 29, 30], to name a few.

Our starting point is the grammar of Dumont for the Eulerian polynomials, namely,

$$G = \{x \rightarrow xy, \quad y \rightarrow xy\},$$

which can be expressed as a differential operator

$$\Delta = xy\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}. \tag{1.6}$$

Ma-Ma-Yeh [20] realized that by a change of variables

$$u = xy, \quad v = x+y,$$

the above grammar is transformed into a new grammar

$$H = \{u \rightarrow uv, \quad v \rightarrow 2u\}, \tag{1.7}$$

ensuring the γ-positivity of the Eulerian polynomials.

Without the setting of a grammar, the above argument can be recast in terms of the differential operator Δ in (1.6). Since Δ is a derivative, $\Delta(x+y) = 2xy$ and $\Delta(xy) = xy(x+y)$, we see that for $n \geq 1$, $A_n(x,y) = \Delta^{n-1}(xy)$ must be a polynomial in $x+y$ and xy with nonnegative coefficients. Nevertheless, the grammar turns out to be informative to the combinatorial understanding of the coefficients.

First, we notice that the above grammar H bears a striking resemblance to the following grammar for 0-1-2 increasing trees and the André polynomials, namely,

$$G = \{x \rightarrow xy, \quad y \rightarrow x\}.$$

Examining the factor of two in the grammar H, we are guided precisely to the structure of 0-1-2 increasing plane trees along with a natural grammatical labeling. This formulation is in agreement with the known interpretation of binary increasing trees on $[n]$ with exactly k leaves and no vertices with left children only, as numerically described by Han-Ma [16] in terms of 0-1-2 increasing trees $[n]$ with k leaves. We find it convenient to work with 0-1-2 increasing plane trees in order to describe the labeling consistent with the grammar H.

The grammatical approach associated with a grammatical labeling of 0-1-2 increasing plane trees offers a test ground for the main result of this paper, which is concerned with the trivariate second-order Eulerian polynomials $C_n(x,y,z)$ on Stirling permutations on the multiset $[n]_2 = \{1,1,2,2,\ldots,n,n\}$, where $n \geq 1$. Stirling permutations were introduced by Gessel and Stanley [14]. A permutation $\sigma = \sigma_1\sigma_2\cdots\sigma_{2n}$ of $[n]_2$ is said to be a Stirling permutation if for any $1 \leq j \leq n$ the elements between the two j's in σ, if any, are greater than j. For $n \geq 1$, the set of Stirling permutations of $[n]_2$ is denoted by Q_n. A descent and an ascent of $\sigma \in Q_n$ can be defined analogously to the case of ordinary permutations. For a Stirling permutations σ, we adopt the convention that σ is patched with a zero both at the beginning and at the end, that is, $\sigma_0 = \sigma_{2n+1} = 0$. The number of Stirling permutations of $[n]_2$ with k descents is called the second-order Stirling number, denoted by $C(n,k)$, or $\left\langle\!\left\langle {n \atop k} \right\rangle\!\right\rangle$.

Bona [2] introduced the notion of a plateau of $\sigma \in Q_n$, which is defined to be a pair of two adjacent elements (σ_i, σ_{i+1}) such that $\sigma_i = \sigma_{i+1}$. More precisely, for $\sigma \in Q_n$, the number plateaux, denoted $\mathrm{plat}(\sigma)$, is defined to be the number of indices $1 \leq i \leq 2n$ such that $\sigma_i = \sigma_{i+1}$. He showed that for $n \geq 1$, the statistics $\mathrm{asc}(\sigma)$, $\mathrm{des}(\sigma)$ and $\mathrm{plat}(\sigma)$ have the same distribution over Q_n.

Janson [17] constructed an urn model to prove the symmetry of the joint distribution of the three statistics. More precisely, for $n \geq 1$, polynomials $C_n(x,y,z)$ are defined by

$$C_n(x,y,z) = \sum_{\sigma \in Q_n} x^{\mathrm{des}(\sigma)} y^{\mathrm{asc}(\sigma)} z^{\mathrm{plat}(\sigma)}.$$

Notice that for $n \geq 1$ and any $\sigma \in Q_n$, we have

$$\mathrm{des}(\sigma) + \mathrm{asc}(\sigma) + \mathrm{plat}(\sigma) = 2n+1. \tag{1.8}$$

The symmetry property also follows from the recurrence relation: For $n \geq 1$,

$$C_{n+1}(x,y,z) = xyz\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right) C_n(x,y,z). \tag{1.9}$$

It should be noted that an equivalent form of the polynomials $C_n(x,y,z)$ was given by Dumont [8] in his study of Stirling permutations and ternary trees. The recurrence relation (1.9) was established by Dumont [8] and by Haglund-Visontai [15]. In the combinatorial definition of $C_n(x,y,z)$ given by Dumont, there are three s-tatistics on ternary trees. While, in principle, such statistics can be transformed into statistics on Stirling permutations, but the corresponding plateau number was not explicitly presented in the work of Dumont.

The differential operator in the recurrence relation (1.9) can be prescribed as a grammar

$$G = \{x \rightarrow xyz, \quad y \rightarrow xyz, \quad z \rightarrow xyz\}. \tag{1.10}$$

Indeed, the grammatical labeling as described in the arXiv version of [6] or in [20] is essentially the same argument as that given in [15], which in turn is in the same vein as the recursive construction formulated for the urn model as given by Janson [17]. So the above grammar G in (1.10) should be attributed to Dumont [8].

The symmetry of $C_n(x,y,z)$ suggests that we may consider the expansion into the elementary symmetric functions, as denoted by

$$u = x+y+z, \quad v = xy+xz+yz, \quad w = xyz.$$

Applying the idea of Ma-Ma-Yeh to the grammar in (1.10), we are led to the following grammar

$$H = \{u \rightarrow 3w, \quad v \rightarrow 2uw, \quad w \rightarrow vw\}. \tag{1.11}$$

Relying on this grammar, we realize that the argument for the γ-expansion of the Eulerian polynomials can be carried over to the expansion of $C_n(x,y,z)$ into the elementary symmetric functions. To be more specific, we prove that for $n \geq 1$, $C_n(x,y,z)$ is a polynomial in u,v,w whose coefficients can be interpreted in terms of 0-1-2-3 increasing plane trees.

The background on the use of context-free grammars for combinatorial enumeration including the notion of a grammatical labeling can be found in [5]. In the next section, we shall give a glimpse of how to compute a generating function based on a context-free grammar. In certain sense, this approach can be thought of as a formal calculus in the spirit of the symbolic method, while we may enjoy the advantage that there is no fear of the lack of rigor.

2 A grammatical calculus for $A_n(x,y)$

A context-free grammar is a set of substitution rules on a set of variables X. A variable can be substituted with a polynomial (or a Laurent polynomials) in X.

A grammar can also be understood as a differential operator. For the purpose of combinatorial enumeration, the variables are attached to combinatorial structures, whereas the rules reflect the recursive construction of combinatorial objects. Computationally speaking, a grammar is a derivative which is often informative for deriving the generating functions.

Let us take the Eulerian polynomials $A_n(x,y)$ to demonstrate the efficiency of the grammatical calculus. Dumont [9] discovered the following grammar for $A_n(x,y)$:

$$G = \{x \rightarrow xy, \quad y \rightarrow xy.\} \tag{2.1}$$

Let D denote the formal derivative with respect to the above grammar G. Dumont showed that $A_n(x,y)$ can be generated by the grammar G, that is, for $n \geq 1$,

$$A_n(x,y) = D^n(x). \tag{2.2}$$

Chen and Fu [5] introduced the notion of a grammatical labeling in the sense that the grammar G reflects the generation of permutations in S_{n+1} from permutations in S_n.

If we express the formal derivative in terms of a differential operator, the above relation (2.2) can be written as $A_1(x,y) = xy$ and for $n \geq 1$,

$$A_{n+1}(x,y) = xy\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)A_n(x,y).$$

The above recurrence relation also appeared in Haglund-Visontai [15]. It is clear that for $n \geq 1$,

$$A_n(x) = A_n(x,y)|_{y=1}. \tag{2.3}$$

Let us proceed to present a derivation of the well-known generating function of $A_n(x)$:

$$\sum_{n\geq 0} A_n(x)\frac{t^n}{n!} = \frac{1-x}{1-xe^{(1-x)t}}. \tag{2.4}$$

As a matter of fact, we find it more convenient to compute the generating function of $A_n(x,y)$ as stated below.

Theorem 2.1. *Set $A_0(x,y) = y$. Then we have*

$$\sum_{n=0}^{\infty} A_n(x,y)\frac{t^n}{n!} = \frac{y-x}{1-xy^{-1}e^{(y-x)t}} \tag{2.5}$$

It is evident that setting $y = 1$ in (2.5) yields (2.4).

To present a grammatical proof of (2.5), recall that for a Laurent polynomial f in x and y, the generating function of f with respect to a grammar G is defined by

$$\mathrm{Gen}(f,t) = \sum_{n\geq 0} D^n(f)\frac{t^n}{n!}. \tag{2.6}$$

Assume that g is also a Laurent polynomial in x and y. The first and foremost property of D is that it is a derivative, that is,

$$D(fg) = D(f)g + fD(g), \tag{2.7}$$

and hence it obeys the Leibnitz rule

$$D^n(fg) = \sum_{k=0}^{n} \binom{n}{k} D^k(f)D^{n-k}(g), \tag{2.8}$$

for any $n \geq 0$. This implies the multiplicative property

$$\mathrm{Gen}(fg,t) = \mathrm{Gen}(f,t)\,\mathrm{Gen}(g,t). \tag{2.9}$$

Proof of Theorem 2.1 by Using the Grammar of Dumont. Under the assumption $A_0(x,y) = y$, we have $A_n(x,y) = D^n(y)$. So our goal is to compute the generating function $\mathrm{Gen}(y,t)$.

For the formal derivative D with respect to the grammar G in (2.1), we have

$$D(y^{-1}) = -y^{-2}D(y) = -xy^{-1} \tag{2.10}$$

and

$$D(x^{-1}y) = x^{-1}y(x-y). \tag{2.11}$$

As noted in [5], since $x-y$ is a constant with respect to D, we deduce that for $n \geq 0$,

$$D^n(xy^{-1}) = xy^{-1}(y-x)^n. \tag{2.12}$$

In light of the property (2.9), it suffices to consider $\mathrm{Gen}(y^{-1},t)$, since

$$\mathrm{Gen}(y,t) = \frac{1}{\mathrm{Gen}(y^{-1},t)}. \tag{2.13}$$

It follows from (2.10) that

$$\mathrm{Gen}(y^{-1},t) = \sum_{n\geq 0} D^n(y^{-1})\frac{t^n}{n!} = y^{-1} - \sum_{n\geq 1} D^{n-1}(xy^{-1})\frac{t^n}{n!}. \tag{2.14}$$

Invoking (2.12), we get

$$\mathrm{Gen}(y^{-1},t) \;=\; y^{-1} - \sum_{n\geq 1} xy^{-1}(y-x)^{n-1}\frac{t^n}{n!}$$

$$= y^{-1} - \frac{xy^{-1}}{y-x}\left(e^{(y-x)t} - 1\right)$$

$$= \frac{1 - xy^{-1}e^{(y-x)t}}{y-x},$$

which completes the proof by utilizing (2.13). ∎

3 The γ-positivity of $A_n(x,y)$

The coefficients $\gamma_{n,k}$ have a number of combinatorial interpretations. For the purpose of this paper, we shall single out the one in connection with 0-1-2 increasing plane trees.

A 0-1-2 increasing plane tree on $[n]$ is an increasing plane tree for which each vertex has at most two children. For a 0-1-2 increasing plane tree T on $[n]$, assume that it has f_0 leaves and $f_2$vertices of degree two, then it is easily seen that

$$f_2 = f_0 - 1. \tag{3.1}$$

Let $s(n,k)$ be the number of 0-1-2 increasing trees on $[n]$ with k leaves, and let $t(n,k)$ be the number of 0-1-2 increasing plane trees on $[n]$ with k leaves. Then we have

$$t(n,k) = 2^{k-1}s(n,k). \tag{3.2}$$

We now come to the a beautiful observation of Ma-Ma-Yeh [20] on a grammatical explanation of the γ-positivity of $A_n(x,y)$. Note that

$$D(xy) = (x+y)xy, \quad D(x+y) = 2xy.$$

If we set

$$u = xy, \quad v = x+y,$$

then we get $D(u) = uv$ and $D(v) = 2u$. In other words, we have a new grammar

$$H = \{u \rightarrow uv, \quad v \rightarrow 2u\}. \tag{3.3}$$

This grammar implies that the coefficients $\gamma_{n,k}$ in (1.4) are nonnegative. As will be seen, we can benefit more from the grammar H. Indeed, it facilitates a combinatorial explanation of the numbers $\gamma_{n,k}$ based on 0-1-2 increasing plane trees, which in turn can be reformulated as the original combinatorial interpretation of

Foata and Schützenberger. On the other hand, the grammar H also plays an active role in the computation of the generating function of the polynomials $\gamma_n(x)$ defined by

$$\gamma_n(x) = \sum_{k=1}^{[(n+1)/2]} \gamma_{n,k} x^k, \tag{3.4}$$

which has been computed by Chow [7].

Let T be a 0-1-2 increasing plane tree on $[n]$, where $n \geq 1$. We define a labeling of T as follows. A leaf is labeled by u, a degree one vertex is labeled by v and a degree two vertex is labeled by 1. The weight of T is defined to be the product of the labels associated with the vertices of T.

For example, Figure 1 is a 0-1-2 increasing plane on $[6]$.

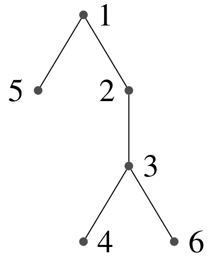

Figure 1: A 0-1-2 increasing plane tree.

A 0-1-2 increasing tree is an increasing tree for which any vertex has at most two children. The 0-1-2 increasing trees on $[n]$ are counted by the Euler number E_n, which also equals the number of alternating permutations on $[n]$. For the grammar

$$G = \{x \to xy,\ y \to x\},$$

the grammatical labeling of a 0-1-2 increasing tree is given as follows. For a 0-1-2 increasing tree T on $[n]$, a leaf is labeled by x, a degree one vertex is labeled by y and a degree two vertex is labeled by 1. A 0-1-2 increasing tree T' on $[n+1]$ can always be obtained from a 0-1-2 increasing tree T on $[n]$ by attaching the vertex $n+1$ to T as a child of a leaf or a degree one vertex.

Increasing plane trees are also called plane recursive trees, see Janson [17]. The above grammatical labeling of 0-1-2 increasing plane trees shows that the γ-coefficients for the Eulerian polynomials can be interpreted based on 0-1-2 increasing plane trees.

Theorem 3.1. *For $n \geq 1$ and $1 \leq k \leq [(n+1)/2]$, the number $\gamma_{n,k}$ equals the number of 0-1-2 increasing plane trees on $\{1,2,\ldots,n\}$ with k leaves.*

It is not hard to transform a 0-1-2 increasing plane tree into a permutation without double descents. There is a one-to-one correspondence ϕ between the set of permutations on $\{1,2,\ldots,n\}$ with j descents and no double descents and the set of 0-1-2 increasing plane trees on $\{1,2,\ldots,n\}$ with k leaves. This is just the classical bijection between permutations and increasing binary trees, restricted to binary trees without vertices having only left children, see Stanley [27].

The structure of 0-1-2 increasing plane trees can be employed to partition the set of permutations of $[n]$ into classes similar to the classification according to the Foata-Strehl group. Let T be a 0-1-2 increasing plane tree on $[n]$. We now consider a labeling of T by assigning a label x or y to a degree one vertex, and a label xy to a leaf, and 1 to a degree two vertex. Let $\alpha(T)$ denote the set of labeled trees obtained from T, and let $w(T)$ denote the sum of weights of all trees in $\alpha(T)$, where the weight of a labeled tree is defined as the product of the labels of all the vertices. Assume that T has k leaves. As we have noticed, T has $k-1$ degree two vertices. Thus it contains $n+1-2k$ degree one vertices. Therefore, the total weight of the trees in $\alpha(T)$ amounts to

$$w(T) = (xy)^k (x+y)^{n+1-2k}. \tag{3.5}$$

The above relation reveals that the set of permutations of $[n]$ can be partitioned into classes with each class corresponding to a 0-1-2 increasing plane tree and the weighted sum as in (3.5), which means that we only need a subgroup of the Foata-Strehl group for the purpose of the γ-expansion of the Eulerian polynomials.

We note that 0-1-2 increasing plane trees with the above labeling scheme can be represented as increasing binary trees. For each labeled tree in $\alpha(T)$, we can represent it by an increasing binary tree on $[n]$. For a degree one vertex v, if it is labeled by x, then we turn its child into a left child as in a binary tree, otherwise, we turn it into a right child as in a binary tree. By the classical bijection between permutations and increasing binary trees, we find that the labeling of a 0-1-2 increasing plane tree is suitable for keeping track of the number of descents of a permutation. That is to say, the grammatical labeling of 0-1-2 increasing plane trees provides combinatorial justification of the relation of Foata-Schützenberger back to the original form.

4 The Second-order Eulerian Polynomials

Gessel and Stanley [14] introduced the notion of Stirling permutations and defined the second-order Eulerian polynomials $C_n(x)$ by $C_0(x) = 1$ and for $n \geq 1$,

$$C_n(x) = \sum_{k=1}^{n} C(n,k)x^k, \tag{4.1}$$

where $C(n,k)$ is the number of Stirling permutations on $[n]_2$ with k descents. A homogeneous version of $C_n(x)$ is given by

$$C_n(x,y) = \sum_{k=1}^{n} C(n,k)x^k y^{2n+1-k}. \tag{4.2}$$

Let Q_n be the set of Stirling permutations of $[n]_2$. Bona [2] introduced the statistic plat, that is the number of plateaux, of a Stirling permutation in Q_n, and proved that the numbers of ascents, descents and plateaux have the same distribution over Q_n for $n \geq 1$.

Janson [17] deduced the symmetry of the joint distribution of the statistics $\mathrm{asc}(\sigma)$, $\mathrm{des}(\sigma)$ and $\mathrm{plat}(\sigma)$ over Q_n for $n \geq 1$. More precisely, they defined the trivariate generating function

$$C_n(x,y,z) = \sum_{\sigma \in Q_n} x^{\mathrm{des}(\sigma)} y^{\mathrm{asc}(\sigma)} z^{\mathrm{plat}(\sigma)}, \tag{4.3}$$

and showed that for $n \geq 1$, $C_n(x,y,z)$ is symmetric in x, y, z.

As a symmetric function, $C_n(x,y,z)$ can be expressed as a polynomial in the elementary symmetric functions in x, y, z. If the coefficients of the polynomial are all nonnegative, we say that the symmetric function is e-positive, see Stanley [28]. We shall show that for $n \geq 1$, $C_n(x,y,z)$ is e-positive along with a combinatorial interpretation of the coefficients. The grammatical approach turns out to be a suitable setting for this task.

Let G be the following grammar

$$G = \{x \rightarrow xyz, \quad y \rightarrow xyz, \quad z \rightarrow xyz\}. \tag{4.4}$$

Let D denote the formal derivative with respect to G. It has been shown by Dumont [8] and Haglund-Visontai [15] that for $n \geq 1$,

$$C_n(x,y,z) = D^n(x). \tag{4.5}$$

For $n \geq 1$, assume that

$$C_n(x,y,z) = \sum_{i+2j+3k=2n+1} \gamma_{i,j,k}(x+y+z)^i(xy+xz+yz)^j(xyz)^k. \tag{4.6}$$

Let

$$u = x + y + z, \quad v = xy + xz + yz, \quad w = xyz.$$

Then we have

$$D(u) = 3w, \quad D(v) = 2uw, \quad D(w) = vw. \tag{4.7}$$

It follows that the γ-coefficients in (4.6) are nonnegative. In other words, $C_n(x,y,z)$ is e-positive.

The main objective of this paper is to give a combinatorial interpretation of the coefficients $\gamma_{i,j,k}$ in (4.6). The relations in (4.7) prompt us to define the grammar

$$H = \{u \to 3w, \quad v \to 2uw, \quad w \to vw\}. \tag{4.8}$$

Theorem 4.1. *For $n \geq 1$ and $i + 2j + 3k = 2n + 1$, the coefficient $\gamma_{i,j,k}$ in the expansion (4.6) of $C_n(x,y,z)$ equals the number of 0-1-2-3 increasing plane trees on $[n]$ with k leaves, j degree one vertices and i degree two vertices.*

Proof. Let T be a 0-1-2-3 increasing plane tree on $[n]$. We first give a labeling of T as follows. Label a leaf by w, a degree one vertex by v, a degree two vertex by u and a degree three vertex by 1. Given any 0-1-2-3 increasing plane tree T on $[n]$ with k leaves, j degree one vertices and i degree two vertices, it has $n - i - j - k$ vertices of degree three. Taking the number of edges into consideration, we get

$$3(n - i - j - k) + 2i + j = n - 1.$$

Thus we have verified that

$$i + 2j + 3k = 2n + 1.$$

Let us examine how to generate a 0-1-2-3 increasing plane tree T' on $[n+1]$ by adding $n+1$ to T as a leaf. We can add $n+1$ to T only as a child of a vertex r that is not of degree three. Hence there are three cases.

Case 1: The vertex r is a leaf with label w. In the resulting tree T', r becomes a degree one vertex with label v and $n+1$ becomes a leaf with label w. This operation corresponds to the substitution $w \to vw$.

Case 2: The vertex r is a degree one vertex with label v. In this case, $n+1$ can be attached to r either as the first child, or the second child. In either case, in the resulting tree T', r becomes a degree two vertex with label u and $n+1$ becomes a leaf with label w. This operation corresponds to the substitution $v \to 2uw$.

Case 3: The vertex r is a degree two vertex with label u. In this case, $n+1$ can be attached to r either as the first child, or the second child, or the third child. In either case, in the resulting tree T', r becomes a degree three vertex with label 1 and $n+1$ becomes a leaf with label w. This operation corresponds to the substitution $u \to 3w$.

The aforementioned three cases exhaust all the possibilities to construct a 0-1-2-3 increasing plane tree T' on $[n+1]$ from a 0-1-2-3 increasing plane tree T on $[n]$ by adding $n+1$ as a leaf. Since each case corresponds to an application of a substitution rule in H, we see that for $n \geq 1$, $D^n(x)$ equals the sum of the weights of 0-1-2-3 increasing plane trees on $[n]$, that is,

$$D^n(w) = \sum_{i+2j+3k=2n+1} \gamma_{i,j,k} u^i v^j w^k. \tag{4.9}$$

Therefore, $\gamma_{i,j,k}$ equals the number of 0-1-2-3 increasing plane trees on $[n]$ with k leaves, j degree one vertices and i degree two vertices. ∎

Figure 2 is an illustration of a 0-1-2-3 increasing plane tree on $[10]$.

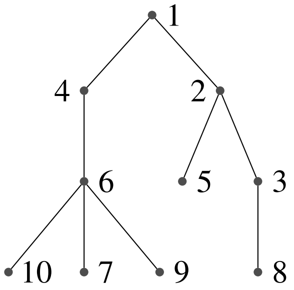

Figure 2: A 0-1-2-3 increasing plane tree.

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